Adaptive Recursive Decentralized Cooperative Localization for Multi-Robot Systems with Time-Varying Measurement Accuracy

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Abstract

Decentralized cooperative localization (DCL) is a promising method to determine accurate multi-robot poses (i.e., positions and orientations) for robot teams operating in an environment without absolute navigation information. Existing DCL methods often use fixed measurement noise covariance matrices for multi-robot pose estimation, however, their performance degrades when the measurement noise covariance matrices are time-varying. To address this problem, in this paper, a novel adaptive recursive DCL method is proposed for multi-robot systems with time-varying measurement accuracy. Each robot estimates its pose and measurement noise covariance matrices simultaneously in a decentralized manner based on the constructed hierarchical Gaussian models using the variational Bayesian approach. Simulation and experimental results show that the proposed method has improved cooperative localization

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accuracy and estimation consistency but slightly heavier computational load than the existing recursive DCL method.

Index Terms

Decentralized cooperative localization, extended Kalman filter, adaptive filter, variational Bayesian, multi-robot systems

I. INTRODUCTION

A. Background

Multi-robot systems have been widely used in many applications, such as area exploration [1], [2], region surveillance [3], and fast search and rescue [4]. Accurate positions and orientations (poses) of all the robots are essential for the success of their operation in performing various tasks [5], [6]. For an environment with accurate and persistent absolute¹ measurements, it is easy to achieve highly accurate estimates of robots' poses. For example, if the global position system (GPS) signal is accessible, the resultant integration of odometry and GPS can provide accurate estimates of robots' poses. However, for an environment with intermittent absolute measurements or without absolute measurements, it is a challenge to achieve accurate estimates of poses [7]. Cooperative localization (CL) is a promising technique to improve the localization accuracy of multi-robot systems for the above scenarios [8], [9]. In the CL, each robot is equipped with the proprioceptive and exteroceptive sensors, and the proprioceptive sensors are used to measure the self-motion of each robot, and the exteroceptive sensors are employed to provide the occasionally accessible absolute measurements and the relative measurements between any two mobile robots. In addition, each robot can share its pose information with other robots through a wireless communication module. With CL, the accurate absolute localization information obtained by some team members can be spread to other members through information exchanges and relative measurements, based on which the multi-robot systems can achieve improved localization accuracy [8], [10].

¹The word "absolute" means that the measurements are taken relative to accurately known reference positions.

B. Related works

The Bayesian filtering technique plays an important role to fuse the self-motion measurements, the occasionally accessible absolute measurements, and the relative measurements in the CL. A large number of Bayesian filters have been specifically designed for the CL, such as the extended Kalman filter (EKF) [8], the particle filter [11], [12], the maximum likelihood estimation algorithm [13], and the maximum *a posteriori* estimation algorithm [14]. Among these filtering methods, the EKF has been a popular choice due to its satisfactory estimation accuracy, ease of implementation, and relatively low computational complexity [15]. Under the assumption that the motion modelling noise and the absolute and relative measurement noises all have Gaussian distributions, the EKF can achieve almost optimal pose estimates of all team members for overall multi-robot systems (except for the first-order linearization errors) [16]. So far, there are two kinds of EKF-based CL (CL-EKF) schemes: the EKF-based centralized CL (CCL-EKF) [8], [10], [17]–[21] and the EKF-based decentralized CL (DCL-EKF) [7], [22]–[25].

In the CCL-EKF method, the self-motion measurements (e.g., linear and rotational velocities) and the occasionally accessible absolute and relative measurements (e.g., relative position, bearing, and range) are all sent to a fusion center, based on which the poses of all robots are jointly estimated using the EKF. Although this CL method is able to achieve almost optimal and completely consistent² pose estimates of all robots under assumptions of Gaussian motion and measurement noises, it requires centralized implementations of the EKF and global communications between the fusion center and all robots in the motion and measurement updates. As a result, the CCL-EKF method is prone to failure caused by the local communication faults and suffers from substantial computational complexities, which may limit its practical applications. To improve the CCL-EKF method, some distributed centralized-equivalent CL (DCECL) methods have been proposed [8], [10], [17]–[21]. Roumeliotis et al. firstly proposed a DCECL method [8], in which the cross-correlation of any two robots is decomposed into two half cross-correlations, and the two robots maintain and update the two half cross-correlations, respectively, and each robot implements the motion update in a distributed manner. This DCECL

²Suppose that $\hat{\mathbf{x}}$ and $\mathbf{P}$ are, respectively, the estimate and estimated error covariance matrix of a random state vector $\mathbf{x}$. The state estimate pair $(\hat{\mathbf{x}}, \mathbf{P})$ is consistent if the difference between the estimated error covariance matrix and the real error covariance matrix is positive semi-definite, i.e., $\mathbf{P} - \mathbb{E}[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T] \geq \mathbf{0}$, and completely consistent if the estimated error covariance matrix is identical to the real error covariance matrix, i.e., $\mathbf{P} = \mathbb{E}[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T]$.

method can save communications and computations if the motion update frequency is greater than the measurement update frequency. However, it still requires global communications among all robots. To further alleviate the communication requirements, Kia et al. proposed an improved DCECL method, where new intermediate local variables are introduced to achieve decentralized implementations of the EKF [10], [21]. Although this method only requires local communications with the adjacent team members, it imposes an additional constraint that the multi-robot systems need to have a communication graph with a spanning tree rooted at the interim master in every time step, which limits the scope of CL for multi-robot systems.

As a popular alternative to the CL-EKF, the DCL-EKF method can solve the above problems inherent in the CCL-EKF methods, in which each robot serves as a local fusion center and the CL is implemented in a decentralized manner. However, the DCL-EKF method suffers from an important problem of double-counting which treats correlated pose estimates as uncorrelated and uses shared measurements repeatedly so that inconsistent pose estimates of all robots are thereby induced [26]. Covariance intersection (CI) is a standard technique to address this problem [27]–[29]. In the CI technique, the effects of unknown cross-correlations are alleviated by inflating the measurement noise variances of all sensors to obtain more consistent state estimates [27], [28]. Many DCL-EKF methods that exploit the CI-technique (DCL-EKF-CI) have been proposed [7], [22], [23]. Although these DCL-EKF-CI methods can guarantee estimation consistency, their estimated error covariance matrices of pose estimates are overly inflated (i.e., overly conservative) due to the inflations of measurement noise variances, which results in poor CL accuracy. Another strategy for addressing the problem of double-counting is to estimate cross-correlations. Zhu et al. proposed an improved DCL-EKF-CI method by estimating and compensating for the unknown cross-correlations [24]. However, this method suffers from heavy computational burdens incurred by a constrained numerical optimization problem. On the other hand, Luft et al. proposed a recursive DCL-EKF (RDCL-EKF) method, in which every cross-correlation is factored as two half cross-correlations³, and each robot estimates its pose and half cross-correlations with the other robots recursively [25]. As compared with the improved DCL-EKF-CI method [24], the RDCL-EKF method has better CL accuracy and smaller computational costs. Although the

³Suppose that Σ^{xy} is the cross-correlation of random variables $\mathbf{x}$ and $\mathbf{y}$. If the cross-correlation Σ^{xy} can be factored as $\Sigma^{xy} = \Sigma^{xy} (\Sigma^{yx})^T$, then both Σ^{xy} and Σ^{yx} are termed half cross-correlations of random variables $\mathbf{x}$ and $\mathbf{y}$.

RDCL-EKF method cannot guarantee that the pose estimates are strictly consistent, it exhibits satisfactory estimation consistency [25]. Unfortunately, its estimation accuracy and consistency degrade substantially when inaccurate measurement noise covariance matrices are exploited, as will be detailed in the next subsection.

C. Motivations for paper

The absolute and relative measurement noise covariance matrices are important parameters for implementing the RDCL-EKF method, which will be shown in our simulation and experimental study. In practical multi-robot CL applications, the absolute and relative measurement noise covariance matrices are commonly selected as fixed nominal values according to engineering experience. However, the measurement noise covariance matrices may be time-varying, which cannot be matched by fixed nominal values. For example, the measurement accuracy of a camera depends heavily on the distance between the camera and the measured target, and the greater the distance, the worse the measurement accuracy that is achieved and vice versa. As a result, the RDCL-EKF will exhibit poor CL accuracy and estimation consistency when inaccurate nominal absolute and relative measurement noise covariance matrices are used. Although some adaptive Kalman filters [30]–[39] have been proposed to estimate inaccurate noise covariance matrices, these methods are not suitable for the DCL of multi-robot systems since they are specially designed for a single linear system. To the best of the authors' knowledge, the adaptive DCL methods for multi-robot systems have not been previously reported.

D. Contributions and organizations of paper

To improve the performance of the existing RDCL-EKF method for multi-robot systems with time-varying measurement accuracy, a novel adaptive RDCL-EKF method is proposed in this paper, in which each robot estimates its pose and absolute and relative measurement noise covariance matrices simultaneously in a decentralized manner. The measurement noise covariance matrices are modelled as inverse-Wishart distributed, from which hierarchical Gaussian models for multi-robot systems can be constructed. For each robot, the posterior probability density functions (PDFs) of its pose and measurement noise covariance matrices are mutually coupled under the constructed hierarchical Gaussian models, and the variational Bayesian (VB) approach is therefore employed to obtain approximate joint estimates. The proposed adaptive RDCL-EKF

method has the same communication mode as the standard RDCL-EKF method, but only requires slightly higher communication overhead to transmit additional parameters in each cooperation. A large number of simulation and experimental results demonstrate that the proposed adaptive RDCL-EKF method has better CL accuracy and estimation consistency but slightly heavier computational load than the existing RDCL-EKF method for multi-robot systems with time-varying measurement accuracy.

This paper is organized as follows. In Section II, the multi-robot CL model and problem statement are presented. In Section III, the proposed adaptive RDCL-EKF method is derived, and its detailed implementations are given. The simulation and experiment studies are, respectively, provided in Sections IV and V. Conclusions are drawn in Section VI.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Multi-robot CL models

Consider a cluster of n_R homogeneous or heterogeneous mobile robots moving in a two dimensional environment with n_L known landmarks. Every mobile robot installs proprioceptive sensors (e.g., wheel-encoders) and exteroceptive sensors (e.g., ultra wide bands (UWB), cameras or laser scanners), in which the proprioceptive sensors are used to measure self-motion of every mobile robot, and the exteroceptive sensors are employed to collect the absolute measurements from the robot to the landmarks or relative measurements from the robot to other robots. The nonlinear discrete-time motion model of robot i can be formulated as follows [7]

$$\begin{cases} x_k^i = x_{k-1}^i + (V_k^i - w_k^{V_i}) \cos(\phi_{k-1}^i) \Delta t \\ y_k^i = y_{k-1}^i + (V_k^i - w_k^{V_i}) \sin(\phi_{k-1}^i) \Delta t \\ \phi_k^i = \phi_{k-1}^i + (\Omega_k^i - w_k^{\Omega_i}) \Delta t \end{cases} \quad (1)$$

where x_k^i , y_k^i and ϕ_k^i are, respectively, the east and north positions and orientation of robot i at time k , and $[x_k^i, y_k^i, \phi_k^i]$ denotes the pose of robot i at time k , and V_k^i and Ω_k^i are the measured linear and rotational velocities of robot i at time k provided by the proprioceptive sensors with measurement noises $w_k^{V_i}$ and $w_k^{\Omega_i}$, respectively, and Δt denotes the discretization time.

In this paper, three kinds of absolute and relative measurements are considered, including the relative position, bearing, and range measurements. For robot i , its three kinds of measurement

models from robot i to landmark l are written as [7], [8]

$$\begin{cases} \mathbf{p}_{ak}^{il} = \mathbf{C}^T(\phi_k^i) \begin{bmatrix} x_L^l - x_k^i \\ y_L^l - y_k^i \end{bmatrix} + \mathbf{v}_{p_{ak}}^{il} \\ \theta_{ak}^{il} = \text{atan2}(y_k^i - y_L^l, x_k^i - x_L^l) - \phi_k^i + v_{\theta_{ak}}^{il} \\ r_{ak}^{il} = \sqrt{(x_k^i - x_L^l)^2 + (y_k^i - y_L^l)^2} + v_{r_{ak}}^{il} \end{cases} \quad (2)$$

where the superscript “T” denotes the transpose operation of a matrix, and the subscript “ ak ” denotes the absolute measurement at time k , and $\mathbf{p}_{ak}^{il}$, θ_{ak}^{il} and r_{ak}^{il} are the relative position⁴, bearing, and range measurements from robot i to landmark l at time k measured by the exteroceptive sensors with measurement noises $\mathbf{v}_{p_{ak}}^{il}$, $v_{\theta_{ak}}^{il}$ and $v_{r_{ak}}^{il}$, respectively, and x_L^l and y_L^l denote the east and north positions of landmark l with the subscript “ L ” denoting the landmark, and $\mathbf{C}(\phi_k^i)$ denotes the direction cosine matrix from the local framework of robot i to the global framework, which is given by

$$\mathbf{C}(\phi_k^i) = \begin{bmatrix} \cos(\phi_k^i) & -\sin(\phi_k^i) \\ \sin(\phi_k^i) & \cos(\phi_k^i) \end{bmatrix} \quad (3)$$

Note that the landmarks are all assumed to be static, and their positions are accurately known in the models (2).

Similarly, for robot i , its three kinds of relative measurement models to robot j are given by [7], [8]

$$\begin{cases} \mathbf{p}_{rk}^{ij} = \mathbf{C}^T(\phi_k^i) \begin{bmatrix} x_k^j - x_k^i \\ y_k^j - y_k^i \end{bmatrix} + \mathbf{v}_{p_{rk}}^{ij} \\ \theta_{rk}^{ij} = \text{atan2}(y_k^i - y_k^j, x_k^i - x_k^j) - \phi_k^i + v_{\theta_{rk}}^{ij} \\ r_{rk}^{ij} = \sqrt{(x_k^i - x_k^j)^2 + (y_k^i - y_k^j)^2} + v_{r_{rk}}^{ij} \end{cases} \quad (4)$$

where the subscript “ rk ” denotes the relative measurement at time k , and the superscript “ ij ” means that robot i is relative to robot j , and $\mathbf{p}_{rk}^{ij}$, θ_{rk}^{ij} and r_{rk}^{ij} are the relative position, bearing, and range measurements of robot i to robot j at time k measured by the exteroceptive sensors with measurement noises $\mathbf{v}_{p_{rk}}^{ij}$, $v_{\theta_{rk}}^{ij}$ and $v_{r_{rk}}^{ij}$, respectively, and x_k^j and y_k^j are, respectively, the east and north positions of robot j at time k .

