Asymptotic uniform boundedness of energy solutions to the Penrose-Fife model

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Abstract. We study a Penrose-Fife phase transition model coupled with homogeneous Neumann boundary conditions. Improving previous results, we show that the initial value problem for this model admits a unique solution under weak conditions on the initial data. Moreover, we prove asymptotic regularization properties of weak solutions.

1. Introduction

The Penrose-Fife system, proposed by Penrose and Fife in [25, 26], represents a thermodynamically consistent model for the description of the kinetics of phase transition and phase separation processes in binary materials. It couples a singular heat equation (cf. (2.3) below) for the *absolute* temperature ϑ with a nonlinear relation describing the evolution of the *phase variable* χ , which represents the local proportion of one of the two components. This can be of the *fourth* order in space (cf. (2.6)–(2.7) below), in case the physical process preserves the total mass of χ (*conserved* Penrose-Fife model, describing phase separation) or of the *second* order in space (cf. (2.5) below), in case the total mass of χ is admitted to vary (*non-conserved* Penrose-Fife model, describing phase transition). In the conserved case, the equation for χ is usually written as a system by introducing an auxiliary variable w called *chemical potential*. We refer to the next section for a detailed presentation of the equation and to the papers [10, 11, 32] for further mathematical background.

Due to physical considerations, it is generally accepted to consider no-flux boundary conditions for χ and, in the conserved case, also for w . On the other hand, various types of boundary conditions (for instance, no-flux, non-homogeneous Dirichlet, or Robin conditions) make sense for the heat equation (2.3), which give rise to different mathematical scenarios. We refer the reader to [11–13, 21, 22, 32] for the case of Robin (or “third type”) conditions, to [14, 15] for the Dirichlet case, and to [10, 17, 34] for the homogeneous Neumann case, which is probably the most difficult one due to lower coercivity properties of the elliptic operator in (2.3).

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Since also the present paper is devoted to the Neumann case, it is worth describing here in some detail the related results proved in [10, 17, 34]. In the pioneering paper [34], the non-conserved model is studied in one space dimension, and well-posedness is proved for smooth initial data (in particular, the author investigates the properties of “classical” solutions). The one-dimensional restriction has been removed in [17], where well-posedness is proved, both in the non-conserved and in the conserved case, for initial data satisfying the equivalent of assumptions (2.11) and (2.18) below, plus the additional summability condition $\vartheta_0 \in L^2(\Omega)$. The authors of [17] can also consider the case when a heat source with *zero spatial mean* is present on the right hand side of (2.3). This restriction is overcome in [10], where a general heat source is considered, paying, however, the price of having a less general class of admissible functions f in (2.5) (excluding in particular the “singular” nonlinearities corresponding to the case of a bounded domain I in (2.9) below). It is also worth mentioning that the heat flux law considered in [10] is slightly different from ours, in the sense that our (2.4) is replaced by a relation like $u \sim \vartheta - \vartheta^{-1}$ (or a generalization of this).

Our aim in this paper is that of extending the above-described results and proving further properties of solutions to the homogeneous Neumann problem for the Penrose-Fife model, both in the non-conserved and in the conserved case. Actually, the two situations can be treated together, with some variants in the proofs (which happen to be slightly more involved for the conserved model). For simplicity, we limit our analysis to the homogeneous problem, but our results can be easily adapted to the case of a (suitably regular) heat source with zero spatial mean (on the other hand, it does not seem possible to remove the zero-mean condition in our setting). As a first property, we will show that the problem admits a unique solution under *weak and physically natural* assumptions on the initial data. In particular, we can remove the requirement $\vartheta_0 \in L^2(\Omega)$ taken in [10, 17]. Actually, noting that the system admits a Liapounov functional representing the total energy, we will prove that a (unique) weak solution exists for any initial datum (ϑ_0, χ_0) having finite energy (this corresponds exactly to hypotheses (2.11) and (2.18) below). As a side effect, we pay the price that, for these solutions (called V' -solutions in the sequel; indeed, we will note $V := H^1(\Omega)$), the heat equation has to be interpreted in the generalized $(H^1)'$ -framework developed by Damlamian and Kenmochi in [13]. Namely, the (thermal part of the) energy has to be intended as a (relaxed) functional operating on the negative order Sobolev space $H^1(\Omega)'$ (cf. (2.20) below), and also, the relation linking ϑ to its inverse has to be stated properly. If, in addition, we assume $\vartheta_0 \in L^1(\Omega)$, then we obtain a class of more regular solutions (called $V' \cap L^1$ -solutions) and we can prove that, in this setting, $\vartheta(t) \in L^1(\Omega)$ for all $t \geq 0$; moreover, both the energy functional and the relation between ϑ and its inverse can be written in the usual (pointwise) sense (cf. (2.19) below). The proof of this property, especially in the conserved case, requires a careful combination of Hilbert and L^1 -techniques.

