The Robustness of Resolvable Block Designs Against the Loss of Whole Blocks or Replicates

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Abstract

This paper considers the robustness of resolvable incomplete block designs in the event of two patterns of missing observations: loss of whole blocks and loss of whole replicates. The approach used to assess designs is based on the concept of block intersection which exploits the resolvability property of the design. This improves on methods using minimal treatment concurrence which have been used previously. It is shown that several classes of designs, including affine resolvable designs, square and rectangular lattice designs and two-category concurrence α -designs and α_n -designs, are maximally robust; some of these classes of designs are also shown to be most replicate robust.

Keywords: Affine resolvable; α -design; α_n -design; Block intersection; Connectivity; Estimability; Lattice design; Maximal robustness; Most replicate robust.

1. Introduction

An incomplete block design for $v = ks$ treatments in blocks of size k is resolvable if the blocks can be partitioned into $r \geq 2$ sets, known as replicates, or super-blocks, such that each replicate consists of $s \geq 2$ blocks and each treatment appears once in each replicate. Square lattice designs, affine resolvable designs and rectangular lattice designs are resolvable designs which were considered by Yates (1936), Bose (1942) and Harshbarger (1949) respectively. The general advantages of resolvable designs for experiments where the replication system describes separate locations or different periods of time, such as agricultural field trials or experiments involving multiple harvests, are well documented; see Patterson and Williams (1976), Bailey *et al.* (1995), John and Williams (1995, chapter 4), Morgan and Reck (2007), Caliński *et al.* (2009) and others. Affine resolvable designs have the property that every pair of blocks from different replicates has k/s treatments in common; Bailey *et al.* (1995) showed that these designs are optimal according to several criteria, including A -, D - and E -optimality, among the class

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of resolvable designs with the same values of v , k and r and these authors gave methods of construction based on orthogonal arrays; see also Morgan (2010). The α -designs (Patterson and Williams, 1976) and α_n -designs (John *et al.*, 2002) are resolvable designs that cover most combinations of v , k and r likely to be required in practice and include several design types, such as square and rectangular lattice designs, as special cases.

It is not unusual for observations to be lost during experimentation. For experiments using incomplete block designs, observation loss can result in an eventual design that is disconnected. If this occurs then, as described by Godolphin (2006), not all treatment contrasts will be estimable which seriously limits the inferences that can be made from the experimental results. Using a criterion introduced by Ghosh (1982), termed Criterion 1 by Dey (1993), a connected incomplete block design is said to be robust against the loss of t observations if the loss of any t observations yields a connected eventual design. The nature of a blocking factor can mean that observation loss tends to involve whole blocks or replicates rather than discrete observations. For example: batches of raw material can be found to be contaminated; areas of agricultural land can be flooded; harvests occasionally fail; trials are ended prematurely due to lack of resources. Baksalary and Tabis (1987), Sathe and Satam (1992) and Godolphin and Warren (2011) considered the robustness of an arbitrary binary block design against the loss of whole blocks. Results obtained by these authors provide lower bounds for λ_* , the minimal treatment concurrence of the design, i.e. the smallest number of blocks that both members of a pair of treatments occur in. These results are therefore only useful for assessing designs with $\lambda_* > 0$ whereas many classes of resolvable designs, such as the α -designs and α_n -designs, have $\lambda_* = 0$. Furthermore, these results do not exploit the property that blocks are partitioned into whole replicates. Recent work by Bailey *et al.* (2013) and Tsai and Liao (2012) concentrate on the particular case of missing blocks in designs with blocks of size two, which have particular relevance in two-colour microarray experiments.

This paper focuses on the robustness of resolvable designs where the likely pattern of observation loss involves whole blocks or replicates. The concept of block intersection considered by Street and Street (1987), Bailey *et al.* (1995) and others is used to identify robustness properties of the designs. Following the loss of a specified number of blocks or replicates during the experiment, the method centres on the connectivity of blocks in the eventual design, rather than on the connectivity of treatments. It is well known that these properties are equivalent for any block design. The approach of determining robustness via block intersection is very different from that based on minimal concurrence; it has useful implications for resolvable designs and enables their robustness properties to be assessed easily using the device of a graph termed the *replicate connectivity graph*.

The paper is organised as follows. Preliminary concepts, not specific to resolvable designs, are introduced in Section 2. Section 3 establishes a number of results giving conditions for resolvable designs to have desirable robustness properties, namely of being *maximally robust* or *most replicate robust*. Concepts from graph theory are introduced and used in this section. For readers unfamiliar with this material, a thorough treatment is given in chapter 1 of Harris *et al.* (2008). In Section 4 conditions are obtained for α -designs and α_n -designs with two replicates to be connected. This enables straightforward construction of the replicate connectivity graph for α -designs and α_n -designs with r

replicates and, hence, use of results from Section 3 to obtain robustness properties of designs in these classes with $r > 2$. Several examples are given to illustrate the main results.