⁴Here, the relative position $\mathbf{p}_{ak}^{il}$ is an absolute measurement since it is taken from robot i to landmark l whose position is accurately known.

TABLE I: Local motion update for robot i [25].

Function 1: $\left(\hat{\mathbf{x}}_{k k-1}^i, \Sigma_{k k-1}^i, \left\{ \sigma_{k k-1}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}} \right) = \text{LMU} \left(\hat{\mathbf{x}}_{k-1 k-1}^i, \Sigma_{k-1 k-1}^i, \left\{ \sigma_{k-1 k-1}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}} , \mathbf{Q}_k^i \right)$
Calculate:
1. Predicted state vector (pose): $\hat{\mathbf{x}}_{k k-1}^i = \mathbf{f}_k^i(\hat{\mathbf{x}}_{k-1 k-1}^i, \mathbf{0})$
2. State Jacobi matrix: $\mathbf{F}_k^i = \frac{\partial \mathbf{f}_k^i(\mathbf{x}_{k-1}^i, \mathbf{w}_k^i)}{\partial \mathbf{x}_{k-1}^i}(\hat{\mathbf{x}}_{k-1 k-1}^i, \mathbf{0})$
3. State noise Jacobi matrix: $\mathbf{G}_k^i = \frac{\partial \mathbf{f}_k^i(\mathbf{x}_{k-1}^i, \mathbf{w}_k^i)}{\partial \mathbf{w}_k^i}(\hat{\mathbf{x}}_{k-1 k-1}^i, \mathbf{0})$
4. Prediction error covariance matrix: $\Sigma_{k k-1}^i = \mathbf{F}_k^i \Sigma_{k-1 k-1}^i (\mathbf{F}_k^i)^\top + \mathbf{G}_k^i \mathbf{Q}_k^i (\mathbf{G}_k^i)^\top$
5. Predicted half cross-correlations:
for $j = 1 : n_R, j \neq i$ do
$\sigma_{k k-1}^{ij} = \mathbf{F}_k^i \sigma_{k-1 k-1}^{ij}$
end for

Define the pose of robot i as its state vector, i.e., $\mathbf{x}_k^i \triangleq [x_k^i, y_k^i, \phi_k^i]^\top$. The motion model and absolute and relative measurement models of robot i can be, respectively, formulated as the following general forms

$$\begin{cases} \mathbf{x}_k^i = \mathbf{f}_k^i(\mathbf{x}_{k-1}^i, \mathbf{w}_k^i) \\ \mathbf{z}_{ak}^{il} = \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l) + \mathbf{v}_{ak}^{il} \\ \mathbf{z}_{rk}^{ij} = \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j) + \mathbf{v}_{rk}^{ij} \end{cases} \quad (5)$$

where $\mathbf{z}_{ak}^{il} \in \mathbf{R}^{n_a}$ and $\mathbf{z}_{rk}^{ij} \in \mathbf{R}^{n_r}$ are, respectively, the absolute and relative measurement vectors to landmark l and robot j , and $\mathbf{x}_L^l \triangleq [x_L^l, y_L^l]^\top$ is the position of landmark l , and $\mathbf{f}_k^i(\cdot)$, $\mathbf{h}_{ak}^{il}(\cdot)$ and $\mathbf{h}_{rk}^{ij}(\cdot)$ are, respectively, the state evolution function and the absolute and relative measurement functions, which are correspondingly defined according to (1), (2) and (4), and $\mathbf{w}_k^i \triangleq [w_k^{V_i}, w_k^{\Omega_i}]^\top$, $\mathbf{v}_{ak}^{il}$ and $\mathbf{v}_{rk}^{ij}$ are, respectively, the state noise and the absolute and relative measurement noises. In this paper, $\mathbf{w}_k^i$, $\mathbf{v}_{ak}^{il}$ and $\mathbf{v}_{rk}^{ij}$ are all assumed to be zero-mean Gaussian white distributed, i.e., $\mathbf{w}_k^i \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k^i)$, $\mathbf{v}_{ak}^{il} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{ak}^{il})$ and $\mathbf{v}_{rk}^{ij} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{rk}^{ij})$, where $\mathbf{Q}_k^i$, $\mathbf{R}_{ak}^{il}$ and $\mathbf{R}_{rk}^{ij}$ are, respectively, the state noise covariance matrix and the absolute and relative measurement noise covariance matrices. Furthermore, $\mathbf{w}_k^i$, $\mathbf{v}_{ak}^{il}$ and $\mathbf{v}_{rk}^{ij}$ are all assumed to be uncorrelated with each other.

TABLE II: Local absolute measurement update for robot i [25].

Function 2: $\left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij(a)} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}} \right) = \text{LAMU} \left(\hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i, \left\{ \sigma_{k|k-1}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}}, l_k^{i*}, \left\{ \mathbf{z}_{ak}^{il}, \mathbf{R}_{ak}^{il}, \mathbf{x}_L^l \right\}_{\{l \in l_k^{i*}\}} \right)$

If the detected landmark set l_k^{i*} of robot i is non-empty:

Construct:

1. Augmented absolute measurement: $\mathbf{z}_{ak}^{il_k^{i*}} = \text{col} \left\{ \mathbf{z}_{ak}^{il} \right\}_{\{l \in l_k^{i*}\}}$
2. Augmented absolute measurement noise covariance matrix: $\mathbf{R}_{ak}^{il_k^{i*}} = \text{diag} \left\{ \mathbf{R}_{ak}^{il} \right\}_{\{l \in l_k^{i*}\}}$

Calculate:

3. Augmented absolute measurement Jacobi matrix: $\mathbf{H}_{ak}^{il_k^{i*}} = \text{row} \left\{ \mathbf{H}_{ak}^{il} \right\}_{\{l \in l_k^{i*}\}}$, where $\mathbf{H}_{ak}^{il} = \frac{\partial \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l)}{\partial \mathbf{x}_k^i} (\hat{\mathbf{x}}_{k|k-1}^i)$
4. Kalman gain of augmented absolute measurement: $\mathbf{K}_{ak}^{il_k^{i*}} = \Sigma_{k|k-1}^i \left(\mathbf{H}_{ak}^{il_k^{i*}} \right)^T \left[\mathbf{H}_{ak}^{il_k^{i*}} \Sigma_{k|k-1}^i \left(\mathbf{H}_{ak}^{il_k^{i*}} \right)^T + \mathbf{R}_{ak}^{il_k^{i*}} \right]^{-1}$
5. Predicted augmented absolute measurement: $\hat{\mathbf{z}}_{ak|k-1}^{il_k^{i*}} = \text{col} \left\{ \hat{\mathbf{z}}_{ak|k-1}^{il} \right\}_{\{l \in l_k^{i*}\}}$, where $\hat{\mathbf{z}}_{ak|k-1}^{il} = \mathbf{h}_{ak}^{il} (\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l)$
6. State estimate: $\hat{\mathbf{x}}_{k|k}^{i(a)} = \hat{\mathbf{x}}_{k|k-1}^i + \mathbf{K}_{ak}^{il_k^{i*}} \left[\mathbf{z}_{ak}^{il_k^{i*}} - \hat{\mathbf{z}}_{ak|k-1}^{il_k^{i*}} \right]$
7. Estimation error covariance matrix of LAMU: $\Sigma_{k|k}^{i(a)} = \left[\mathbf{I} - \mathbf{K}_{ak}^{il_k^{i*}} \mathbf{H}_{ak}^{il_k^{i*}} \right] \Sigma_{k|k-1}^i$
8. Estimated half cross-correlations:

for $j = 1 : n_R, j \neq i$ do

$$\sigma_{k|k}^{ij(a)} = \left[\mathbf{I} - \mathbf{K}_{ak}^{il_k^{i*}} \mathbf{H}_{ak}^{il_k^{i*}} \right] \sigma_{k|k-1}^{ij}$$

end for

If the detected landmark set l_k^{i*} of robot i is empty: $\hat{\mathbf{x}}_{k|k}^{i(a)} = \hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k}^{i(a)} = \Sigma_{k|k-1}^i, \left\{ \sigma_{k|k}^{ij(a)} = \sigma_{k|k-1}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}}$

B. Problem statement

Before presenting the main motivation of this paper, the existing RDCL-EKF algorithm [25] is firstly reviewed. In the RDCL-EKF method, the heart is to estimate beliefs $\left\{ \hat{\mathbf{x}}_{k|k}^i, \Sigma_{k|k}^i \right\}_{\{1 \leq i \leq n_R\}}$ and cross-correlations $\left\{ \sigma_{k|k}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}}$ of all robots based on a decentralized strategy, where the cross-correlation $\Sigma_{k|k}^{ij}$ is factored as two half cross-correlations, i.e., $\Sigma_{k|k}^{ij} = \sigma_{k|k}^{ij} \left(\sigma_{k|k}^{ji} \right)^T$, and the half cross-correlations $\sigma_{k|k}^{ij}$ and $\sigma_{k|k}^{ji}$ are recursively estimated and maintained by robots i and j , respectively. For robot i , it requires to estimate its own belief $\left\{ \hat{\mathbf{x}}_{k|k}^i, \Sigma_{k|k}^i \right\}$ and half cross-correlations $\left\{ \sigma_{k|k}^{ij} \right\}_{\substack{\{j \neq i\} \\ \{1 \leq j \leq n_R\}}}$ with the remaining $(n_R - 1)$ robots. The RDCL-EKF algorithm consists of three parts: the local motion update (LMU), the local absolute measurement update (LAMU), and the cooperative relative measurement update (CRMU), which are, respectively, summarized as three functions in Table I–Table III. In Table II–Table III, l_k^{i*} and j_k^{i*} , respectively,

TABLE III: Cooperative relative measurement update for robot i [25].

Function 3: $\left(\hat{\mathbf{x}}_{k|k}^i, \Sigma_{k|k}^i, \left\{ \sigma_{k|k}^{ij} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}} \right) = \text{CRMU} \left(\hat{\mathbf{x}}_{k|k}^{(a)}, \Sigma_{k|k}^{(a)}, \left\{ \sigma_{k|k}^{ij(a)} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, j_k^{i*}, \left\{ \mathbf{z}_{rk}^{ij}, \mathbf{R}_{rk}^{ij} \right\}_{j \in j_k^{i*}} \right)$

If the detected robot set j_k^{i*} of robot i is non-empty:

1. Receive information from robots in j_k^{i*} : $\left\{ \hat{\mathbf{x}}_{k|k}^{j(a)}, \Sigma_{k|k}^{j(a)}, \sigma_{k|k}^{js(a)} \right\}_{\substack{s \neq j \in j_k^{i*} \\ 1 \leq s \leq n_R}}$

Construct:

2. Augmented state estimate of robots in j_k^{i*} : $\hat{\mathbf{x}}_{k|k}^{ij_k^{i*}(a)} = \text{col} \left\{ \hat{\mathbf{x}}_{k|k}^{j(a)} \right\}_{j \in j_k^{i*}}$
3. Augmented estimation error covariance matrix of robots in j_k^{i*} : $\Sigma_{k|k}^{ij_k^{i*}(a)} = \text{block} \left\{ \Sigma_{k|k}^{j(a)}, \Sigma_{k|k}^{js(a)} \right\}_{\substack{s \neq j \\ \{s, j \in j_k^{i*}\}}}$,
where $\Sigma_{k|k}^{js(a)} = \sigma_{k|k}^{js(a)} \left(\sigma_{k|k}^{js(a)} \right)^T$.
4. Augmented relative measurement: $\mathbf{z}_{rk}^{ij_k^{i*}} = \text{col} \left\{ \mathbf{z}_{rk}^{ij} \right\}_{j \in j_k^{i*}}$
5. Augmented relative measurement noise covariance matrix: $\mathbf{R}_{rk}^{ij_k^{i*}} = \text{diag} \left\{ \mathbf{R}_{rk}^{ij} \right\}_{j \in j_k^{i*}}$

Calculate:

6. Cross-correlation of robot i and robots in j_k^{i*} : $\Sigma_{k|k}^{ij_k^{i*}(a)} = \text{row} \left\{ \Sigma_{k|k}^{ij(a)} \right\}_{j \in j_k^{i*}}$, where $\Sigma_{k|k}^{ij(a)} = \sigma_{k|k}^{ij(a)} \left(\sigma_{k|k}^{ji(a)} \right)^T$
7. Augmented estimation error covariance matrix of robot i and robots in j_k^{i*} : $\Sigma_{k|k}^{(a)} = \begin{bmatrix} \Sigma_{k|k}^i & \Sigma_{k|k}^{ij_k^{i*}(a)} \\ \left(\Sigma_{k|k}^{ij_k^{i*}(a)} \right)^T & \Sigma_{k|k}^{ij_k^{i*}(a)} \end{bmatrix}$
8. Augmented measurement Jacobi matrix of robot i and robots in j_k^{i*} : $\mathbf{H}_{rk}^{ij_k^{i*}} = \text{row} \left\{ \mathbf{H}_{rk}^i, \mathbf{H}_{rk}^j \right\}_{j \in j_k^{i*}}$,
where $\mathbf{H}_{rk}^i = \frac{\partial \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j)}{\partial \mathbf{x}_k^i} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right)$ and $\mathbf{H}_{rk}^j = \frac{\partial \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j)}{\partial \mathbf{x}_k^j} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right)$
9. Kalman gain of augmented relative measurement: $\mathbf{K}_{rk}^{ij_k^{i*}} = \Sigma_{k|k}^{(a)} \left(\mathbf{H}_{rk}^{ij_k^{i*}} \right)^T \left[\mathbf{H}_{rk}^{ij_k^{i*}} \Sigma_{k|k}^{(a)} \left(\mathbf{H}_{rk}^{ij_k^{i*}} \right)^T + \mathbf{R}_{rk}^{ij_k^{i*}} \right]^{-1}$
10. Predicted augmented relative measurement: $\hat{\mathbf{z}}_{rk|k-1}^{ij_k^{i*}} = \text{col} \left\{ \hat{\mathbf{z}}_{rk|k-1}^{ij} \right\}_{j \in j_k^{i*}}$, where $\hat{\mathbf{z}}_{rk|k-1}^{ij} = \mathbf{h}_{rk}^{ij} \left(\hat{\mathbf{x}}_{k|k-1}^{i(a)}, \hat{\mathbf{x}}_{k|k-1}^{j(a)} \right)$
11. State estimate: $\begin{bmatrix} \hat{\mathbf{x}}_{k|k}^i \\ \hat{\mathbf{x}}_{k|k}^{ij_k^{i*}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{ij_k^{i*}(a)} \end{bmatrix} + \mathbf{K}_{rk}^{ij_k^{i*}} \left[\mathbf{z}_{rk}^{ij_k^{i*}} - \hat{\mathbf{z}}_{rk|k-1}^{ij_k^{i*}} \right]$
12. Estimation error covariance matrix: $\begin{bmatrix} \Sigma_{k|k}^i & \Sigma_{k|k}^{ij_k^{i*}} \\ \left(\Sigma_{k|k}^{ij_k^{i*}} \right)^T & \Sigma_{k|k}^{ij_k^{i*}} \end{bmatrix} = \left(\mathbf{I} - \mathbf{K}_{rk}^{ij_k^{i*}} \mathbf{H}_{rk}^{ij_k^{i*}} \right) \Sigma_{k|k}^{(a)}$
13. Estimated half cross-correlations of robot i : $\left\{ \sigma_{k|k}^{ij} = \sigma_{k|k}^{ij} \right\}_{j \in j_k^{i*}}$