After analyzing well-posedness in the weak setting, we pass to what we can consider the main results of the paper, regarding uniform time-regularization properties of weak

solutions. In this frame, we present two theorems. First, we show that, for any $T > 0$, there exists a constant $\underline{\vartheta} > 0$ depending only on the “energy” of the initial data and on T such that $\vartheta(x, t) \geq \underline{\vartheta}$ for a.e. $(x, t) \in (T, +\infty) \times \Omega$. This holds in the class of V' -solutions. Second, we prove that an analogous bound from above (i.e., $\vartheta(x, t) \leq \bar{\vartheta}$ for a.e. $(x, t) \in (T, +\infty) \times \Omega$ and a suitable $\bar{\vartheta} > 0$) holds provided that the initial temperature ϑ_0 , in addition to the “energy” regularity, satisfies the additional hypothesis $\vartheta_0 \in L^{3+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ (thus, we need slightly stronger conditions on data with respect, for instance, to [17]). This additional assumption is natural (and probably optimal) and appears also in other works concerning L^∞ -regularization properties of the solutions to *very fast diffusion* equations like (2.3) on $\mathbb{R}^3$ (see, e.g., [6, 33] and the references therein). In particular, in three space dimensions, the exponent $p = 3$ happens to be critical for the boundedness (for strictly positive times) of the solutions to (2.3): starting from initial data in $L^p(\mathbb{R}^3)$, $p > 3$, implies boundedness of the solutions for strictly positive times (see [6]). For $p < 3$, the situation is drastically different, as the self-similar solution (2.40) shows.

In the frame of the Penrose-Fife model, the regularizations result we prove are new, at least up to are knowledge; moreover, they appear to be optimal, in the sense that they hold under minimal conditions on the initial data. In particular, the smoothing effect for $1/\vartheta$ holds for V' -solutions, namely the weakest form of solution we deal with. Finally, these results are non-obvious, first because they regard the very singular equation (2.3), and second because of the coupling with the phase equation (2.5). More precisely, a source of difficulties (especially when proving the smoothing effect for ϑ) comes from the regularity of χ_t (for initial data having the “energy” regularity, plus $\vartheta_0 \in L^{3+\varepsilon}$) that is not sufficient to implement the Moser iteration scheme directly from $t = 0$. As a consequence, we have to control in a sharp way the behavior of the norm $\|\chi_t\|_{L^{3+\varepsilon}(\Omega)}$ when $t \searrow 0$ (see the proof of Theorem 2.7 below).

We finally note that the proved uniform bounds permit, by standard methods, to improve further the regularity of solutions for strictly positive times. In particular, our estimates complement a recent paper by Prüss and Wilke [27], who showed maximal regularity estimates for the conserved Penrose-Fife model under the conditional (i.e., unproved in their paper) assumption that the temperature ϑ satisfies the uniform bounds $\underline{\vartheta} \leq \vartheta(x, t) \leq \bar{\vartheta}$ almost everywhere. Thanks to our results, the maximal regularity estimates of Prüss and Wilke hold for all strictly positive times and all weak solutions emanating from initial data satisfying the “energy regularity” plus the condition $\vartheta_0 \in L^{3+\varepsilon}(\Omega)$. It is also worth noting that the uniform bound $\vartheta \geq \underline{\vartheta}$ implies, in the non-conserved case, the so-called “separation from singularities” property for χ in case the configuration potential of the system (i.e., the function $\widehat{b}$ defined in (2.10)) has a bounded domain. For instance, in the physically relevant case of the *logarithmic potential*

$$\widehat{b}(r) = (1+r) \log(1+r) + (1-r) \log(1-r), \quad (1.1)$$

whose domain is $[-1, 1]$, we obtain that $-1 + \delta \leq \chi(x, t) \leq 1 - \delta$ for some $\delta > 0$ and for all times $t \geq 1$. Unfortunately, we cannot prove such a property in the case of

the conserved model. Actually, up to our knowledge, this is unknown, at least in the three-dimensional case, also for the (simpler) conserved phase-field model of Caginalp type (cf. the related discussion in [23]).

The remainder of the paper is organized as follows. In the next section, we will present our hypotheses and state our results. The proofs will be detailed in the subsequent Sect. 3.

2. Main results

Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \in \{2, 3\}$. For the sake of simplicity, let us assume $|\Omega| = 1$ so that $\|v\|_{L^p(\Omega)} \leq \|v\|_{L^q(\Omega)}$ for all $1 \leq p \leq q \leq +\infty$, $v \in L^q(\Omega)$. For simplicity, we will often write $\|\cdot\|_p$ in place of $\|\cdot\|_{L^p(\Omega)}$. Let $H := L^2(\Omega)$, endowed with the standard scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let also $V := H^1(\Omega)$. We note by $\|\cdot\|_X$ the norm in the generic Banach space X , and by $\langle \cdot, \cdot \rangle_X$ the duality between X' and X .