2. Preliminaries

Let D be an incomplete block design, not necessarily resolvable, in which v treatments are arranged in blocks $B_1, \dots, B_b$, all of size k . Further, let D be binary, so no treatment occurs more than once in a single block. The design D is specified by the $v \times b$ treatment-block incidence matrix $N = [N_1 \dots N_b]$, where the $v \times 1$ vector N_i has i th term 1 if treatment i occurs in block B_i and zero otherwise. The block concurrence matrix, or concurrence matrix of the dual design, is $N'N$ which has (i, j) th element $N'_i N_j = N'_j N_i$ specifying the number of treatments common to blocks B_i and B_j . When examining the connectivity status of the design, it matters which elements of $N'N$ are non-zero. Following Street and Street (1987, chapter 2), B_i intersects with B_j if $N'_i N_j \neq 0$, i.e. at least one treatment is common to both blocks. Blocks B_i and B_j are connected if they intersect or if there is a chain of blocks of D linking B_i to B_j such that each adjoining pair of blocks in the chain intersect.

If b_* blocks are lost during an experiment based on a planned design D , where $b_* \leq r - 1$, the eventual design consists of v treatments arranged in $b - b_*$ blocks. Design D is said to be robust to the loss of b_* blocks when both D and this eventual design are connected sets of blocks, irrespective of which b_* blocks are lost. When $b_* = r - 1$ the eventual design is written as $D_\#$ and Ghosh (1982) gave the following definition.

Definition A binary design D is maximally robust if D and $D_\#$ are connected sets of blocks, irrespective of which $r - 1$ blocks are lost from D to yield $D_\#$.

It should be noted that a design D which is robust to the loss of any b_* blocks will also be robust to the loss of any b_* observations.

3. General Conditions for Resolvable Designs

Let $D_1, \dots, D_r$ denote the r replicates of a resolvable design D . The following result gives a simple condition for maximal robustness which applies to an arbitrary resolvable design, provided that k , the size of the blocks of D , is not too small compared to s , the number of blocks in a replicate.

Theorem 1. *Let D be a resolvable design such that $k > \frac{1}{2}s$. If each block of replicate D_i has a treatment in common with more than half of the blocks in D_j ($i, j = 1, \dots, r; i \neq j$), then D is maximally robust.*

Proof The initial requirement that $k > \frac{1}{2}s$ is necessary for the condition of the theorem to be achievable. Suppose that $r - 1$ blocks are lost from D during the course of the experiment. The eventual design, $D_\#$, retains at least one whole replicate, which can

be assumed to be D_1 and, furthermore, at least one other replicate, D_2 say, has lost at most one block.

Suppose that $D_\#$ is a disconnected design. It follows that the blocks of $D_\#$ can be arranged into two nonempty sets S_1, S_2 such that all replicates of a proper subset of the treatments are contained in blocks of S_1 and all replicates of the remaining treatments are contained in blocks of S_2 (see Godolphin (2004)). At least one block from D_1 will be in S_1 , say block B_1 , and at least one block from D_1 will be in S_2 , say block B_2 . From the condition of the theorem there is at least one block, B_s say, in the remnant of D_2 in $D_\#$ which intersects with B_1 . But B_s intersects with more than half the blocks in D_1 , therefore S_1 contains more than half of the blocks from D_1 . By the same argument S_2 contains more than half of the blocks from D_1 . Consequently, disjoint sets S_1, S_2 do not exist, which implies that $D_\#$ is connected. $\square$

The condition of Theorem 1 is tight for some designs as illustrated by the first example.

Example 1 Let D be a resolvable design such that three whole replicates of fifteen treatments are arranged in fifteen blocks of size 3:

0	1	2	3	4	0	1	2	3	4	0	2	4	9	12
5	6	7	8	9	6	5	9	7	8	1	3	7	10	13
10	11	12	13	14	11	10	13	14	12	6	5	8	11	14
replicate D_1					replicate D_2					replicate D_3				

where columns show the blocks. Replicate D_1 , in which the i th block has treatments labelled $\{i + js - 1 : j = 0, \dots, k - 1\}$ is said to be in standard form. For this design $r = 3$ and $3 = k > \frac{1}{2}s = \frac{5}{2}$; also, treatments labelled 0 and 2 do not occur together in any block and so the minimum concurrence $\lambda_* = 0$. Most of the blocks in each replicate intersect with three blocks in each of the other two replicates; however, there are exceptions, e.g block $[0 \ 5 \ 10]'$ in replicate D_1 intersects with only two blocks in D_2 , so the condition of Theorem 1 fails and maximal robustness is not guaranteed. In fact D is robust against the loss of any one block but it is not maximally robust: if blocks $[2 \ 3 \ 5]'$ and $[9 \ 10 \ 11]'$ of D_3 are missing then the eventual design $D_\#$ is disconnected since its blocks can be arranged into two sets with blocks in one set containing all replicates of treatments labelled 0, 1, 5, 6, 10, 11 and no occurrences of the other treatments.