for $t = 1 : n_R, t \notin \{i, j_k^{i*}\}$ do

$$\sigma_{k|k}^{it} = \Sigma_{k|k}^i \left(\Sigma_{k|k}^{i(a)} \right)^{-1} \sigma_{k|k}^{it(a)}$$

end for

If robot i is detected by robot j : Send information to robot j : $\hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}, \sigma_{k|k}^{ij(a)}$; Receive information from robot j : $\hat{\mathbf{x}}_{k|k}^j, \Sigma_{k|k}^j, \sigma_{k|k}^{ij} = \mathbf{I}$; Calculate the estimated half cross-correlations of robot i :

for $t = 1 : n_R, t \notin \{i, j\}$ do

$$\sigma_{k|k}^{it} = \Sigma_{k|k}^i \left(\Sigma_{k|k}^{i(a)} \right)^{-1} \sigma_{k|k}^{it(a)}$$

end for

If robot i doesn't cooperate with any robots: $\hat{\mathbf{x}}_{k|k}^i = \hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^i = \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij} = \sigma_{k|k}^{ij(a)} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}$

Fig. 1: Diagram of existing RDCL-EKF algorithm for robot i .

denote the landmark set and the robot set detected by robot i at time k , and $\text{row}\{\mathbf{a}_l\}_{l \in \Gamma}$, $\text{col}\{\mathbf{a}_l\}_{l \in \Gamma}$, and $\text{diag}\{\mathbf{a}_l\}_{l \in \Gamma}$ denote the block row matrix, the block column matrix, and the block diagonal matrix, respectively, which are composed of all matrices in the set $\{\mathbf{a}_l | l \in \Gamma\}$.

It is seen from Table II–Table III that the absolute and relative measurement noise covariance matrices $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$ are, respectively, the inputs of the LAMU and the CRMU functions. Both the state estimate $\hat{\mathbf{x}}_{k|k}^i$ and estimation error covariance matrix $\Sigma_{k|k}^i$ depend on $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$, and then the state estimation accuracy relies on the accuracies of $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$. Meanwhile, the half cross-correlations $\{\sigma_{k|k}^{ij}\}_{1 \leq j \leq n_R}^{j \neq i}$ depend on $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$, which means that the accuracies of $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$ also have effects on the estimation consistency. In practical multi-robot CL applications, the measurement accuracy of exteroceptive sensors may be time-varying. For example, the measurement accuracy of a camera depends heavily on the distance between the camera and the measured target, and the greater the distance, the worse the measurement accuracy that is achieved and vice versa. However, the absolute and relative measurement noise covariance matrices $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$ are commonly selected as fixed nominal values $\{\bar{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\bar{\mathbf{R}}_{rk}^{ij}\}_{j \in j_k^{i*}}$ according to engineering experience, as shown in Fig. 1. As a result, poor CL accuracy and serious estimation inconsistency will be induced by the use of inaccurate nominal values $\{\bar{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\bar{\mathbf{R}}_{rk}^{ij}\}_{j \in j_k^{i*}}$ in the LAMU and CRMU functions, which represents the main motivation of this paper.

Fig. 2: All absolute measurement noise covariance matrices for the entire system.

Fig. 3: All relative measurement noise covariance matrices for the entire system.

III. A NOVEL ADAPTIVE RDCL-EKF METHOD

A. Design idea of the proposed method

In this paper, to improve the CL accuracy and estimation consistency, a novel adaptive RDCL-EKF method is proposed, in which the absolute and relative measurement noise covariance matrices are adaptively estimated based on the VB approach. For multi-robot systems, any mobile robot i may detect any landmark l , and the absolute measurement noise covariance matrix $\mathbf{R}_{ak}^{il}$ is required in the LAMU if the mobile robot i detects the landmark l , where $1 \leq i \leq n_R$ and $1 \leq l \leq n_L$. Since each mobile robot has different running statuses and each landmark has different positions, $n_R \times n_L$ absolute measurement noise covariance matrices must be adaptively estimated for the entire system, which are shown in Fig. 2. It is seen from Fig. 2 that all

Fig. 4: Diagram of the proposed decentralized estimation strategy for $\{\mathbf{R}_{rk}^{ij}, \mathbf{R}_{rk}^{ji}\}$.

absolute measurement noise covariance matrices are mutually independent so that they can be separately estimated. On the other hand, any mobile robot i may detect any mobile robot j , and the relative measurement noise covariance matrix $\mathbf{R}_{rk}^{ij}$ is required in the CRMU if the mobile robot i detects the mobile robot j , where $1 \leq i, j \leq n_R$ and $i \neq j$. Considering that $\mathbf{R}_{rk}^{ij}$ and $\mathbf{R}_{rk}^{ji}$ are identical, then only $n_R(n_R - 1)/2$ relative measurement noise covariance matrices are unknown for the entire system, which are illustrated in Fig. 3. An intuitive method is that the robots i and j estimate the relative measurement noise covariance matrix $\mathbf{R}_{rk}^{ij}$ ($i < j$) together based on the measurements $\mathbf{z}_{rk}^{ij}$ and $\mathbf{z}_{rk}^{ji}$. However, a fusion center is required to collect the relative measurements $\mathbf{z}_{rk}^{ij}$ and $\mathbf{z}_{rk}^{ji}$ for estimating $\mathbf{R}_{rk}^{ij}$, which violates the original intention of recursive DCL (RDCL).

To address the above problem, in this paper, a decentralized estimation strategy is proposed to estimate the relative measurement noise covariance matrices, in which each pair of relative measurement noise covariance matrices $\{\mathbf{R}_{rk}^{ij}, \mathbf{R}_{rk}^{ji}\}$ ($i < j$) is sequentially estimated and updated. That is to say, if robot j is first detected by robot i , then $\mathbf{R}_{rk}^{ij}$ is estimated based on the relative measurement $\mathbf{z}_{rk}^{ij}$ and the estimation information of $\mathbf{R}_{rk}^{ij}$ is sent from robot i to robot j . Next, if robot i is detected by robot j , then $\mathbf{R}_{rk}^{ji}$ is estimated based on the relative measurement $\mathbf{z}_{rk}^{ji}$ and the received estimation information of $\mathbf{R}_{rk}^{ij}$. Finally, the estimation information of $\mathbf{R}_{rk}^{ji}$ is sent from robot j to robot i to replace the previous estimation information of $\mathbf{R}_{rk}^{ij}$, and the estimate of $\mathbf{R}_{rk}^{ij}$ is further refined. The diagram of the proposed decentralized estimation strategy for $\{\mathbf{R}_{rk}^{ij}, \mathbf{R}_{rk}^{ji}\}$ is shown in Fig. 4. It can be seen from Fig. 4 that the proposed decentralized estimation strategy

Fig. 5: Diagram of the proposed RDCL-EKF method for robot i .

can guarantee that the constraint $\mathbf{R}_{rk}^{ij} = \mathbf{R}_{rk}^{ji}$ holds, and both the relative measurements $\mathbf{z}_{rk}^{ij}$ and $\mathbf{z}_{rk}^{ji}$ are used to estimate the relative measurement noise covariance matrices $\{\mathbf{R}_{rk}^{ij}, \mathbf{R}_{rk}^{ji}\}$.

The diagram of the proposed RDCL-EKF method for robot i is illustrated in Fig. 5, where $\{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}$ and $\{\hat{\mathbf{R}}_{rk}^{ij}\}_{j \in j_k^{i*}}$ denote the estimated values of the absolute and relative measurement noise covariance matrices, respectively. We can observe from Fig. 5 that, for the proposed method, the absolute and relative measurement noise covariance matrices are adaptively estimated in a sequential manner. Firstly, the state vector (pose) $\mathbf{x}_k^i$ of robot i and the absolute measurement noise covariance matrix $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i*}}$ are jointly estimated based on the previous measurements $\mathbf{z}_{1:k-1}$ and the current absolute measurement $\{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}$ using the VB approach in the local absolute measurement update, where $\mathbf{z}_{1:k-1}$ denotes the set of all available absolute and relative measurements from time 1 to time $k-1$. Then, the state vector (pose) $\mathbf{x}_k^i$ of robot i and the relative measurement noise covariance matrix $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$ are jointly estimated based on the previous measurements $\mathbf{z}_{1:k-1}$, the current absolute measurement $\{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}$, and the current relative measurement $\{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}}$ using the VB approach in the cooperative relative measurement update. As compared with the existing RDCL-EKF method, the proposed method can achieve better CL accuracy and estimation consistency via adaptively estimating the absolute and relative measurement noise covariance matrices. Next, we will present the statistical modelling of the state vector and the measurement noise covariance matrices.

B. Statistical modelling

In the RDCL-EKF framework, the one-step prediction PDF of robot i is approximated as Gaussian, i.e.,

$$p(\mathbf{x}_k^i | \mathbf{z}_{1:k-1}) = \mathcal{N}(\mathbf{x}_k^i; \hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i) \quad (6)$$

where $\hat{\mathbf{x}}_{k|k-1}^i$ and $\Sigma_{k|k-1}^i$ can be obtained using the LMU function in Table I.

According to the absolute measurement model in (5), the absolute measurement likelihood PDF of robot i relative to landmark l can be formulated as

$$p(\mathbf{z}_{ak}^{il} | \mathbf{x}_k^i, \mathbf{R}_{ak}^{il}) = \mathcal{N}(\mathbf{z}_{ak}^{il}; \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l), \mathbf{R}_{ak}^{il}), \quad l \in l_k^{i*} \quad (7)$$

where the absolute measurement noise covariance matrix $\mathbf{R}_{ak}^{il}$ is assumed as a random variable. In Bayesian statistics, the inverse-Wishart distribution is often used as a conjugate prior distribution of the covariance matrix of a Gaussian distribution with known mean value, which can guarantee the conjugate inference under the variational inference framework [40]. Motivated by this fact, the absolute measurement noise covariance matrix $\mathbf{R}_{ak}^{il}$ is modelled as inverse-Wishart distributed

$$p(\mathbf{R}_{ak}^{il} | \mathbf{z}_{1:k-1}) = \text{IW}(\mathbf{R}_{ak}^{il}; u_{ak|k-1}^{il}, \mathbf{U}_{ak|k-1}^{il}), \quad l \in l_k^{i*} \quad (8)$$

and $u_{ak|k-1}^{il}$ and $\mathbf{U}_{ak|k-1}^{il}$ denote the degrees of freedom (dof) parameter and scale matrix of $p(\mathbf{R}_{ak}^{il} | \mathbf{z}_{1:k-1})$, respectively.

In practical DCL for multi-robot systems, the working environment of an exteroceptive sensor is often slowly time-varying. For example, the distance between the camera and the measured target is a slowly time-varying quantity because of the slow robot movement speed. As a result, the absolute measurement noise covariance matrix $\mathbf{R}_{ak}^{il}$, which depends on the distance between the camera and the measured target, is also a slowly time-varying parameter. Motivated by this fact, $u_{ak|k-1}^{il}$ and $\mathbf{U}_{ak|k-1}^{il}$ can be approximately obtained via the following linear propagation model [38]

$$u_{ak|k-1}^{il} = \rho u_{ak-1|k-1}^{il}, \quad \mathbf{U}_{ak|k-1}^{il} = \rho \mathbf{U}_{ak-1|k-1}^{il} \quad (9)$$

where $u_{ak-1|k-1}^{il}$ and $\mathbf{U}_{ak-1|k-1}^{il}$ denote the dof parameter and scale matrix of $p(\mathbf{R}_{ak-1}^{il} | \mathbf{z}_{1:k-1})$, respectively, and $\rho \in (0, 1]$ denotes the forgetting factor. Such linear propagation guarantees that $p(\mathbf{R}_{ak}^{il} | \mathbf{z}_{1:k-1})$ has the same mean value but increased variance (uncertainty) as compared with $p(\mathbf{R}_{ak-1}^{il} | \mathbf{z}_{1:k-1})$, which accounts for the slow time-variation and unavailable evolution model of $\mathbf{R}_{ak}^{il}$.

Fig. 6: A graphical illustration of the constructed hierarchical Gaussian model for robot i .

According to the relative measurement model in (5), the relative measurement likelihood PDF of robots i and j can be written as

$$p(\mathbf{z}_{rk}^{ij} | \mathbf{x}_k^i, \mathbf{x}_k^j, \mathbf{R}_{rk}^{ij}) = \mathcal{N}(\mathbf{z}_{rk}^{ij}; \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j), \mathbf{R}_{rk}^{ij}), \quad j \in j_k^{i*} \quad (10)$$

where the relative measurement noise covariance matrix $\mathbf{R}_{rk}^{ij}$ is assumed as a random variable and modelled as inverse-Wishart distributed

$$p(\mathbf{R}_{rk}^{ij} | \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}) = \text{IW}(\mathbf{R}_{rk}^{ij}; u_{rk|k-1}^{ij}, \mathbf{U}_{rk|k-1}^{ij}), \quad j \in j_k^{i*} \quad (11)$$

and $u_{rk|k-1}^{ij}$ and $\mathbf{U}_{rk|k-1}^{ij}$ denote the dof parameter and scale matrix of $p(\mathbf{R}_{rk}^{ij} | \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}})$, respectively, which can be obtained using a similar way to (9) as follows

$$u_{rk|k-1}^{ij} = \rho u_{rk-1|k-1}^{ij}, \quad \mathbf{U}_{rk|k-1}^{ij} = \rho \mathbf{U}_{rk-1|k-1}^{ij} \quad (12)$$

where $u_{rk-1|k-1}^{ij}$ and $\mathbf{U}_{rk-1|k-1}^{ij}$ denote the dof parameter and scale matrix of $p(\mathbf{R}_{rk-1}^{ij} | \mathbf{z}_{1:k-1})$, respectively. Note that the absolute measurements $\{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}$ have been obtained when $\mathbf{R}_{rk}^{ij}$ is adaptively estimated.