For any function, or functional z , defined on Ω , we can then set

$$z_\Omega := \frac{1}{|\Omega|} \int_\Omega z = \int_\Omega z, \quad (2.1)$$

where the integral is substituted with the duality $\langle z, 1 \rangle$ in case, e.g., $z \in V'$.

We also define the elliptic operator

$$A : V \rightarrow V', \quad \langle Av, z \rangle := \int_\Omega \nabla v \cdot \nabla z. \quad (2.2)$$

Then, for $t \in (0, +\infty)$, we consider the singular heat equation

$$\vartheta_t + Au = -\chi_t, \quad \text{in } V', \quad (2.3)$$

$$u = -\frac{1}{\vartheta}, \quad \text{almost everywhere in } \Omega. \quad (2.4)$$

In the *non-conserved* case, this is coupled with the equation

$$\chi_t + A\chi + b(\chi) - \chi = u, \quad \text{in } H, \quad (2.5)$$

while in the *conserved* case, (2.3)–(2.4) is coupled with the system

$$\chi_t + Aw = 0, \quad \text{in } V', \quad (2.6)$$

$$w = A\chi + b(\chi) - \chi - u, \quad \text{in } V'. \quad (2.7)$$

In both cases, we will take the initial conditions

$$\vartheta|_{t=0} = \vartheta_0, \quad \chi|_{t=0} = \chi_0, \quad \text{in } \Omega. \quad (2.8)$$

The nonlinear function b is assumed to satisfy

$$\begin{aligned} b &\in C_{\text{loc}}^{0,1}(I; \mathbb{R}), \quad b(0) = 0, \quad b'(r) \geq 0 \text{ a.e. in } I, \\ \lim_{r \rightarrow \partial I} b(r) \text{ sign } r &= \lim_{r \rightarrow \partial I} b'(r) = +\infty, \end{aligned} \quad (2.9)$$

where I , the *domain* of b , is an open, possibly bounded, interval of $\mathbb{R}$ containing 0. We also set

$$\widehat{b}(r) := \int_0^r b(s) \, ds \quad (2.10)$$

in such a way that $\widehat{b}$ is a convex function satisfying $\widehat{b}(0) = 0$.

Concerning the initial datum χ_0 , we will always assume

$$\chi_0 \in V, \quad \widehat{b}(\chi_0) \in L^1(\Omega). \quad (2.11)$$

To specify the regularity required for the initial temperature ϑ_0 , we first need to introduce some convex analysis machinery. First of all, we set

$$j(v) := -\log(-v), \quad j^*(z) := -1 - \log z, \quad (2.12)$$

respectively, for $v < 0$ and $z > 0$. Then, j and j^* can be seen as a couple of *conjugate functions* according to the standard theory (cf., e.g., [7]). This permits to introduce the convex functional

$$J : V \rightarrow [0, +\infty], \quad J(v) := \int_{\Omega} j(v), \quad (2.13)$$

where we have implicitly set $J(v) = +\infty$ for those $v \in V$ such that $j(v)$ is not summable (this happens, for instance, when v is strictly positive on a set of strictly positive measure). The set $\{v \in V : J(v) < +\infty\}$ is called *domain* of J . Then, the conjugate of J is given by

$$J^* : V' \rightarrow (-\infty, +\infty], \quad J^*(\zeta) := \sup \{ \langle \zeta, v \rangle - J(v), \, v \in V \}. \quad (2.14)$$

We can also introduce the subdifferential of J with respect to the duality between V' and V . Namely, given $v \in V$, we set

$$\zeta \in \partial_{V,V'} J(v) \iff \langle \zeta, z - v \rangle \leq J(z) - J(v) \quad \forall z \in V. \quad (2.15)$$

In general, the “weak” subdifferential $\partial_{V,V'} J$ is a multivalued maximal-monotone operator. Its structure is analyzed in several papers (see, e.g., [5]; see also [8] and [16] for the slightly different situation where V is substituted by $H_0^1(\Omega)$). Actually, it is not difficult to prove that

$$\zeta \in L^1(\Omega), \, \zeta = j'(v) = -\frac{1}{v} \text{ a.e. in } \Omega \implies \zeta \in \partial_{V,V'} J(v). \quad (2.16)$$