A theorem of Ghosh (1982) states that all balanced incomplete block designs are maximally robust; however, this appears to be the only result in the literature that identifies a particular class of designs as having the maximal robustness property. A useful consequence of Theorem 1 is to extend Ghosh's result to include the class of affine resolvable designs given by Bose (1942).

Proposition 1. *All affine resolvable designs are maximally robust.*

Proof If D is affine resolvable then k/s is an integer, so $k > \frac{1}{2}s$ and Theorem 1 applies. The result follows from the affine resolvability property which implies that each block in replicate D_i intersects with all blocks in the remaining $r - 1$ replicates ($i = 1, \dots, r$). $\square$

The defining property of affine resolvability is a condition on pairs of blocks of $\tilde{D}$ which is comparable to the condition of common concurrence on pairs of treatments of a balanced design; hence Proposition 1 can be regarded as a ‘mirror-image’ of Ghosh’s theorem. It is noted that maximal robustness is a property of the square lattice designs of Yates (1936) and the rectangular lattice designs of Harshbarger (1949).

Proposition 2. *Square lattice designs and rectangular lattice designs are maximally robust.*

Proof Square lattice designs are affine resolvable. Rectangular lattice designs have $k > \frac{1}{2}s$ and are constructed so that each block has treatments in common with all but one block in every other replicate and there are at least three blocks in each replicate. Therefore the maximal robustness of both design types follows from Theorem 1. $\square$

For the class of connected resolvable designs with $r = 2$, the following result establishes maximal robustness of every member of the class. This will be used extensively in the remainder of the paper to determine robustness properties of designs with $r > 2$.

Proposition 3. *Every connected resolvable design in two replicates is maximally robust.*

Proof Let D be a connected resolvable design in two replicates D_1 and D_2 and suppose that $D_{\#}$ is the eventual design after losing a block, B say, from D_2 . Suppose that $D_{\#}$ is disconnected; then the blocks of $D_{\#}$ separate into two nonempty sets S_1, S_2 such that no block of S_1 intersects with a block of S_2 . Clearly S_i contains at least as many blocks from D_1 as from the remnant of D_2 ($i = 1, 2$) since each treatment occurring in the remnant of D_2 has the same matching treatment in D_1 . Furthermore one set, S_1 say, has an even number of blocks. Therefore half of the blocks of S_1 are from D_1 and half are from the remnant of D_2 , which implies that each treatment occurring in blocks of S_1 is replicated twice. Hence B does not intersect with any of the blocks of S_1 . This can only arise if design D is disconnected, which is a contradiction. $\square$

Proposition 3 focuses attention on the need to determine whether a two-replicate resolvable design D is connected or not. A simple method for this is to consider the block concurrence matrix $N'N$, which has the form

$$N'N = \begin{bmatrix} kI & U \\ U' & kI \end{bmatrix}, \quad (3.1)$$

where I denotes the identity matrix and all four components in (3.1) have dimension $s \times s$. If U cannot be made block diagonal by permuting its rows and columns then D is connected, otherwise D is disconnected. This follows since U is the incidence matrix of a symmetric block design in s treatments, designated the contraction of D by Williams *et al.* (1976), and the design D is connected if and only if the contraction of D is connected. Eccleston and Hedayat (1974) give a necessary and sufficient condition for the latter to be the stated condition on U .

In the general case, where $r \geq 2$, the block concurrence matrix $N'N$ of D can be represented as a block matrix of r^2 components of size $s \times s$ such that the r diagonal

terms are all kI and the $\frac{1}{2}r(r-1)$ upper triangular terms U_{ij} represent incidence matrices⁶ for the contractions generated by pairs of replicates D_i, D_j ($1 \leq i \leq r-1; i+1 \leq j \leq r$). Each of the U_{ij} has the same role as U in (3.1); i.e. replicates D_i and D_j are a connected set of blocks, denoted by $D_i \sim D_j$, if and only if U_{ij} cannot be made block diagonal by permuting rows and columns. This result can be used to establish robustness properties of D which do not depend on conditions concerning the relative sizes of k and s . The information on pairs of replicates that comprise connected sets of blocks is represented as a graph.

Definition Let D be a resolvable design. The replicate connectivity graph, $\mathcal{R}$, for D is a simple graph with vertices $1, \dots, r$. Two distinct vertices i and j are joined by an edge if and only if replicates D_i and D_j together comprise a connected set of $2s$ blocks.

This representation of the properties of pairs of replicates enables concepts from graph theory to be used in determining connectivity properties of resolvable designs. In particular, the definition of a graph as being Hamiltonian if and only if it contains a cycle that passes through each vertex exactly once, see Harris *et al.* (2008, chapter 1), is used in the following result.