Equations (6)-(12) make up an hierarchical Gaussian model of robot i whose graphical illustration is shown in Fig. 6. By using a similar way, n_R hierarchical Gaussian models of the entire multi-robot systems can be constructed. Next, we will present how to jointly estimate the state vector (pose) of robot i and measurement noise covariance matrices based on these n_R hierarchical Gaussian models using the VB approach.

C. Joint estimates of $\mathbf{x}_k^i$ and $\{\mathbf{R}_{ak}^{il}\}_{l \in l_k^{i}}$*

To estimate $\mathbf{x}_k^i$ and $\{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}$ simultaneously, we require to calculate the joint posterior PDF $p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right)$. Unfortunately, a recursive analytical solution of this joint posterior PDF is unavailable since the Gaussian-inverse-Wishart joint PDF does not have a closed form under the Bayesian estimation framework. To solve this problem, the standard VB approach [40], [41] is employed to obtain a free factored approximation of the joint posterior PDF as follows

$$p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \approx q^{(a)}(\mathbf{x}_k^i) \prod_{l \in l_k^{i*}} q^{(a)}(\mathbf{R}_{ak}^{il}) \quad (13)$$

where $q^{(a)}(\mathbf{x}_k^i)$ and $\{q^{(a)}(\mathbf{R}_{ak}^{il})\}_{l \in l_k^{i*}}$ are the approximate posterior PDFs of the local absolute measurement update, which are given by

$$\log q^{(a)}(\mathbf{x}_k^i) = \mathbb{E}_{\{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}} \left[\log p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \right] \quad (14)$$

$$\log q^{(a)}(\mathbf{R}_{ak}^{il}) = \mathbb{E}_{\left\{\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{is}\}_{s \in l_k^{i*}, s \neq l}\right\}} \left[\log p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{l \in l_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \right] \quad (15)$$

where $\mathbb{E}_y[\cdot]$ denotes the expectation operation with respect to the approximate posterior PDF $q(y)$.

Proposition 1. *Exploiting (14), the approximate posterior PDF $q^{(a)}(\mathbf{x}_k^i)$ can be updated as Gaussian, i.e.,*

$$q^{(a)}(\mathbf{x}_k^i) = \mathcal{N}(\mathbf{x}_k^i; \hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}) \quad (16)$$

where $\hat{\mathbf{x}}_{k|k}^{i(a)}$ and $\Sigma_{k|k}^{i(a)}$ can be obtained by running the LAMU function with $\{\mathbf{z}_{ak}^{il}, \hat{\mathbf{R}}_{ak}^{il}, \mathbf{x}_L^l\}_{l \in l_k^{i*}}$, and $\hat{\mathbf{R}}_{ak}^{il}$ denotes the estimated absolute measurement noise covariance matrix and is given by

$$\hat{\mathbf{R}}_{ak}^{il} = \left\{ \mathbb{E} \left[(\mathbf{R}_{ak}^{il})^{-1} \right] \right\}^{-1}, \quad l \in l_k^{i*} \quad (17)$$

Proposition 2. *Using (15), the approximate posterior PDF $q^{(a)}(\mathbf{R}_{ak}^{il})$ can be updated as an inverse-Wishart PDF, i.e.,*

$$q^{(a)}(\mathbf{R}_{ak}^{il}) = \text{IW}(\mathbf{R}_{ak}^{il}; u_{ak|k}^{il}, \mathbf{U}_{ak|k}^{il}) \quad (18)$$

where the posterior dof parameter $u_{ak|k}^{il}$ and scale matrix $\mathbf{U}_{ak|k}^{il}$ are, respectively, given by

$$\begin{cases} u_{ak|k}^{il} = u_{ak|k-1}^{il} + 1 \\ \mathbf{U}_{ak|k}^{il} = \mathbf{U}_{ak|k-1}^{il} + \mathbf{A}_k^{il} \end{cases} \quad (19)$$

and the auxiliary matrix $\mathbf{A}_k^{il}$ is written as

$$\mathbf{A}_k^{il} = \mathbb{E} \left\{ [\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l)] [\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l)]^T \right\} \quad (20)$$

Proof. See Appendix A for the proofs of Propositions 1–2. $\square$

Finally, we calculate the required expectations in (17) and (20). Since the scale matrix $\mathbf{R}_{ak}^{il}$ is updated as an inverse-Wishart PDF according to (18), the estimate of $\mathbf{R}_{ak}^{il}$ is calculated as

$$\hat{\mathbf{R}}_{ak}^{il} = \left\{ \mathbb{E} \left[(\mathbf{R}_{ak}^{il})^{-1} \right] \right\}^{-1} = \mathbf{U}_{ak|k}^{il} / u_{ak|k}^{il} \quad (21)$$

Utilizing the first-order Taylor approximation of $\mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l)$ at $\mathbf{x}_k^i = \hat{\mathbf{x}}_{k|k-1}^i$, the auxiliary matrix $\mathbf{A}_k^{il}$ can be approximately calculated as

$$\begin{aligned} \mathbf{A}_k^{il} &= \mathbb{E} \left\{ [\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \mathbf{H}_{ak}^{il}(\mathbf{x}_k^i - \hat{\mathbf{x}}_{k|k-1}^i)] [\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \right. \\ &\quad \left. \mathbf{H}_{ak}^{il}(\mathbf{x}_k^i - \hat{\mathbf{x}}_{k|k-1}^i)]^T \right\} \\ &= \mathbf{H}_{ak}^{il} \Sigma_{k|k}^{i(a)} (\mathbf{H}_{ak}^{il})^T + \left[\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \mathbf{H}_{ak}^{il}(\hat{\mathbf{x}}_{k|k}^{i(a)} - \hat{\mathbf{x}}_{k|k-1}^i) \right] \times \\ &\quad \left[\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \mathbf{H}_{ak}^{il}(\hat{\mathbf{x}}_{k|k}^{i(a)} - \hat{\mathbf{x}}_{k|k-1}^i) \right]^T \end{aligned} \quad (22)$$

where $\mathbf{H}_{ak}^{il}$ is given in Table II.

Equations (16)-(22) constitute the proposed VB based local absolute measurement update. It can be seen from (16)-(22) that the calculations of the approximate posterior PDFs $q^{(a)}(\mathbf{x}_k^i)$ and $\{q^{(a)}(\mathbf{R}_{ak}^{il})\}_{\{l \in l_k^{i*}\}}$ are mutually coupled. As a result, the analytical solutions of these approximate posterior PDFs are unavailable. To address this problem, in this paper, the fixed-point iteration method is employed to achieve approximate solutions of these posterior PDFs, in which the posterior PDFs $q^{(a)}(\mathbf{x}_k^i)$ and $\{q^{(a)}(\mathbf{R}_{ak}^{il})\}_{\{l \in l_k^{i*}\}}$ are alternately updated. That is to say, at the $(m+1)$ -th iteration, the posterior PDF $q^{(a)}(\mathbf{x}_k^i)$ is firstly updated as $q^{(a)(m+1)}(\mathbf{x}_k^i)$ by fixing $\{q^{(a)}(\mathbf{R}_{ak}^{il}) = q^{(a)(m)}(\mathbf{R}_{ak}^{il})\}_{\{l \in l_k^{i*}\}}$, and then the posterior PDFs $\{q^{(a)}(\mathbf{R}_{ak}^{il})\}_{\{l \in l_k^{i*}\}}$ are updated as $\{q^{(a)(m+1)}(\mathbf{R}_{ak}^{il})\}_{\{l \in l_k^{i*}\}}$ based on the updated posterior PDF $q^{(a)(m+1)}(\mathbf{x}_k^i)$. Such fixed-point iterations are performed until convergence in the proposed VB based local absolute measurement update. The fixed-point iteration of the proposed VB based local absolute measurement update for robot i is summarized as the VBLAMU function in Table IV, where m^* , M and ϵ denote the actual number of iterations, the maximum number of iterations, and the iterative threshold, respectively.

TABLE IV: Fixed-point iteration of the proposed VB based local absolute measurement update for robot i

Function 4: $\left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij(a)} \right\}, \left\{ u_{ak|k}^{il}, \mathbf{U}_{ak|k}^{il} \right\}_{l \in l_k^{i*}} \right) = \text{VBLAMU} \left(\hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i, \left\{ \sigma_{k|k-1}^{ij} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, l_k^{i*}, \left\{ \mathbf{z}_{ak}^{il}, u_{ak|k-1}^{il}, \mathbf{U}_{ak|k-1}^{il}, \mathbf{x}_L^l \right\}_{l \in l_k^{i*}}, M, \epsilon \right)$

If the detected landmark set l_k^{i*} of robot i is non-empty:

1. Initialization: $u_{ak|k}^{il(0)} = u_{ak|k-1}^{il}, \mathbf{U}_{ak|k}^{il(0)} = \mathbf{U}_{ak|k-1}^{il}, \hat{\mathbf{R}}_{ak}^{il(0)} = \mathbf{U}_{ak|k}^{il(0)} / u_{ak|k}^{il(0)}$ for all $l \in l_k^{i*}$
2. Fixed-point iteration:

for $m = 0 : M - 1$ **do**

Update posterior PDF $q^{(a)(m+1)}(\mathbf{x}_k^i)$

Run $\left(\hat{\mathbf{x}}_{k|k}^{i(a)(m+1)}, \Sigma_{k|k}^{i(a)(m+1)}, \left\{ \sigma_{k|k}^{ij(a)(m+1)} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \left\{ \mathbf{z}_{ak}^{il}, \hat{\mathbf{R}}_{ak}^{il(m)}, \mathbf{x}_L^l \right\}_{l \in l_k^{i*}} \right) = \text{LAMU} \left(\hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i, \left\{ \sigma_{k|k-1}^{ij} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, l_k^{i*}, \left\{ \mathbf{z}_{ak}^{il}, \hat{\mathbf{R}}_{ak}^{il(m)}, \mathbf{x}_L^l \right\}_{l \in l_k^{i*}} \right)$

Calculate $\mathbf{A}_k^{il(m+1)} = \mathbf{H}_{ak}^{il} \Sigma_{k|k}^{i(a)(m+1)} (\mathbf{H}_{ak}^{il})^T + \left[\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \mathbf{H}_{ak}^{il}(\hat{\mathbf{x}}_{k|k}^{i(a)(m+1)} - \hat{\mathbf{x}}_{k|k-1}^i) \right] \times \left[\mathbf{z}_{ak}^{il} - \mathbf{h}_{ak}^{il}(\hat{\mathbf{x}}_{k|k-1}^i, \mathbf{x}_L^l) - \mathbf{H}_{ak}^{il}(\hat{\mathbf{x}}_{k|k}^{i(a)(m+1)} - \hat{\mathbf{x}}_{k|k-1}^i) \right]^T$

Update posterior PDFs $q^{(a)(m+1)}(\mathbf{R}_{ak}^{il})$

Calculate $u_{ak|k}^{il(m+1)} = u_{ak|k-1}^{il} + 1, \mathbf{U}_{ak|k}^{il(m+1)} = \mathbf{U}_{ak|k-1}^{il} + \mathbf{A}_k^{il(m+1)}, \hat{\mathbf{R}}_{ak}^{il(m+1)} = \mathbf{U}_{ak|k}^{il(m+1)} / u_{ak|k}^{il(m+1)}$

Check convergence: if $\frac{\|\hat{\mathbf{x}}_{k|k}^{i(a)(m+1)} - \hat{\mathbf{x}}_{k|k}^{i(a)(m)}\|}{\|\hat{\mathbf{x}}_{k|k}^{i(a)(m+1)}\|} \leq \epsilon$, terminate iteration

end for

3. Save $\hat{\mathbf{x}}_{k|k}^{i(a)} = \hat{\mathbf{x}}_{k|k}^{i(a)(m^*)}, \Sigma_{k|k}^{i(a)} = \Sigma_{k|k}^{i(a)(m^*)}, \left\{ \sigma_{k|k}^{ij(a)} = \sigma_{k|k}^{ij(a)(m^*)} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \left\{ u_{ak|k}^{il} = u_{ak|k}^{il(m^*)}, \mathbf{U}_{ak|k}^{il} = \mathbf{U}_{ak|k}^{il(m^*)} \right\}_{l \in l_k^{i*}}$

If the detected landmark set l_k^{i*} of robot i is empty: $\hat{\mathbf{x}}_{k|k}^{i(a)} = \hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k}^{i(a)} = \Sigma_{k|k-1}^i, \left\{ \sigma_{k|k}^{ij(a)} = \sigma_{k|k-1}^{ij} \right\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \left\{ u_{ak|k}^{il} = u_{ak|k-1}^{il}, \mathbf{U}_{ak|k}^{il} = \mathbf{U}_{ak|k-1}^{il} \right\}_{l \in l_k^{i*}}$

D. Joint estimates of $\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$

To jointly estimate $\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}$ and $\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}$, we need to calculate the joint posterior PDF $p \left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}} \right)$. Similar to Section III. C, we also employ the standard VB approach [41] to obtain a free factored approximate solution of this joint posterior PDF as follows

$$p \left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}} \right) \approx q \left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}} \right) \times \prod_{j \in j_k^{i*}} q(\mathbf{R}_{rk}^{ij}) \quad (23)$$

where $q(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}})$ and $\{q(\mathbf{R}_{rk}^{ij})\}_{j \in j_k^{i*}}$ are the approximate posterior PDFs of the cooperative relative measurement update, which are given by

$$\log q(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}) = \mathbb{E}_{\{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}} \left[\log p(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}}) \right] \quad (24)$$

$$\log q(\mathbf{R}_{rk}^{ij}) = \mathbb{E}_{\left\{ \mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{is}\}_{s \neq j}, \{\mathbf{R}_{rk}^{js}\}_{s \in j_k^{i*}} \right\}} \left[\log p(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}}) \right] \quad (25)$$

Proposition 3. Exploiting (24), the approximate posterior PDF $q(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}})$ can be updated as Gaussian, i.e.,

$$q(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}) = \mathcal{N} \left(\begin{bmatrix} \mathbf{x}_k^i \\ \mathbf{x}_k^{j_k^{i*}} \end{bmatrix}; \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^i \\ \hat{\mathbf{x}}_{k|k}^{j_k^{i*}} \end{bmatrix}, \begin{bmatrix} \Sigma_{k|k}^i & \Sigma_{k|k}^{ij_k^{i*}} \\ (\Sigma_{k|k}^{ij_k^{i*}})^T & \Sigma_{k|k}^{j_k^{i*}} \end{bmatrix} \right) \quad (26)$$

where the state estimates $\hat{\mathbf{x}}_{k|k}^i$ and $\hat{\mathbf{x}}_{k|k}^{j_k^{i*}}$, the estimation error covariance matrix $\Sigma_{k|k}^i$ and $\Sigma_{k|k}^{ij_k^{i*}}$, and the cross-correlation $\Sigma_{k|k}^{j_k^{i*}}$ can be obtained using the CRMU function with $\{\mathbf{z}_{rk}^{ij}, \hat{\mathbf{R}}_{rk}^{ij}\}_{j \in j_k^{i*}}$, and the estimated relative measurement noise covariance matrix $\hat{\mathbf{R}}_{rk}^{ij}$ is written as

$$\hat{\mathbf{R}}_{rk}^{ij} = \left\{ \mathbb{E} \left[(\mathbf{R}_{rk}^{ij})^{-1} \right] \right\}^{-1} \quad (27)$$

Proposition 4. According to (25), the approximate posterior PDF $q(\mathbf{R}_{rk}^{ij})$ can be updated as an inverse-Wishart PDF as follows

$$q(\mathbf{R}_{rk}^{ij}) = \text{IW}(\mathbf{R}_{rk}^{ij}; u_{rk|k}^{ij}, \mathbf{U}_{rk|k}^{ij}) \quad (28)$$

where the posterior dof parameter $u_{rk|k}^{ij}$ and scale matrix $\mathbf{U}_{rk|k}^{ij}$ are, respectively, given by

$$\begin{cases} u_{rk|k}^{ij} = u_{rk|k-1}^{ij} + 1 \\ \mathbf{U}_{rk|k}^{ij} = \mathbf{U}_{rk|k-1}^{ij} + \mathbf{B}_k^{ij} \end{cases} \quad (29)$$

and the auxiliary matrix $\mathbf{B}_k^{ij}$ is formulated as

$$\mathbf{B}_k^{ij} = \mathbb{E} \left\{ [\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j)] [\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j)]^T \right\} \quad (30)$$

Proof. See Appendix B for the proofs of Propositions 3–4. $\square$

TABLE V: Fixed-point iteration of the proposed VB based cooperative relative measurement update for robot i

Function 5: $\left(\hat{\mathbf{x}}_{k|k}^i, \Sigma_{k|k}^i, \left\{ \sigma_{k|k}^{ij} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}}, \left\{ u_{rk|k}^{ij}, \mathbf{U}_{rk|k}^{ij} \right\}_{\{j \in j_k^{i*}\}} \right) = \text{VBCRMU} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij(a)} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}}, j_k^{i*}, \left\{ \mathbf{z}_{rk}^{ij}, \hat{\mathbf{R}}_{rk}^{ij(m)} \right\}_{\{j \in j_k^{i*}\}} \right)$

If the detected robot set j_k^{i*} of robot i is non-empty:

1. Initialization: $u_{rk|k}^{ij(0)} = u_{rk|k-1}^{ij}, \mathbf{U}_{rk|k}^{ij(0)} = \mathbf{U}_{rk|k-1}^{ij}, \hat{\mathbf{R}}_{rk}^{ij(0)} = \mathbf{U}_{rk|k}^{ij(0)} / u_{rk|k}^{ij(0)}$ for all $j \in j_k^{i*}$
2. Fixed-point iteration:
for $m = 0 : M - 1$ **do**
 - Update posterior PDF $q^{(m+1)}(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{\{j \in j_k^{i*}\}})$
 - Run $\left(\hat{\mathbf{x}}_{k|k}^{i(m+1)}, \Sigma_{k|k}^{i(m+1)}, \left\{ \sigma_{k|k}^{ij(m+1)} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}} \right) = \text{CRMU} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij(a)} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}}, j_k^{i*}, \left\{ \mathbf{z}_{rk}^{ij}, \hat{\mathbf{R}}_{rk}^{ij(m)} \right\}_{\{j \in j_k^{i*}\}} \right)$
 - Save $\hat{\mathbf{x}}_{k|k}^{j_k^{i*}(m+1)}, \Sigma_{k|k}^{j_k^{i*}(m+1)}, \Sigma_{k|k}^{j_k^{i*}(m+1)}$ produced in the CRMU function
 - Extract $\hat{\mathbf{x}}_{k|k}^{j(m+1)}, \Sigma_{k|k}^{ij(m+1)}$, and $\Sigma_{k|k}^{j(m+1)}$ from $\hat{\mathbf{x}}_{k|k}^{j_k^{i*}(m+1)}, \Sigma_{k|k}^{j_k^{i*}(m+1)}, \Sigma_{k|k}^{j_k^{i*}(m+1)}$, respectively.
 - Construct $\Sigma_{k|k}^{\xi^{ij} \xi^{ij(m+1)}} = \begin{bmatrix} \Sigma_{k|k}^{i(m+1)} & \Sigma_{k|k}^{ij(m+1)} \\ (\Sigma_{k|k}^{ij(m+1)})^T & \Sigma_{k|k}^{j(m+1)} \end{bmatrix}, \xi_{k|k}^{ij(m+1)} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(m+1)} \\ \hat{\mathbf{x}}_{k|k}^{j(m+1)} \end{bmatrix}, \xi_{k|k}^{ij(a)} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{j(a)} \end{bmatrix}$
 - Calculate $\mathbf{B}_k^{ij(m+1)} = \mathbf{H}_{rk}^{ij} \Sigma_{k|k}^{\xi^{ij} \xi^{ij(m+1)}} (\mathbf{H}_{rk}^{ij})^T + \left[\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right) - \mathbf{H}_{rk}^{ij} (\xi_{k|k}^{ij(m+1)} - \xi_{k|k}^{ij(a)}) \right] \times \left[\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right) - \mathbf{H}_{rk}^{ij} (\xi_{k|k}^{ij(m+1)} - \xi_{k|k}^{ij(a)}) \right]^T$
 - Update posterior PDFs $q^{(m+1)}(\mathbf{R}_{rk}^{ij})$ for all $j \in j_k^{i*}$
 - Calculate $u_{rk|k}^{ij(m+1)} = u_{rk|k-1}^{ij} + 1, \mathbf{U}_{rk|k}^{ij(m+1)} = \mathbf{U}_{rk|k-1}^{ij} + \mathbf{B}_k^{ij(m+1)}, \hat{\mathbf{R}}_{rk}^{ij(m+1)} = \mathbf{U}_{rk|k}^{ij(m+1)} / u_{rk|k}^{ij(m+1)}$
 - Check convergence: if $\frac{\|\hat{\mathbf{x}}_{k|k}^{i(m+1)} - \hat{\mathbf{x}}_{k|k}^{i(m)}\|}{\|\hat{\mathbf{x}}_{k|k}^{i(m+1)}\|} \leq \epsilon$, terminate iteration

end for

3. Save $\hat{\mathbf{x}}_{k|k}^i = \hat{\mathbf{x}}_{k|k}^{i(m^*)}, \Sigma_{k|k}^i = \Sigma_{k|k}^{i(m^*)}, \left\{ \sigma_{k|k}^{ij} = \sigma_{k|k}^{ij(m^*)} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}}, \left\{ u_{rk|k}^{ij} = u_{rk|k}^{ij(m^*)}, \mathbf{U}_{rk|k}^{ij} = \mathbf{U}_{rk|k}^{ij(m^*)} \right\}_{\{j \in j_k^{i*}\}}$

If the detected robot set j_k^{i*} of robot i is empty: $\hat{\mathbf{x}}_{k|k}^i = \hat{\mathbf{x}}_{k|k}^{i(a)}, \Sigma_{k|k}^i = \Sigma_{k|k}^{i(a)}, \left\{ \sigma_{k|k}^{ij} = \sigma_{k|k}^{ij(a)} \right\}_{\{1 \leq j \leq n_R\}}^{\{j \neq i\}}, \left\{ u_{rk|k}^{ij} = u_{rk|k-1}^{ij}, \mathbf{U}_{rk|k}^{ij} = \mathbf{U}_{rk|k-1}^{ij} \right\}_{\{j \in j_k^{i*}\}}$

Similar to Section III. C, the expectations in (27) and (30) can be, respectively, calculated as

$$\hat{\mathbf{R}}_{rk}^{ij} = \mathbf{U}_{rk|k}^{ij} / u_{rk|k}^{ij} \quad (31)$$

$$\begin{aligned} \mathbf{B}_k^{ij} = & \mathbf{H}_{rk}^{ij} \Sigma_{k|k}^{\xi^{ij} \xi^{ij}} (\mathbf{H}_{rk}^{ij})^T + \left[\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right) - \mathbf{H}_{rk}^{ij} (\xi_{k|k}^{ij} - \xi_{k|k}^{ij(a)}) \right] \times \\ & \left[\mathbf{z}_{rk}^{ij} - \mathbf{h}_{rk}^{ij} \left(\hat{\mathbf{x}}_{k|k}^{i(a)}, \hat{\mathbf{x}}_{k|k}^{j(a)} \right) - \mathbf{H}_{rk}^{ij} (\xi_{k|k}^{ij} - \xi_{k|k}^{ij(a)}) \right]^T \end{aligned} \quad (32)$$

where the auxiliary parameters are given by

$$\begin{cases} \Sigma_{k|k}^{\xi^{ij}\xi^{ij}} = \begin{bmatrix} \Sigma_{k|k}^i & \Sigma_{k|k}^{ij} \\ \left(\Sigma_{k|k}^{ij}\right)^T & \Sigma_{k|k}^j \end{bmatrix} \\ \xi_{k|k}^{ij} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^i \\ \hat{\mathbf{x}}_{k|k}^j \end{bmatrix}, \quad \xi_{k|k}^{ij(a)} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{j(a)} \end{bmatrix} \end{cases} \quad (33)$$

and $\hat{\mathbf{x}}_{k|k}^j$, $\hat{\mathbf{x}}_{k|k}^{j(a)}$, $\Sigma_{k|k}^{ij}$ and $\Sigma_{k|k}^j$ can be, respectively, extracted from $\hat{\mathbf{x}}_{k|k}^{j*}$, $\hat{\mathbf{x}}_{k|k}^{j*(a)}$, $\Sigma_{k|k}^{ij*}$ and $\Sigma_{k|k}^{j*}$ accordingly.

Equations (27)-(33) constitute the proposed VB based cooperative relative measurement update. We can observe from (27)-(33) that the calculations of the approximate posterior PDFs $q\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}\right)$ and $\{q(\mathbf{R}_{rk}^{ij})\}_{j \in j_k^{i*}}$ are also mutually coupled so that the analytical solutions of these posterior PDFs are unavailable. Similar to the proposed VB based local absolute measurement update, the approximate posterior PDFs $q\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}\right)$ and $\{q(\mathbf{R}_{rk}^{ij})\}_{j \in j_k^{i*}}$ are also alternately updated based on the fixed-point iteration in the cooperative relative measurement update. The fixed-point iteration of the proposed VB based cooperative relative measurement update for robot i is summarized as VBCRMU function in Table V.

E. The proposed adaptive RDCL-EKF algorithm

The proposed adaptive RDCL-EKF algorithm is composed of four parts: the local motion updates of all robots, the local time updates of measurement noise covariance matrices of all robots, the VB based local absolute measurement updates of all robots, and the VB based cooperative relative measurement updates of all robots. The detailed update process is as follows.

- In the local motion update, each robot propagates its state estimate, estimation error covariance matrix, and cross-correlations independently.
- In the local time updates of measurement noise covariance matrices, each robot propagates its dof parameters and scale matrices of absolute and relative measurement noise covariance matrices with respect to all landmarks and all the other robots.
- In the VB based local absolute measurement update, each robot updates its state estimate, estimation error covariance matrix, cross-correlations, and the dof parameters and scale matrices of the absolute measurement noise covariance matrices relative to all detected landmarks independently, while the dof parameters and scale matrices of absolute and