Theorem 2. *Let D be a resolvable design with the property that its replicate connectivity graph is Hamiltonian. Then D is maximally robust.*

Proof By the condition of the theorem, the r replicates of D can be arranged in a cycle of length r , say $D_1 \sim \dots \sim D_r \sim D_1$, so the adjacent replicate pairs $\{D_i, D_{i+1}\}$ ($i = 1, \dots, r-1$) and $\{D_r, D_1\}$ each form a connected set of $2s$ blocks. Let $r-1$ blocks be lost from D and let $D_{\#}$ be the eventual design. Regardless of which blocks are lost there will be at least one replicate pair, say $\{D_1, D_2\}$, that loses no more than one block between them. It follows from Proposition 3 that the set of blocks belonging to $\{D_1, D_2\}$ that are not lost is a connected set and contains all v treatments. Thus, $D_{\#}$ is connected and the theorem is proved. $\square$

Corollary 1. *If D is a resolvable design such that each of the $\frac{1}{2}r(r-1)$ pairs of replicates is a connected set of blocks then D is maximally robust.*

Example 2 These results are used to consider again design D from Example 1. The three U matrices (3.1) for the replicate pairs $\{D_1, D_2\}$, $\{D_1, D_3\}$ and $\{D_2, D_3\}$ are the 5×5 upper components of the block concurrence matrix $N'N$ given, respectively, by

$$U_{12} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad U_{13} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad U_{23} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix};$$

so replicate pair $\{D_1, D_2\}$ is a disconnected set of $2s = 10$ blocks and pairs $\{D_1, D_3\}$ and $\{D_2, D_3\}$ are each connected sets. Consequently $\mathcal{R}$ is not Hamiltonian. As noted

previously, D is robust against the loss of one block but the design is not maximally robust.

Example 3 The resolvable design $T1$ given by Clatworthy (1973, p. 234) is a two-associate partially balanced design based on a triangular association scheme such that $r = 6$ replicates of $v = 10$ treatments are arranged in $s = 5$ blocks per replicate, each block of size $k = 2$:

0 1	0 2	0 3	0 4	0 5	0 6
2 3	1 8	1 2	1 3	1 4	1 7
4 8	6 3	4 6	2 5	2 7	2 9
5 6	4 7	5 7	7 8	3 9	3 8
7 9	5 9	8 9	6 9	6 8	4 5
D_1	D_2	D_3	D_4	D_5	D_6

where rows show the blocks. Theorem 1 does not apply since $k < \frac{1}{2}s$; also it is clear that the minimum concurrence $\lambda_* = 0$. It is straightforward to see by inspection, or by forming the block components U_{ij} of $N'N$, that replicate pairs $\{D_1, D_3\}$, $\{D_1, D_5\}$ and $\{D_4, D_6\}$ are disconnected sets of blocks and the remaining twelve pairs are connected. Thus Corollary 1 does not apply to $T1$. However there is a Hamiltonian cycle in $\mathcal{R}$, for example $D_1 \sim D_2 \sim D_3 \sim D_4 \sim D_5 \sim D_6 \sim D_1$, showing that $T1$ is maximally robust.

3.1. Robustness against loss of whole replicates

The replicate connectivity graph is a useful tool for assessing whether a resolvable design D is invulnerable to the loss of r_* whole replicates, i.e. whether the eventual design is connected irrespective of which r_* replicates are removed from D , where $1 \leq r_* \leq r-2$. Note that the removal of any $r-1$ replicates to leave a single replicate will always give a disconnected eventual design. When $r_* = r-2$ the eventual design is written as $D_{\dagger}$. The following definition parallels that for maximal robustness, but for observation loss occurring as replicates rather than individual blocks.

Definition A resolvable design D is most replicate robust if D and $D_{\dagger}$ are connected sets of blocks, irrespective of which $r-2$ replicates are lost from D to yield $D_{\dagger}$.

Let $\mathcal{R}_*$ be the replicate connectivity graph of the eventual design, then vertices in $\mathcal{R}_*$ comprise $r-r_*$ vertices from $\mathcal{R}$ and edges between pairs of vertices in $\mathcal{R}_*$ are exactly those between the same pairs of vertices in $\mathcal{R}$. Let two vertices in $\mathcal{R}$ be joined by an edge then the corresponding replicates, D_1 and D_2 say, are a connected set. Any design which includes these two replicates is also connected, since no blocks additional to those in D_1 and D_2 will introduce any new treatments. Hence a sufficient condition for the eventual design to be connected is that at least one edge of $\mathcal{R}$ remains in $\mathcal{R}_*$, i.e. that $\mathcal{R}_*$ contains at least one edge.

The graph $\mathcal{R}$ will consist of connected subgraphs $\mathcal{R}_1, \dots, \mathcal{R}_q$ for some q , with $1 \leq q \leq r$. Each subgraph $\mathcal{R}_i$ has r_i vertices and e_i edges, $i = 1, \dots, q$. The following result depends on being able to relate properties of $\mathcal{R}$ and $\mathcal{R}_*$ and this is achieved using the concepts of *independent set* and *independence number* from graph theory. For an

arbitrary simple graph $\mathcal{G}$, an independent set is a set of vertices with the property that no pair are joined by an edge. The independence number of $\mathcal{G}$ is the size of the largest independent set, see Harris *et al.* (2008, chapter 1). The independence number of the subgraph $\mathcal{R}_i$ is denoted $\mathcal{I}_i$, $i = 1, \dots, q$.

Theorem 3. *Let D be a resolvable design with r replicates such that the replicate connectivity graph has $q \geq 1$ connected component subgraphs, as described above. D is robust against the loss of r_* whole replicates if either of the following conditions is satisfied:*

$$(i) \quad r_* \leq (r - 1) - \sum_{i=1}^q \mathcal{I}_i, \quad (ii) \quad r_* \leq r - \frac{q+2}{2} - \sum_{i=1}^q \sqrt{\frac{1}{4} + r_i(r_i - 1) - 2e_i}. \quad (3.2)$$

Proof Assume for simplicity that $q = 1$. Any subgraph of $\mathcal{R}_1$ that contains at least $\mathcal{I}_1 + 1$ vertices has at least one edge and therefore corresponds to a connected set of blocks that contains all v treatments. The loss of some number r_* replicates from D is equivalent to the removal of r_* vertices from $\mathcal{R}$ to give $\mathcal{R}_*$, the replicate connectivity graph of the eventual design. It follows that if $r_* \leq r - (\mathcal{I}_1 + 1)$ then $\mathcal{R}_*$ contains at least one edge and the eventual design will be connected. The corresponding argument when $\mathcal{R}$ has $q \geq 2$ connected component subgraphs is straightforward, so condition (3.2) (i) holds. Condition (3.2) (ii) follows from the upper bound for the independence number of a simple undirected connected subgraph, given by Harant *et al.* (2001) as

$$\mathcal{I}_i \leq \frac{1 + \sqrt{1 + 4r_i(r_i - 1) - 8e_i}}{2}. \quad (3.3)$$

□

Corollary 2. *If D is a resolvable design such that each of the $\frac{1}{2}r(r-1)$ pairs of replicates is a connected set of blocks then D is a most replicate robust design.*

Proof When $\mathcal{R}$ is complete then $q = 1$ and $\mathcal{I}_1 = 1$, so it follows from condition (3.2) (i) of Theorem 3 that D is robust against the loss of $r - 2$ complete replicates. □

Proposition 4. *All affine resolvable designs and all square and rectangular lattice designs are most replicate robust.*

Proof These results hold because $\mathcal{R}$ is complete for each design. □

Example 4 The replicate connectivity graph, $\mathcal{R}$, for Example 3 has six vertices and twelve edges, the only vertex pairs not joined by an edge correspond to replicate pairs $\{D_1, D_3\}$, $\{D_1, D_5\}$ and $\{D_4, D_6\}$. These three vertex pairs form the largest independent sets so $q = 1$ and $\mathcal{I}_1 = 2$; Theorem 3 shows that $r_* \leq 5 - 2 = 3$, so the resolvable partially balanced design of triangular type, cited as $T1$ by Clatworthy, is robust against the loss of three replicates. There are, of course, three ways in which the eventual design is disconnected if as many as four replicates are lost during the experiment.

Example 5 The replicate connectivity graph of a resolvable design D , with treatments⁹ arranged in $r = 9$ replicates, consists of $q = 3$ connected subgraphs $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, as follows:

It is seen by inspection that $\mathcal{I}_1 = 2, \mathcal{I}_2 = 1$ and $\mathcal{I}_3 = 2$, which are the same as the bounds given by (3.3). It follows from condition (3.2) (i) that $r_* \leq 8 - 5 = 3$, i.e. D is robust against the loss of three complete replicates.

Designs with the same parameters do not necessarily have common robustness properties. This is demonstrated in Example 6 where, despite having the same parameters and being equally efficient, the designs under consideration have different robustness properties.

Example 6 The resolvable design $SR39$ given by Clatworthy (1973, p. 149) is a semi-regular group divisible design such that $r = 8$ replicates of $v = 8$ treatments are arranged in $s = 2$ blocks of size $k = 4$ per replicate. Mitra *et al.* (2002) give a non-isomorphic semi-regular group divisible design with the same parameters. The replicate connectivity graph of design $SR39$ contains a Hamiltonian cycle and has four independent sets of size 2; hence by Theorem 2 the design is maximally robust and by Theorem 3 it is robust against the loss of $7 - 2 = 5$ complete replicates. It transpires that there are four ways in which the eventual design is disconnected if six replicates are lost during the experiment so design $SR39$ is not most replicate robust. In comparison, the replicate connectivity graph for the design of Mitra *et al.* (2002) is complete, i.e. each pair of vertices is joined by an edge. Hence this design is maximally robust from Corollary 1 and it is most replicate robust because of Corollary 2.

4. Conditions for α -Designs and α_n -Designs

4.1. The α -Designs

There are many values of k, r and s for which neither affine resolvable nor lattice designs exist. Patterson and Williams (1976) introduced the class of α -designs which is a wide class of resolvable designs that caters for all values of k, r and s likely to be needed in practice. These designs are useful in applications and some are highly efficient; however, unlike the affine resolvable and lattice designs, neither robustness against the loss of blocks or whole replicates cannot be guaranteed: indeed some α -designs are not connected. All α -designs are such that the minimum concurrence $\lambda_* = 0$.