TABLE VI: Implementations of the proposed adaptive RDCL-EKF algorithm

Inputs: $\{\hat{\mathbf{x}}_{k-1 k-1}^i, \Sigma_{k-1 k-1}^i, \mathbf{Q}_k^i, l_k^{i*}, j_k^{i*}\}_{1 \leq i \leq n_R}, \{\sigma_{k-1 k-1}^{ij}\}_{\substack{j \neq i \\ 1 \leq i, j \leq n_R}}, \{u_{ak k-1}^{il}, \mathbf{U}_{ak k-1}^{il}\}_{\substack{1 \leq l \leq n_L \\ 1 \leq i \leq n_R}},$ $\{\mathbf{z}_{ak}^{il}\}_{\substack{l \in l_k^{i*} \\ 1 \leq i \leq n_R}}, \{\mathbf{x}_L^l\}_{1 \leq l \leq n_L}, \{u_{rk k-1}^{ij}, \mathbf{U}_{rk k-1}^{ij}\}_{\substack{i \neq j \\ 1 \leq i, j \leq n_R}}, \{\mathbf{z}_{rk}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq i \leq n_R}}, \rho, M, \epsilon$
Local motion updates of all robots: For all $1 \leq i \leq n_R$, robot i runs the LMU function independently: $\left(\hat{\mathbf{x}}_{k k-1}^i, \Sigma_{k k-1}^i, \{\sigma_{k k-1}^{ij}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}} \right) = \text{LMU} \left(\hat{\mathbf{x}}_{k-1 k-1}^i, \Sigma_{k-1 k-1}^i, \{\sigma_{k-1 k-1}^{ij}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \mathbf{Q}_k^i \right)$
Local time updates of measurement noise covariance matrices of all robots: For all $1 \leq i \leq n_R$, robot i calculates the dof parameters and scale matrices independently: for $l = 1 : n_L$ do $u_{ak k-1}^{il} = \rho u_{ak-1 k-1}^{il}, \quad \mathbf{U}_{ak k-1}^{il} = \rho \mathbf{U}_{ak-1 k-1}^{il}$ end for for $j = 1 : n_R$ do If $j \neq i$: $u_{rk k-1}^{ij} = \rho u_{rk-1 k-1}^{ij}, \quad \mathbf{U}_{rk k-1}^{ij} = \rho \mathbf{U}_{rk-1 k-1}^{ij}$ end for VB based local absolute measurement updates of all robots: For all $1 \leq i \leq n_R$, robot i runs the VBLAMU function independently: $\left(\hat{\mathbf{x}}_{k k}^{i(a)}, \Sigma_{k k}^{i(a)}, \{\sigma_{k k}^{ij(a)}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \{u_{ak k}^{il}, \mathbf{U}_{ak k}^{il}\}_{\substack{l \in l_k^{i*} \\ 1 \leq l \leq n_L}}, \{\mathbf{z}_{ak}^{il}, u_{ak k-1}^{il}, \mathbf{U}_{ak k-1}^{il}, \mathbf{x}_L^l\}_{\substack{l \in l_k^{i*} \\ 1 \leq l \leq n_L}}, M, \epsilon \right) = \text{VBLAMU} \left(\hat{\mathbf{x}}_{k k-1}^i, \Sigma_{k k-1}^i, \{\sigma_{k k-1}^{ij}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, l_k^{i*}, \right.$ $\left. \{\mathbf{z}_{ak}^{il}, u_{ak k-1}^{il}, \mathbf{U}_{ak k-1}^{il}, \mathbf{x}_L^l\}_{\substack{l \in l_k^{i*} \\ 1 \leq l \leq n_L}}, M, \epsilon \right)$ If $l \notin l_k^{i*}$: $\{u_{ak k}^{il} = u_{ak k-1}^{il}, \mathbf{U}_{ak k}^{il} = \mathbf{U}_{ak k-1}^{il}\}_{\substack{l \notin l_k^{i*} \\ 1 \leq l \leq n_L}}$ VB based cooperative relative measurement updates of all robots: For all $1 \leq i \leq n_R$, robot i runs the VBCRMU function: $\left(\hat{\mathbf{x}}_{k k}^i, \Sigma_{k k}^i, \{\sigma_{k k}^{ij}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, \{u_{rk k}^{ij}, \mathbf{U}_{rk k}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq j \leq n_R}}, \{\mathbf{z}_{rk}^{ij}, u_{rk k-1}^{ij}, \mathbf{U}_{rk k-1}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq j \leq n_R}}, M, \epsilon \right) = \text{VBCRMU} \left(\hat{\mathbf{x}}_{k k}^{i(a)}, \Sigma_{k k}^{i(a)}, \{\sigma_{k k}^{ij(a)}\}_{\substack{j \neq i \\ 1 \leq j \leq n_R}}, j_k^{i*}, \right.$ $\left. \{u_{rk k}^{ij}, \mathbf{U}_{rk k}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq j \leq n_R}}, M, \epsilon \right)$ If the robot i has been detected by the robot $j \in j_k^{i*}$: send $\{u_{rk k}^{ji} = u_{rk k}^{ij}, \mathbf{U}_{rk k}^{ji} = \mathbf{U}_{rk k}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq j \leq n_R}}$ to robot j Else: send $\{u_{rk k-1}^{ji} = u_{rk k-1}^{ij}, \mathbf{U}_{rk k-1}^{ji} = \mathbf{U}_{rk k-1}^{ij}\}_{\substack{j \in j_k^{i*} \\ 1 \leq j \leq n_R}}$ to robot j If $j \notin j_k^{i*}$: $\{u_{rk k}^{ij} = u_{rk k-1}^{ij}, \mathbf{U}_{rk k}^{ij} = \mathbf{U}_{rk k-1}^{ij}\}_{\substack{j \notin j_k^{i*} \\ 1 \leq j \leq n_R}}$ Outputs: $\{\hat{\mathbf{x}}_{k-1 k-1}^i, \Sigma_{k-1 k-1}^i\}_{1 \leq i \leq n_R}, \{\sigma_{k k}^{ij}\}_{\substack{j \neq i \\ 1 \leq i, j \leq n_R}}, \{u_{ak k}^{il}, \mathbf{U}_{ak k}^{il}\}_{\substack{1 \leq l \leq n_L \\ 1 \leq i \leq n_R}}, \{u_{rk k}^{ij}, \mathbf{U}_{rk k}^{ij}\}_{\substack{i \neq j \\ 1 \leq i, j \leq n_R}}$

relative measurement noise covariance matrices relative to the undetected landmarks all

ing RDCL-EKF method, but only requires slightly higher communication overhead to transmit additional parameters in each cooperation. It is worth noting that we have assumed perfect communications between the local robots, and the problems with regard to the delays and dropouts in communication will be further considered in our future work. Meanwhile, variational Bayesian learning allows the robot to adaptively estimate the measurement noise covariance matrices, contributing to better performance for the problem of DCL with time-varying accuracy, but at the expense of slightly higher computational complexity.

IV. SIMULATION STUDY

A. Simulation setups and descriptions

In this section, the performance of the proposed adaptive RDCL-EKF algorithm is validated using a series of simulation tests. A team of 6 homogeneous mobile robots move randomly according to the motion model (1) in a two dimensional environment with 5 known landmarks, i.e., $n_R = 6$ and $n_L = 5$, where the true linear velocity of each robot is $V_k = 0.5$ (m/s), and the true rotational velocity Ω_k of each robot is randomly drawn from $[-0.5, 0.5]$ (rad/s), and the discretization time is set as $\Delta t = 0.5$ (sec). The true trajectories of six mobile robots are shown in Fig. 8. The proprioceptive sensor is selected as the wheel encoder which is used to measure the linear and rotational velocities. The wheel encoders installed on every robot have the same measurement accuracy, and the standard deviation is set as $\sigma_k = 5\%V_k$ which is proportional to the true linear velocity. According to the velocity measuring principle of wheel encoders, the standard deviations of linear and rotational velocity measurements can be, respectively, set as $\sigma_{V_k} = \frac{\sqrt{2}}{2}\sigma_k$ and $\sigma_{\Omega_k} = \frac{\sqrt{2}}{a}\sigma_k$, where a denotes the distance between two drive wheels of every robot [15].

In this simulation study, the bearing and range measurement models in (2) and (4) are used to validate the performance of the proposed method. Note that similar simulation results and conclusions can be also obtained using the relative position measurement models in (2) and (4), and they are omitted for brevity. The standard deviations of absolute measurements are all proportional to the true distance between the exteroceptive sensor and the reference landmark, and the standard deviations of relative measurements are also all proportional to the true distance between the exteroceptive sensor and the mobile robot. Considering that every exteroceptive sensor has a limiting measurement accuracy, the standard deviations of absolute and relative

Fig. 8: True trajectories of six mobile robots.

Fig. 9: RMSEs of position of all the algorithms compared for six mobile robots.

measurements all have the minimum values, and the minimum standard deviations of bearing and range are, respectively, chosen as 1 (deg) and 0.1 (m). Specifically, for the absolute and relative measurements, the standard deviations of bearing and range are, respectively, set as the five percent of the distance between the exteroceptive sensor and the measured target, i.e., $\sigma_{\theta_k} = 5\%d_k$

Fig. 10: RMSEs of orientation of all the algorithms compared for six mobile robots.

Fig. 11: NEESs of all the algorithms compared for six mobile robots.

(deg) and $\sigma_{rk} = 5\%d_k$ (m), where $\sigma_{\theta k} \geq 1$ (deg) and $\sigma_{rk} \geq 1$ (m), and d_k denotes the true distance between the exteroceptive sensor and the measured target (reference landmark or mobile robot). Since every mobile robot has limited sensing scope, it can only detect the landmarks and other mobile robots occasionally. To be more consistent with the actual situation, it is assumed that

each mobile robot detects every landmark and mobile robot both with the probability of 20%. To better show the advantages of the proposed method, four CL-EKF algorithms are compared in the simulation tests, in which the true state noise covariance matrix $\mathbf{Q}_k = \text{diag} \{[\sigma_{V_k}^2, \sigma_{\Omega_r}^2]\}$ of each robot is used. The detailed descriptions and parameter settings of the four CL-EKF algorithms are listed as follows.

- CCL-EKF [8]: The existing CCL-EKF algorithm is the optimal centralized algorithm employing the true absolute and relative measurement noise covariance matrices $\mathbf{R}_{ak}$ and $\mathbf{R}_{rk}$ to estimate the poses of all robots, which is used as the most accurate reference.
- RDCL-EKF-T [25]: The existing RDCL-EKF algorithm employs the true absolute and relative measurement noise covariance matrices $\mathbf{R}_{ak}$ and $\mathbf{R}_{rk}$ to estimate the poses of all robots, which acts as the benchmark for the proposed adaptive DCL method.
- RDCL-EKF-N [25]: The existing RDCL-EKF algorithm utilizes the nominal absolute and relative measurement noise covariance matrices $\bar{\mathbf{R}}_{ak}$ and $\bar{\mathbf{R}}_{rk}$ to estimate the poses of all robots and will be compared with the proposed algorithm in terms of estimation accuracy and consistency.
- RDCL-EKF-VB: The proposed adaptive RDCL-EKF algorithm employs the standard VB approach to estimate the absolute and relative measurement noise covariance matrices, in which the nominal absolute and relative measurement noise covariance matrices $\bar{\mathbf{R}}_{ak}$ and $\bar{\mathbf{R}}_{rk}$ are used as initial prior values, and the initial dof parameters and scale matrices are set as: $\left\{ u_{a0|0}^{il} = 5, \mathbf{U}_{a0|0}^{il} = 5\bar{\mathbf{R}}_{a0} \right\}_{\substack{1 \leq l \leq n_L \\ 1 \leq i \leq n_R}}$ and $\left\{ u_{r0|0}^{ij} = 5, \mathbf{U}_{r0|0}^{ij} = 5\bar{\mathbf{R}}_{r0} \right\}_{\substack{i \neq j \\ 1 \leq i, j \leq n_R}}$, and the other parameters are selected as $\rho = 1 - 10^{-5}$, $M = 50$, $\epsilon = 10^{-4}$.

where $\mathbf{R}_{ak} = \mathbf{R}_{rk} = \text{diag} \{[\sigma_{r_k}^2, \sigma_{\theta_k}^2]\}$ and $\bar{\mathbf{R}}_{ak} = \bar{\mathbf{R}}_{rk} = \text{diag} \{[\bar{\sigma}_r^2, \bar{\sigma}_\theta^2]\}$, and $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ denote the nominal standard deviations of range and bearing measurements, which are, respectively, selected as 0.5 (m) and 5 (deg). The CCL-EKF algorithm can obtain the optimal CL accuracy, and it is used as the performance benchmark for all CL-EKF algorithms. On the other hand, the RDCL-EKF-T algorithm can achieve the best CL accuracy in the RDCL-EKF framework, and it is used as the accuracy benchmark for the RDCL-EKF-N and RDCL-EKF-VB algorithms. The codes of all algorithms are performed on a computer with Intel Core i7-6500U CPU @ 2.50 GHz.

In all comparative algorithms, the initial state estimates of all robots are set as their initial

true states, and the corresponding estimation error covariance matrices are all set as $10^{-10}\mathbf{I}$. Simulation time is set as 200 (sec) and 1000 Monte Carlo run simulation trials are performed. To compare the CL accuracy, the root mean square errors (RMSEs) of position and orientation are used as performance metrics, which are defined as

$$\begin{cases} \text{RMSE}_p^i(k) = \sqrt{\frac{1}{M_n} \sum_{s=1}^{M_n} \left[\left(x_k^{i(s)} - \hat{x}_{k|k}^{i(s)} \right)^2 + \left(y_k^{i(s)} - \hat{y}_{k|k}^{i(s)} \right)^2 \right]} \\ \text{RMSE}_o^i(k) = \sqrt{\frac{1}{M_n} \sum_{s=1}^{M_n} \left(\phi_k^{i(s)} - \hat{\phi}_{k|k}^{i(s)} \right)^2} \end{cases} \quad (34)$$

where RMSE_p^i and RMSE_o^i denote the RMSEs of position and orientation of robot i , respectively, and $x_k^{i(s)}$, $y_k^{i(s)}$ and $\phi_k^{i(s)}$ are, respectively, the true values of east position, north position and orientation of robot i at the s -th Monte Carlo simulation, and $\hat{x}_{k|k}^{i(s)}$, $\hat{y}_{k|k}^{i(s)}$ and $\hat{\phi}_{k|k}^{i(s)}$ are, respectively, the estimates of east position, north position and orientation of robot i at the s -th Monte Carlo simulation, and M_n denotes the total numbers of Monte Carlo simulations.

To compare the estimation consistency, the normalized estimation error squared (NEES) is selected as the evaluation index, which is defined as follows

$$\text{NEES}^i(k) = \frac{1}{M_n} \sum_{s=1}^{M_n} \left(\mathbf{x}_k^{i(s)} - \hat{\mathbf{x}}_{k|k}^{i(s)} \right)^T \left(\Sigma_{k|k}^{i(s)} \right)^{-1} \left(\mathbf{x}_k^{i(s)} - \hat{\mathbf{x}}_{k|k}^{i(s)} \right) \quad (35)$$

where NEES^i denotes the NEES of robot i , and $\mathbf{x}_k^{i(s)}$ is the true state vector of robot i at the s -th Monte Carlo simulation, and $\hat{\mathbf{x}}_{k|k}^{i(s)}$ and $\Sigma_{k|k}^{i(s)}$ are, respectively, the state estimate and estimation error covariance matrix of robot i at the s -th Monte Carlo simulation. In theory, if the state estimate is consistent, then the NEES value is not greater than the state dimension (i.e., $\text{NEES} \leq 3$), and the state estimate is strictly consistent when the NEES value is identical to state dimension (i.e., $\text{NEES} = 3$).