The r replicates of an α -design, D , are generated by the columns of a $k \times r$ reduced array A_0 given by

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{1,1} & \cdots & a_{1,r-1} \\ \vdots & \vdots & & \vdots \\ 0 & a_{k-1,1} & \cdots & a_{k-1,r-1} \end{bmatrix}, \quad (4.1)$$

such that $a_{i,j} \in \Sigma$, where $\Sigma = \{0, 1, \dots, s-1\}$ denotes the set of integer-valued residues modulo (s) . Other arrays A with some non-zero residues in the first row and first column are possible: their use is unnecessary since a design obtained with such an array will be isomorphic to one generated from a reduced array (4.1). In the particular case when $r = 2$ the residues in the second column of (4.1) are written $0, a_1, \dots, a_{k-1}$ for simplicity. Details of the derivation and properties of D are given by Patterson and Williams (1976), Street and Street (1987, §8.7) and John and Williams (1995, chapter 6). Use of the reduced array generates one replicate, here designated D_1 , which is in standard form as described in Example 1. The i th replicate D_i , for $i = 2, \dots, r$, is obtained using the i th column of A_0 and has blocks the s columns of the $k \times s$ matrix sum:

$$C_i + S = \begin{bmatrix} 0 & 1 & \cdots & s-1 \\ a_{1,i} & a_{1,i} + 1 & \cdots & a_{1,i} + s-1 \\ \vdots & \vdots & & \vdots \\ a_{k-1,i} & a_{k-1,i} + 1 & \cdots & a_{k-1,i} + s-1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ s & s & \cdots & s \\ \vdots & \vdots & & \vdots \\ s-1 & s-1 & \cdots & s-1 \end{bmatrix}, \quad (4.2)$$

where the terms in C_i are reduced module (s) . Following Williams *et al.* (1976), the array C_i is the contraction for the two replicate sub-design comprising D_1 and D_i .

In the particular case of designs with $r = 2$, the question of whether an α -design, D , is connected or not depends entirely on an interesting condition governing the choice of residues $a_1, \dots, a_{k-1}$ in the reduced array.

Theorem 4. *Let D be the α -design for two replicates generated by a reduced array A_0 with residues $0, a_1, \dots, a_{k-1}$ in the second column. Design D is maximally robust if and only if the greatest common divisor of $\{s, a_1, \dots, a_{k-1}\}$ is unity.*

The proof of Theorem 4 is given after the proof of Theorem 5. An equivalent condition for D to be a connected design is stated by Williams *et al.* (2011). Note that the two-replicate design must be generated by a reduced array in order to apply the condition of the theorem.

Now suppose that D is an α -design in $r \geq 3$ replicates and let D_i and D_j be two replicates of D . Consider the sub-design comprising just these two replicates. This sub-design is isomorphic to a two replicate design with residues in the second column of the reduced array:

$$0, a_1 = a_{1,i} - a_{1,j}, a_2 = a_{2,i} - a_{2,j}, \dots, a_{k-1} = a_{k-1,i} - a_{k-1,j},$$

where the residues are evaluated modulo (s) . It follows from Theorem 4 that a necessary and sufficient condition for an edge to exist between vertices i and j of the replicate

connectivity graph $\mathcal{R}$ of D is that the terms in $\{s, a_1, \dots, a_{k-1}\}$ have greatest common divisor unity. This condition can be applied $\frac{1}{2}r(r-1)$ times, i.e. to each pair of replicates, as part of a straightforward and immediate procedure of assessing the connectivity of all two replicate sub-designs and hence, enabling easy construction of $\mathcal{R}$. Robustness properties of D are then obtained by examination of the conditions of Theorems 2 and 3. The procedure is illustrated by an example.

Example 7 Patterson and Williams (1976) give two highly efficient four-replicate α -designs in Table 4 of their paper, which are suggested by the authors to cover situations where lattice designs do not exist. Their α -design which is alternative to a square lattice has $v = 36$ treatments in $s = 6$ blocks per replicate of size $k = 6$, and the reduced array (4.1) is given by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 2 & 5 & 4 \\ 0 & 3 & 1 & 2 \\ 0 & 4 & 3 & 1 \\ 0 & 5 & 4 & 0 \end{bmatrix}.$$

To determine the robustness properties of this design, consider the sets $\{a_1, \dots, a_5\}$ from the reduced arrays for the replicate pairs (D_1, D_2) , (D_1, D_3) , (D_1, D_4) , (D_2, D_3) , (D_2, D_4) , (D_3, D_4) . These are:

$\{1, 2, 3, 4, 5\}$, $\{2, 5, 1, 3, 4\}$, $\{5, 4, 2, 1, 0\}$, $\{1, 3, 4, 5, 5\}$, $\{4, 2, 5, 3, 1\}$ and $\{3, 5, 1, 4, 2\}$.

From Theorem 4 all six pairs of replicates are connected. Hence $\mathcal{R}$ is complete, so Corollaries 1 and 2 apply. D is maximally robust and it is most replicate robust.

The Patterson-Williams α -design which is alternative to a rectangular lattice has $v = 30$ treatments in $s = 6$ blocks per replicate of size $k = 5$, with reduced array given by the first five rows of the array A_0 cited above; clearly, the same conclusions also apply to this design.

4.2. The α_n -Designs

The α_n -designs were introduced by John *et al.* (2002) to cater for experiments in which $r \geq 2$ complete replicates of v treatment combinations are set out as a resolvable factorial design in n treatment factors. The α -designs correspond to α_1 -designs and thus the α_n -designs provide a class of resolvable designs at least as wide as the class of §4.1. In general, α_n -designs are available for many experiments involving two or more treatment factors although they do require s and k to be composite numbers: in both cases a product of n integers not necessarily prime. All α_n -designs have minimum concurrence $\lambda_* = 0$ and neither robustness nor connectedness is guaranteed.

Each α_n -design is generated from a $k \times r$ array of residues modulo (s) . John *et al.* (2002) do not consider the use of a reduced array A_0 , as specified by (4.1), to generate the design but opt for an alternative array A with some nonzero residues in the first row and first column. However A_0 is preferred here since any given α_n -design is isomorphic

to an α_n -design that is generated by a reduced array, and identification of the robustness properties of an α_n -design obtained in this way is more straightforward.¹²

The method of construction of α_n -designs is outlined below and the process is illustrated in Example 8. Let $s = s_1 \cdots s_n$ and consider the abelian group $\mathbb{G} = \mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_n}$ which is the direct sum of n cyclic subgroups $\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_n}$ of orders $s_1, \dots, s_n$ respectively. The group table for $\mathbb{G}$ is a Latin Square L of order s in the residues $\Sigma = \{0, 1, \dots, s-1\}$, where the group table headings and marginal elements to the left of L are the elements of Σ in numerical order. Denoting the i th row of L by the $1 \times s$ vector $\ell(i)$, for $i = 0, \dots, k-1$, with the ordering specified it follows that $\ell(i)$ has residue i in the first position. The α_n -design generated by A_0 of (4.1) has replicate D_1 in standard form and, for $i = 2, \dots, r$, the blocks of D_i are given by the s columns of

$$C_i + S = \begin{bmatrix} \ell(0) \\ \ell(a_{1,i}) \\ \vdots \\ \ell(a_{k-1,i}) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ s & s & \cdots & s \\ \vdots & \vdots & & \vdots \\ s-1 & s-1 & \cdots & s-1 \end{bmatrix}, \quad (4.3)$$

i.e. C_i , the contraction for the two replicate sub-design containing D_1 and D_i , comprises rows of L , not necessarily distinct.

Example 8 A square lattice design involving $r = 4$ replicates of sixteen treatments can be derived as an α_2 -design in sixteen blocks of size 4, by taking $s_1 = s_2 = 2$. The reduced array and group table Latin Square are given by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

The α_2 -design in four replicates makes use of (4.3) to obtain D_2 , D_3 and D_4 and is given by

0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
4	5	6	7	5	4	7	6	6	7	4	5	7	6	5	4
8	9	10	11	10	11	8	9	11	10	9	8	9	8	11	10
12	13	14	15	15	14	13	12	13	12	15	14	14	15	12	13
replicate D_1				replicate D_2				replicate D_3				replicate D_4			

where columns show the blocks. Square lattice designs in two (or three) replicates are obtained by taking any two (or three) of $D_1, \dots, D_4$; these designs are shown to be A -, D - and E -optimal by Cheng and Bailey (1991), in addition they are maximally robust and most replicate robust from Propositions 2 and 4. None of these designs are obtained as α -designs so α_2 -designs provide additional designs in these cases.

To obtain conditions for robustness of α_n -designs, let $r = 2$ with replicates D_1 , D_2 and let the reduced array A_0 for the design have residues $0, a_1, \dots, a_{k-1}$ in the second column.

Theorem 5. *Let D be the α_n -design for two replicates with residues $0, a_1, \dots, a_{k-1}$ in the reduced array (4.1). D is maximally robust if and only if $a_1, \dots, a_{k-1}$ do not belong to the same proper subgroup of $\mathbb{G}$.*

Proof In view of Proposition 3 it suffices to show that the $2k$ blocks of D comprise a disconnected set if and only if $a_1, \dots, a_{k-1}$ belong to a proper subgroup of $\mathbb{G}$. By the method of construction, the i th block of D_1 contains the k treatments with labels corresponding to $i - 1$ modulo (s) . Thus, it is required to establish that a necessary and sufficient condition for the columns of the contraction C_2 of 4.3 to be a disconnected set, when viewed as a set of s blocks, is that $a_1, \dots, a_{k-1}$ belong to a proper subgroup of $\mathbb{G}$. Let $a_1, \dots, a_{k-1}$ be distinct, although this does not affect the argument, and assume that $\mathbb{G}_0$ is a subgroup of $\mathbb{G}$ containing the residues $\Sigma_0 = \{0, a_1, \dots, a_p\}$, where $k - 1 \leq p \leq \frac{1}{2}s$. Consider a portion of the group table consisting of rows of L given by $\ell(0), \ell(a_1), \dots, \ell(a_{k-1})$, let S_1 be the $p + 1$ columns of this portion headed by $0, a_1, \dots, a_p$ and let S_2 be the remaining $s - p - 1$ columns. From the group closure property of $\mathbb{G}_0$ every entry in the columns of S_1 is an element of Σ_0 and every entry in the columns of S_2 is an element of Σ but not of Σ_0 ; hence the contraction is disconnected.