To compare the estimation accuracy of absolute and relative measurement noise covariance matrices, the RMSEs of standard deviations of range and bearing measurements are used as evaluation metrics, which are defined as

$$\begin{cases} \text{RMSE}_{\sigma_r} = \sqrt{\frac{1}{M_n} \sum_{s=1}^{M_n} \sum_{i=1}^{n_R} \left[\frac{\sum_{l \in I_k^{i*(s)}} \left(\sigma_r - \hat{\sigma}_{r,ak|k}^{il(s)} \right)^2}{\sum_{i=1}^{n_R} |I_k^{i*(s)}|} + \frac{\sum_{j \in J_k^{i*(s)}} \left(\sigma_r - \hat{\sigma}_{r,rk|k}^{ij(s)} \right)^2}{\sum_{i=1}^{n_R} |J_k^{i*(s)}|} \right]} \\ \text{RMSE}_{\sigma_\theta} = \sqrt{\frac{1}{M_n} \sum_{s=1}^{M_n} \sum_{i=1}^{n_R} \left[\frac{\sum_{l \in I_k^{i*(s)}} \left(\sigma_\theta - \hat{\sigma}_{\theta,ak|k}^{il(s)} \right)^2}{\sum_{i=1}^{n_R} |I_k^{i*(s)}|} + \frac{\sum_{j \in J_k^{i*(s)}} \left(\sigma_\theta - \hat{\sigma}_{\theta,rk|k}^{ij(s)} \right)^2}{\sum_{i=1}^{n_R} |J_k^{i*(s)}|} \right]} \end{cases} \quad (36)$$

TABLE VII: ARMSEs of position and orientation, ANEESs, and single step run times of all the algorithms compared for six mobile robots.

CL Methods	Metrics	Robot 1	Robot 2	Robot 3	Robot 4	Robot 5	Robot 6
CCL-EKF	ARMSE _p (m)	0.09	0.11	0.12	0.09	0.10	0.09
	ARMSE _o (deg)	1.34	1.42	1.43	1.29	1.30	1.31
	ANEES	3.00	3.00	3.04	2.98	3.00	2.99
	Time (ms)	0.26	0.26	0.26	0.26	0.26	0.26
RDCL-EKF-T	ARMSE _p (m)	0.10	0.11	0.12	0.09	0.10	0.09
	ARMSE _o (deg)	1.34	1.42	1.43	1.29	1.30	1.32
	ANEES	3.54	3.48	3.44	3.50	3.20	3.47
	Time (ms)	0.10	0.10	0.10	0.10	0.10	0.10
RDCL-EKF-N	ARMSE _p (m)	0.22	0.27	0.29	0.20	0.25	0.21
	ARMSE _o (deg)	1.82	1.86	1.87	1.79	1.83	1.81
	ANEES	10.79	13.80	15.85	10.07	12.53	9.89
	Time (ms)	0.10	0.10	0.10	0.10	0.10	0.10
RDCL-EKF-VB	ARMSE _p (m)	0.12	0.13	0.14	0.11	0.13	0.11
	ARMSE _o (deg)	1.51	1.57	1.59	1.48	1.52	1.50
	ANEES	3.23	3.37	3.54	3.34	3.35	3.27
	Time (ms)	0.28	0.28	0.28	0.28	0.28	0.28

where RMSE_{σ_r} and $\text{RMSE}_{\sigma_\theta}$ denote, respectively, the RMSEs of standard deviations of range and bearing measurements, and $l_k^{i*(s)}$ and $j_k^{i*(s)}$ denote, respectively, the detected landmark and robot sets by robot i at time k of the s -th Monte Carlo simulation, and $|l_k^{i*(s)}|$ denotes the number of landmarks within the set $l_k^{i*(s)}$, and $|j_k^{i*(s)}|$ denotes the number of mobile robots within the set $j_k^{i*(s)}$, and $\hat{\sigma}_{r,ak|k}^{il(s)}$ and $\hat{\sigma}_{\theta,ak|k}^{il(s)}$ denote the estimates of standard deviations of absolute range and bearing measurements at time k of the s -th Monte Carlo simulation, respectively, and $\hat{\sigma}_{r,rk|k}^{ij(s)}$ and $\hat{\sigma}_{\theta,rk|k}^{ij(s)}$ denote the estimates of standard deviations of relative range and bearing measurements at time k of the s -th Monte Carlo simulation, respectively.

Fig. 12: RMSEs of standard deviations of range and bearing measurements.

Fig. 13: ARMSEs of position and orientation and ANEESs when $\bar{\sigma}_\theta = 5$ (deg) and $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m).

B. Simulation results and analyses

Fig. 9–Fig. 11 show the RMSEs of position and orientation and the NEESs of all the algorithms compared for six mobile robots, respectively, and Table VII lists the averaged RMSEs (ARMSEs)

Fig. 14: ARMSEs of standard deviations of range and bearing measurements when $\bar{\sigma}_\theta = 5$ (deg) and $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m).

Fig. 15: ARMSEs of position and orientation and ANEESs when $\bar{\sigma}_r = 0.5$ (m) and $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg).

of position and orientation, the averaged NEESs (ANEESs), and the single step run times of all the algorithms compared for six mobile robots, respectively. It is seen from Fig. 9–Fig. 10

Fig. 16: ARMSEs of standard deviations of range and bearing measurements when $\bar{\sigma}_r = 0.5$ (m) and $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg).

and Table VII that the proposed RDCL-EKF-VB method has similar RMSEs and ARMSEs of position and orientation to the RDCL-EKF-T but smaller RMSEs and ARMSEs of position and orientation than the existing RDCL-EKF-N method. It can be seen from Fig. 11 and Table VII that the NEEs and ANEEs of the proposed method and the RDCL-EKF-T are almost identical and all close to the benchmark 3, however, the NEEs and ANEEs of the existing RDCL-EKF-N method are all far from the benchmark 3. We can also see from Table VII that the proposed RDCL-EKF-VB method requires more single step run times than the existing RDCL-EKF-N method. Thus, the proposed RDCL-EKF-VB method has similar CL accuracy and estimation consistency as the RDCL-EKF-T method but better CL accuracy and estimation consistency than the existing RDCL-EKF-N method at the cost of the increased computational complexity.

Fig. 12 illustrates the RMSEs of standard deviations of range and bearing measurements. We can see from Fig. 12 that the proposed RDCL-EKF-VB method has smaller $RMSE_{\sigma_r}$ and $RMSE_{\sigma_\theta}$ than the existing RDCL-EKF-N method. The $ARMSE_{\sigma_r}$ and $ARMSE_{\sigma_\theta}$ of the existing RDCL-EKF-N method and the proposed RDCL-EKF-VB method are, respectively, 1.22 (m), 3.21 (deg), 0.54 (m), and 1.27 (deg). Thus, the proposed RDCL-EKF-VB method can better estimate the absolute and relative measurement noise covariance matrices than the existing RDCL-EKF-N

method, which results in better CL accuracy and estimation consistency.

To better show the advantages of the proposed RDCL-EKF-VB method, the performances of all methods are further compared when different nominal standard deviations of range and bearing measurements are selected. Fig. 13 and Fig. 14 show the ARMSEs of position and orientation, ANEESs, and the ARMSEs of standard deviations of range and bearing measurements, respectively when the nominal standard deviation of bearing measurement is chosen as $\bar{\sigma}_\theta = 5$ (deg) and the nominal standard deviation of range measurement is set as $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m). Fig. 15 and Fig. 16 give the ARMSEs of position and orientation, ANEESs, and the ARMSEs of standard deviations of range and bearing measurements, respectively when the nominal standard deviation of range measurement is chosen as $\bar{\sigma}_r = 0.5$ (m) and the nominal standard deviation of bearing measurement is set as $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg). It is observed from Fig. 13 and Fig. 15 that the proposed RDCL-EKF-VB method always has better CL accuracy and estimation consistency than the existing RDCL-EKF-N method when different standard deviations of range and bearing measurements are selected. It can be also observed from Fig. 14 and Fig. 16 that the proposed RDCL-EKF-VB method can always estimate the absolute and relative measurement noise covariance matrices better than the existing RDCL-EKF-N method when different standard deviations of range and bearing measurements are selected. Consequently, for the case of time-varying measurement accuracy, the CL accuracy and estimation consistency of the existing RDCL-EKF method can be further improved by the proposed method, which is induced by the recursively adaptive learning of the absolute and relative measurement noise covariance matrices based on online pose estimates and measurement data using the VB approach.

V. EXPERIMENT STUDY

A. Experimental setups and descriptions

In this section, the performance of the proposed adaptive RDCL-EKF algorithm is tested with an open access *UTIAS multi-robot cooperative localization and mapping dataset* that was provided by Leung et al. [42]. In the experiment, a team of five identical mobile robots moved on a flat lab with fifteen landmarks, where the fifteen cylindrical tubes with distinguishable barcode identifiers served as landmarks. Each robot was equipped with a wheel encoder and a monocular camera, in which the wheel encoder was used to measure the linear and rotational velocities of each robot with output frequency 67 (Hz), and the monocular camera was employed to obtain

Fig. 17: Experimental field for UTIAS multi-robot cooperative localization and mapping dataset [42].

Fig. 18: Experimental platforms for UTIAS multi-robot cooperative localization and mapping dataset [42].

the range and bearing measurements between the robot and the landmark or the other robot. The reference positions of landmarks and the reference poses of all mobile robots were all provided by a 10-camera Vicon motion capture system at 100 (Hz) whose position accuracy is on the order of 1 (mm). The experimental field and platforms are, respectively, shown in Fig. 17 and Fig. 18.

Fig. 19: PEs of all the algorithms compared for five mobile robots using the data 1.

Fig. 20: OEs of all the algorithms compared for five mobile robots using the data 1.

The motion model of each robot is described by (1), and the discretization time of the motion model is set as $\Delta t = 0.02$ (sec). The range and bearing measurements between the robot and the landmark are described by the absolute measurement models in (2), and the range and bearing measurements between the robot and the other robot are modelled by the relative measurement

Fig. 21: NEESs of all the algorithms compared for five mobile robots using the data 1.

models in (4). Eight sets of real-world data (i.e., data 1 to data 8) are used in this experiment study, which are, respectively, collected in eight different runs with different test trajectories, moving velocities and running times ranging from 15 (min) to 70 (min). To demonstrate the effectiveness and superiority of the proposed method, three CL-EKF algorithms are compared in the experimental tests, including CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB, where CCL-EKF-N is the existing CCL-EKF algorithm using the nominal absolute and relative measurement noise covariance matrices $\mathbf{R}_{ak}$ and $\mathbf{R}_{rk}$ to estimate the poses of all robots as reference, and the descriptions and parameter settings of the RDCL-EKF-N and RDCL-EKF-VB in the experiment study are the same as those in the simulation study. Note that the CCL-EKF-N algorithm is used as an optimal benchmark for CL. Since the true state and measurement noise covariance matrices are unknown in the test, their nominal values are used in the CCL-EKF-N, RDCL-EKF-N, and RDCL-EKF-VB. In the experimental comparisons, the nominal state noise covariance matrix is selected as $\bar{\mathbf{Q}}_k = \text{diag}\{\bar{\sigma}_V^2, \bar{\sigma}_\Omega^2\}$, where $\bar{\sigma}_V = \frac{\sqrt{2}}{2}\bar{\sigma}$ and $\bar{\sigma}_\Omega = \frac{2\sqrt{2}}{\bar{\sigma}}$ with $\bar{\sigma} = 0.1$, and the nominal absolute and relative measurement noise covariance matrices are chosen as $\bar{\mathbf{R}}_{ak} = \bar{\mathbf{R}}_{rk} = \text{diag}\{\bar{\sigma}_r^2, \bar{\sigma}_\theta^2\}$ with nominal standard deviations $\bar{\sigma}_r = 0.5$ (m) and $\bar{\sigma}_\theta = 3$ (deg). The codes of all algorithms are performed on a computer with Intel Core i7-6500U CPU @ 2.50 GHz.

TABLE VIII: APEs, AOE, ANEESs, and single step run times of all the algorithms compared for five mobile robots using the data 1.

CL Methods	Metrics	Robot 1	Robot 2	Robot 3	Robot 4	Robot 5
CCL-EKF-N	APE (m)	0.10	0.09	0.08	0.10	0.10
	AOE (deg)	3.25	2.31	2.06	3.54	2.55
	ANEES	12.15	12.03	12.50	9.72	13.72
	Time (ms)	0.008	0.008	0.008	0.008	0.008
RDCL-EKF-N	APE (m)	0.35	0.40	0.25	0.56	0.31
	AOE (deg)	6.73	8.13	5.31	13.83	6.99
	ANEES	222.7	490.4	118.0	1462.5	214.0
	Time (ms)	0.008	0.008	0.008	0.008	0.008
RDCL-EKF-VB	APE (m)	0.15	0.14	0.11	0.23	0.13
	AOE (deg)	4.42	4.16	3.27	6.80	3.85
	ANEES	23.01	27.71	21.80	37.07	22.44
	Time (ms)	0.014	0.014	0.014	0.014	0.014

To compare the CL accuracy, the position error (PE) and the orientation error (OE) are selected as performance metrics, which are defined as follows

$$\begin{cases} \text{PE}^i(k) = \sqrt{\left(x_{rk}^i - \hat{x}_{k|k}^i\right)^2 + \left(y_{rk}^i - \hat{y}_{k|k}^i\right)^2} \\ \text{OE}^i(k) = \left|\phi_{rk}^i - \hat{\phi}_{k|k}^i\right| \end{cases} \quad (37)$$

where PE^i and OE^i denote the PE and OE of robot i , respectively, and x_{rk}^i , y_{rk}^i and ϕ_{rk}^i are, respectively, the reference values of east position, north position and orientation of robot i , and $\hat{x}_{k|k}^i$, $\hat{y}_{k|k}^i$ and $\hat{\phi}_{k|k}^i$ are, respectively, the estimates of east position, north position and orientation of robot i . Similar to as in the simulation study, the NEES is used to evaluate the estimation consistency, but M_n is set as 1 in the experiment study. To better show the results of experimental comparisons, the PEs, OEs and NEESs of all the algorithms compared are smoothed with a span of 20 (sec).

Fig. 22: OAPEs, OAOEs and OANEEs when $\bar{\sigma}_\theta = 5$ (deg) and $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m) using the data 1.

B. Experimental results and analyses

Firstly, we compare the PEs, OEs and NEEs of the CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB using the data 1. Note that, similar results can be also obtained using data 2 to data 8, so they are omitted for brevity. The PEs, OEs and NEEs of all the algorithms compared for five mobile robots are, respectively, shown in Fig. 19, Fig. 20 and Fig. 21. The averaged PEs (APEs), averaged OEs (AOEs), ANEEs, and single step run times of all the algorithms compared for five mobile robots are given in Table VIII. It can be seen from Fig. 19–Fig. 20 and Table VIII that the proposed RDCL-EKF-VB algorithm has smaller PEs, OEs, APEs and AOE than the existing RDCL-EKF-N algorithm. We can also see from Fig. 21 and Table VIII that the NEEs and ANEEs of the proposed RDCL-EKF-VB algorithm are closer to 3 as compared with those of the existing RDCL-EKF-N algorithm. Meanwhile, as shown in Table VIII, the proposed RDCL-EKF-VB algorithm has greater single step run times than the existing RDCL-EKF-N algorithm. It can be seen that the proposed RDCL-EKF-VB algorithm has better CL accuracy and estimation consistency but slightly greater implementation load than the existing RDCL-EKF-N algorithm for the real-world data 1, which results from the use of the more accurate measurement noise covariance matrices.