Let the portion of group table consisting of rows $\ell(0), \ell(a_1), \dots, \ell(a_{k-1})$ be disconnected, i.e. columns of this portion split into two sets such that no columns of one set intersect with those of the other. Let S_1 be a connected set of columns of the portion containing the column headed by 0, and let S_2 be the nonempty set of the remaining columns. Clearly S_1 contains columns headed by $0, a_1, \dots, a_{k-1}$; suppose that other columns, if any, in S_1 are headed by $a_k^*, \dots, a_p^*$. The set $\Sigma_0 = \{0, a_1, \dots, a_{k-1}, a_k^*, \dots, a_p^*\}$ forms a subgroup of $\mathbb{G}$ if it is closed with respect to the group operation $\oplus$. Now $a_i \oplus a_j \in \Sigma_0$ and $a_i \oplus a_j^* \in \Sigma_0$; furthermore $a_i^* \oplus a_j^* \in \Sigma_0$ since a_i^* can be written as a direct sum of a finite number of a_i terms, which establishes the theorem. $\square$

As noted with α -designs, if D is an α_n -design in $r \geq 3$ replicates then a sub-design comprising any two replicates, D_i and D_j say, is isomorphic to the α_n -design with residues in the second column of the reduced array for this sub-design:

$$0, a_1 = a_{1,i} - a_{1,j}, a_2 = a_{2,i} - a_{2,j}, \dots, a_{k-1} = a_{k-1,i} - a_{k-1,j},$$

all evaluated modulo (s) . Thus, it is straightforward to apply the condition of Theorem 5 to each pair of replicates of an α_n -design D . This enables construction of the replicate connectivity graph and hence use of Theorems 2 and 3. The process can be used to assess the robustness properties of a design and as an aid in constructing robust designs. For example when $s = 4$ and $s_1 = s_2 = 2$ the only proper subgroups are $\{0, 1\}$, $\{0, 2\}$ and $\{0, 3\}$ so it is straightforward to ensure that the columns of the reduced array are selected to generate a design, such as the lattice designs of Example 8, which is invulnerable to loss of whole blocks or whole replicates.

Theorem 4 is a particular case of Theorem 5 where $\mathbb{G} = \mathbb{Z}_s$, the cyclic group of order s , and the condition that elements of $\{s, a_1, \dots, a_{k-1}\}$ have greatest common divisor greater than unity is necessary and sufficient for $a_1, \dots, a_{k-1}$ to belong to a subgroup of $\mathbb{Z}_s$.

John and Williams (1995, §4.5) and Williams *et al.* (2011) recommend on grounds¹⁴ of efficiency the use of two-category concurrence designs, i.e. the $\alpha_n(0,1)$ -designs, in which every pair of treatments appears once in a block or not at all. It is easy to show that an $\alpha_n(0,1)$ -design cannot exist unless the reduced array for every pair of replicates has $a_1, \dots, a_{k-1}$ distinct and different from zero. This enables the following general conclusions about robustness to be made after noting the possible subgroups of $\mathbb{G}$:

Proposition 5. (i) *If s is a prime number then each $\alpha(0,1)$ -design is maximally robust and most replicate robust.*

(ii) *If s is a composite integer and s_1 is the largest factor of s then each $\alpha_n(0,1)$ -design with block size $k \geq s_1 + 1$ is maximally robust and most replicate robust.*

Proof These two conditions are sufficient to ensure that $\mathcal{R}$ is complete in each case, where Lagrange's theorem has been applied in (ii).

The values of $s < 20$ for which the α_n -designs form a larger class than the class of α -designs alone are $s = 4, 8, 9, 12, 16$ and 18 . No α_n -designs exist for $n \geq 2$ when s is prime. Furthermore, all α_2 -designs based on groups $\mathbb{Z}_p \oplus \mathbb{Z}_q$ are isomorphic to α -designs when p and q are co-prime. Thus the α_2 -design of John *et al.* (2002) for three replicates of 24 treatments in blocks of size 4, which is based on group $\mathbb{Z}_3 \oplus \mathbb{Z}_2$, is isomorphic to an α -design based on group $\mathbb{Z}_6$, as can be seen directly. A similar comment applies to the α_2 -design of Table 4 of Williams *et al.* (2011), based on group $\mathbb{Z}_2 \oplus \mathbb{Z}_7$, which is used to demonstrate partially replicated designs, and which is isomorphic to an α -design based on group $\mathbb{Z}_{14}$.

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