Fig. 23: OAPEs, OAOEs and OANEEs when $\bar{\sigma}_r = 0.5$ (m) and $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg) using the data 1.

TABLE IX: OAPEs, OAOEs and OANEEs of all the algorithms compared using Data 1–8.

CL Methods	Metrics	Data 1	Data 2	Data 3	Data 4	Data 5	Data 6	Data 7	Data 8
CCL-EKF-N	APE (m)	0.09	0.10	0.08	0.08	0.11	0.09	0.12	0.11
	AOE (deg)	2.74	3.04	2.46	2.21	3.89	3.30	3.18	4.65
	ANEES	12.03	14.55	11.87	13.65	9.98	9.80	18.59	14.04
RDCL-EKF-N	APE (m)	0.37	0.41	0.31	0.32	0.98	0.35	0.38	0.47
	AOE (deg)	8.20	9.07	6.67	6.25	24.77	8.89	8.12	13.39
	ANEES	501.5	365.8	233.2	286.7	1108.1	233.1	247.3	535.35
RDCL-EKF-VB	APE (m)	0.15	0.17	0.14	0.12	0.39	0.18	0.19	0.24
	AOE (deg)	4.50	5.15	4.02	3.49	9.89	5.29	4.92	8.53
	ANEES	26.41	32.40	27.95	27.31	75.48	73.05	47.68	42.02

Secondly, we aim to test the dependency of the proposed method on the selections of nominal standard deviations of range and bearing measurements. To this end, the overall APEs (OAPEs), overall AOE (OAOEs) and overall ANEEs (OANEEs) among all robots are used to compare

the performance, which are defined as

$$\text{OAPE} = \frac{1}{n_R} \sum_{i=1}^{n_R} \text{APE}^i, \quad \text{OAOE} = \frac{1}{n_R} \sum_{i=1}^{n_R} \text{AOE}^i, \quad \text{OANEES} = \frac{1}{n_R} \sum_{i=1}^{n_R} \text{ANEES}^i \quad (38)$$

where APE^i , AOE^i and ANEES^i denote, respectively, the APE, AOE and ANEES of robot i , and OAPE, OAOE and OANEES denote, respectively, the overall APE, AOE and ANEES among all robots.

We compare the OAPEs, OAOEs and OANEESs of the CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB using the data 1 when different nominal standard deviations of range and bearing measurements are selected. Fig. 22 illustrates the OAPEs, OAOEs and OANEESs of the CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB when the nominal standard deviation of bearing measurement is chosen as $\bar{\sigma}_\theta = 5$ (deg) and the nominal standard deviation of range measurement is set as $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m). Fig. 23 exhibits the OAPEs, OAOEs and OANEESs of the CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB when the nominal standard deviation of range measurement is chosen as $\bar{\sigma}_r = 0.5$ (m) and the nominal standard deviation of bearing measurement is set as $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg). It can be observed from Fig. 22 and Fig. 23 that the proposed RDCL-EKF-VB algorithm always has better CL accuracy and estimation consistency than the existing RDCL-EKF-N algorithm when $\bar{\sigma}_r = 0.1 : 0.1 : 1.0$ (m) and $\bar{\sigma}_\theta = 1 : 1 : 10$ (deg), which exhibits the weak dependency of the proposed method on the selections of nominal standard deviations of range and bearing measurements. The underlying cause is that the dependency on the nominal values of the measurement noise covariance matrices will decrease as the number of time-updates for them increases.

To better exhibit the practicability of the proposed method, we compare the OAPEs, OAOEs and OANEESs of the CCL-EKF-N, RDCL-EKF-N and RDCL-EKF-VB using the data 1 to data 8. Table IX lists the OAPEs, OAOEs and OANEESs of all the algorithms compared using data 1–8. We can observe from Table IX that the proposed RDCL-EKF-VB algorithm always has better CL accuracy and estimation consistency than the existing RDCL-EKF-N algorithm for the data 1–8, which demonstrates the superiority of the proposed method in practical CL application.

VI. CONCLUSIONS

In this paper, a novel adaptive RDCL-EKF method was proposed for multi-robot systems with time-varying measurement accuracy, where a novel decentralized estimation strategy was

proposed for estimating the unknown absolute and relative measurement noise covariance matrices in a decentralized manner. For each robot, its pose and absolute and relative measurement noise covariance matrices were jointly inferred based on the constructed hierarchical Gaussian model using the VB approach. The proposed adaptive RDCL-EKF method has the same mode and number of communications as the standard RDCL-EKF method, but only requires slightly higher communication capacities to transmit additional parameters in each cooperation. A large number of simulations and real-world data sets were used to compare the proposed adaptive RDCL-EKF method and the existing RDCL-EKF method. Simulation and experimental results have demonstrated that the proposed adaptive RDCL-EKF method has better CL accuracy and estimation consistency but slightly heavier computational burden than the existing RDCL-EKF method for multi-robot systems with time-varying measurement accuracy.

VII. APPENDIX

A. Proofs of Propositions 1–2

Before solving $q^{(a)}(\mathbf{x}_k^i)$ and $q^{(a)}(\mathbf{R}_{ak}^{il})$, the joint PDF $p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{\{l \in l_k^{i*}\}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{\{l \in l_k^{i*}\}}\right)$ in (14) and (15) is firstly calculated. According to Bayes' theorem and using (6)-(8) yields

$$\begin{aligned} p\left(\mathbf{x}_k^i, \{\hat{\mathbf{R}}_{ak}^{il}\}_{\{l \in l_k^{i*}\}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{\{l \in l_k^{i*}\}}\right) &= p(\mathbf{z}_{1:k-1})p(\mathbf{x}_k^i|\mathbf{z}_{1:k-1}) \prod_{l \in l_k^{i*}} p(\mathbf{z}_{ak}^{il}|\mathbf{x}_k^i, \mathbf{R}_{ak}^{il}) \times \\ p(\mathbf{R}_{ak}^{il}|\mathbf{z}_{1:k-1}) &= p(\mathbf{z}_{1:k-1})N(\mathbf{x}_k^i; \hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i) \prod_{l \in l_k^{i*}} N(\mathbf{z}_{ak}^{il}; \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l), \mathbf{R}_{ak}^{il}) \times \\ &\text{IW}(\mathbf{R}_{ak}^{il}; u_{ak|k-1}^{il}, \mathbf{U}_{ak|k-1}^{il}) \end{aligned} \quad (39)$$

Next, we solve $q^{(a)}(\mathbf{x}_k^i)$ and $q^{(a)}(\mathbf{R}_{ak}^{il})$. Substituting (39) in (14), we have

$$\log q^{(a)}(\mathbf{x}_k^i) = \log N(\mathbf{x}_k^i; \hat{\mathbf{x}}_{k|k-1}^i, \Sigma_{k|k-1}^i) + \sum_{l \in l_k^{i*}} \log N(\mathbf{z}_{ak}^{il}; \mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l), \hat{\mathbf{R}}_{ak}^{il}) + c_{\mathbf{x}_k^i} \quad (40)$$

By linearizing the absolute measurement functions $\{\mathbf{h}_{ak}^{il}(\mathbf{x}_k^i, \mathbf{x}_L^l)\}_{\{l \in l_k^{i*}\}}$ at $\mathbf{x}_k^i = \hat{\mathbf{x}}_{k|k-1}^i$ and exploiting (40), $q^{(a)}(\mathbf{x}_k^i)$ can be approximately updated as (16).

Substituting (39) in (15) gives

$$\log q^{(a)}(\mathbf{R}_{ak}^{il}) = -\frac{1}{2}(n_a + u_{ak|k-1}^{il} + 2) \log |\mathbf{R}_{ak}^{il}| - \frac{1}{2} \text{tr} \left\{ (\mathbf{U}_{ak|k-1}^{il} + \mathbf{A}_k^{il}) (\mathbf{R}_{ak}^{il})^{-1} \right\} + c_{\mathbf{R}_{ak}^{il}} \quad (41)$$

where $\text{tr}(\cdot)$ denotes the trace operation of a matrix.

According to the definition of the inverse-Wishart PDF, $q^{(a)}(\mathbf{R}_{ak}^{il})$ can be updated as (18).

B. Proofs of Propositions 3–4

Firstly, we calculate the joint PDF in (24)-(25) as follows. According to the conditional independence and the Bayes' theorem, the joint PDF can be factored as

$$p\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}}\right) = p\left(\mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \times p\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \prod_{j \in j_k^{i*}} p\left(\mathbf{z}_{rk}^{ij} | \mathbf{x}_k^i, \mathbf{x}_k^j, \mathbf{R}_{rk}^{ij}\right) p\left(\mathbf{R}_{rk}^{ij} | \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \quad (42)$$

Exploiting the variational approximation to $p\left(\mathbf{x}_k^i | \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right)$ in (13) and (16), the joint posterior PDF $p\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right)$ can be approximated as Gaussian, i.e.,

$$p\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}} \middle| \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) = \mathcal{N}\left(\begin{bmatrix} \mathbf{x}_k^i \\ \mathbf{x}_k^{j_k^{i*}} \end{bmatrix}; \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{j_k^{i*}(a)} \end{bmatrix}, \begin{bmatrix} \Sigma_{k|k}^{i(a)} & \Sigma_{k|k}^{ij_k^{i*}(a)} \\ \left(\Sigma_{k|k}^{ij_k^{i*}(a)}\right)^T & \Sigma_{k|k}^{j_k^{i*}(a)} \end{bmatrix}\right) \quad (43)$$

where the augmented state vector is defined as $\mathbf{x}_k^{j_k^{i*}} \triangleq \text{col}\{\mathbf{x}_k^j\}_{j \in j_k^{i*}}$, and the state estimates $\hat{\mathbf{x}}_{k|k}^{i(a)}$ and $\hat{\mathbf{x}}_{k|k}^{j_k^{i*}(a)}$, the error covariance matrices $\Sigma_{k|k}^{i(a)}$ and $\Sigma_{k|k}^{j_k^{i*}(a)}$, and the cross-correlation $\Sigma_{k|k}^{ij_k^{i*}(a)}$ are all defined in Table III, which can be obtained by running the LAMU function with the estimated absolute measurement noise covariance matrices.

Using (10)-(11) and (43) in (42), we have

$$\begin{aligned} & p\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}, \{\mathbf{R}_{rk}^{ij}\}_{j \in j_k^{i*}}, \mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}, \{\mathbf{z}_{rk}^{ij}\}_{j \in j_k^{i*}}\right) \\ &= \mathcal{N}\left(\begin{bmatrix} \mathbf{x}_k^i \\ \mathbf{x}_k^{j_k^{i*}} \end{bmatrix}; \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{j_k^{i*}(a)} \end{bmatrix}, \begin{bmatrix} \Sigma_{k|k}^{i(a)} & \Sigma_{k|k}^{ij_k^{i*}(a)} \\ \left(\Sigma_{k|k}^{ij_k^{i*}(a)}\right)^T & \Sigma_{k|k}^{j_k^{i*}(a)} \end{bmatrix}\right) p\left(\mathbf{z}_{1:k-1}, \{\mathbf{z}_{ak}^{il}\}_{l \in l_k^{i*}}\right) \times \\ & \prod_{j \in j_k^{i*}} \mathcal{N}\left(\mathbf{z}_{rk}^{ij}; \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j), \mathbf{R}_{rk}^{ij}\right) \text{IW}(\mathbf{R}_{rk}^{ij}; u_{rk|k-1}^{ij}, \mathbf{U}_{rk|k-1}^{ij}) \end{aligned} \quad (44)$$

Secondly, we calculate the approximate posterior PDFs $q\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}\right)$ and $\{q(\mathbf{R}_{rk}^{ij})\}_{j \in j_k^{i*}}$. Substituting (44) in (24) yields

$$\begin{aligned} \log q\left(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}\right) &= \log \mathcal{N}\left(\begin{bmatrix} \mathbf{x}_k^i \\ \mathbf{x}_k^{j_k^{i*}} \end{bmatrix}; \begin{bmatrix} \hat{\mathbf{x}}_{k|k}^{i(a)} \\ \hat{\mathbf{x}}_{k|k}^{j_k^{i*}(a)} \end{bmatrix}, \begin{bmatrix} \Sigma_{k|k}^{i(a)} & \Sigma_{k|k}^{ij_k^{i*}(a)} \\ \left(\Sigma_{k|k}^{ij_k^{i*}(a)}\right)^T & \Sigma_{k|k}^{j_k^{i*}(a)} \end{bmatrix}\right) + \\ & \sum_{j \in j_k^{i*}} \log \mathcal{N}\left(\mathbf{z}_{rk}^{ij}; \mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j), \hat{\mathbf{R}}_{rk}^{ij}\right) + c_{\{\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}}\}} \end{aligned} \quad (45)$$

By linearizing the relative measurement function $\mathbf{h}_{rk}^{ij}(\mathbf{x}_k^i, \mathbf{x}_k^j)$ at $\{\mathbf{x}_k^i = \hat{\mathbf{x}}_{k|k}^{i(a)}, \mathbf{x}_k^j = \hat{\mathbf{x}}_{k|k}^{j(a)}\}$ and exploiting (45), $q(\mathbf{x}_k^i, \{\mathbf{x}_k^j\}_{j \in j_k^{i*}})$ can be approximately updated as (26).

Substituting (44) in (25) gives

$$\log q(\mathbf{R}_{rk}^{ij}) = -\frac{1}{2}(n_r + u_{rk|k-1}^{ij} + 2) \log |\mathbf{R}_{rk}^{ij}| - \frac{1}{2} \text{tr} \left\{ \left(\mathbf{U}_{rk|k-1}^{ij} + \mathbf{B}_k^{ij} \right) (\mathbf{R}_{rk}^{ij})^{-1} \right\} + c_{\mathbf{R}_{rk}^{ij}} \quad (46)$$

According to the definition of the inverse-Wishart distribution, the posterior PDF $q(\mathbf{R}_{rk}^{ij})$ is updated as (28).

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