On the other hand, for given $v \in V$, the set $\partial_{V,V'} J(v)$ needs not be contained in $L^1(\Omega)$. More precisely, a generic element $\zeta \in \partial_{V,V'} J(v)$ is a bounded measure which may have a singular part ζ_s . A precise characterization is given in [8, Thm. 3] for the H_0^1 -case. However, it is easy to realize that such a characterization extends to the case of $V = H^1(\Omega)$ at least provided Ω is smooth; the only difference being that ζ_s can be

supported also on the boundary $\partial\Omega$. On the other hand, if we are able to prove that the singular part of some element ζ is 0, then we still have pointwise inclusion. Namely, there holds that

$$\zeta \in \partial_{V,V'} J(v) \cap L^1(\Omega) \implies \zeta = -\frac{1}{v} \text{ a.e. in } \Omega. \quad (2.17)$$

That said, the minimal regularity required on ϑ_0 is given by

$$\vartheta_0 \in V', \quad J^*(\vartheta_0) < +\infty. \quad (2.18)$$

Then, we can define the “strong” energy functional of the system as

$$\mathcal{E}_s(\vartheta, \chi) := \int_{\Omega} \left(\vartheta + \chi - 1 - \log \vartheta + \frac{1}{2} |\nabla \chi|^2 + \widehat{b}(\chi) - \frac{1}{2} \chi^2 \right), \quad (2.19)$$

However, since in general ϑ_0 is just an element of V' , it does not make sense to compute $\mathcal{E}_s(\vartheta_0, \chi_0)$. Actually, we have to relax $\mathcal{E}_s$, by defining

$$\mathcal{E}(\vartheta, \chi) := \langle \vartheta, 1 \rangle + J^*(\vartheta) + \int_{\Omega} \left(\chi + \frac{1}{2} |\nabla \chi|^2 + \widehat{b}(\chi) - \frac{1}{2} \chi^2 \right). \quad (2.20)$$

Actually, assumptions (2.18) and (2.11) are equivalent to asking that the “relaxed” energy $\mathbb{E}_0 := \mathcal{E}(0)$ is finite. In the sequel, we shall often write $\mathcal{E}(t)$ in place of $\mathcal{E}(\vartheta(t), \chi(t))$. Notice that the last (2.9) ensures the coercivity of $\mathcal{E}$. Notice also that the no-flux conditions entail some conservation properties. Actually, for the conserved system, testing (2.3) and (2.6) by 1, we immediately get

$$\vartheta_{\Omega}(t) = (\vartheta_0)_{\Omega}, \quad \chi_{\Omega}(t) = (\chi_0)_{\Omega}, \quad \forall t \geq 0, \quad (2.21)$$

while of course for the non-conserved system, we only have

$$\vartheta_{\Omega}(t) + \chi_{\Omega}(t) = (\vartheta_0)_{\Omega} + (\chi_0)_{\Omega}, \quad \forall t \geq 0. \quad (2.22)$$

The following result, stating existence of strong solutions, is well known (see, for instance, [17] for a proof):

THEOREM 2.1. *Let us assume (2.9), (2.18), (2.11) and, additionally, let*

$$\vartheta_0 \in H. \quad (2.23)$$

Then, in the non-conserved case, there exists a unique triplet (ϑ, u, χ) satisfying, for all $T > 0$,

$$\vartheta - \log \vartheta \in L^{\infty}(0, T; L^1(\Omega)), \quad \vartheta \in L^2(0, T; H), \quad (2.24)$$

$$u \in L^2(0, T; V), \quad (2.25)$$

$$\chi \in H^1(0, T; H) \cap L^{\infty}(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad b(\chi) \in L^2(0, T; H), \quad (2.26)$$

and solving (2.3)–(2.4) and (2.5) a.e. in $(0, T)$, together with the initial conditions (2.8).

Analogously, in the non-conserved case, we have a unique triplet (ϑ, u, χ) satisfying, for all $T > 0$, (2.24)–(2.25), together with

$$\chi \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad b(\chi) \in L^2(0, T; H), \quad (2.27)$$

solving (2.3)–(2.4) and (2.6)–(2.7) a.e. in $(0, T)$, and satisfying (2.8).

REMARK 2.2. It is worth noting (cf. [5] for more details) that (2.23) entails

$$J^*(\vartheta_0) = \int_{\Omega} (-1 - \log \vartheta_0). \quad (2.28)$$

Thus, in fact, no functional on V' appears in the above “strong” formulation.

In the next result, we will provide an existence theorem working without the additional regularity (2.23). We will pay the price of the occurrence of relaxed functionals. Moreover, we will also need to intend relation (2.4) in the relaxed sense of [13].

THEOREM 2.3 ((V')-solutions). *Let (2.9), (2.18) and (2.11) hold.*

Then, there exists a unique triplet (ϑ, u, χ) , satisfying, for all $T > 0$,

$$\vartheta \in H^1(0, T; V'), \quad J^*(\vartheta) \in L^\infty(0, T), \quad (2.29)$$

together with (2.25) and, a.e. in $(0, T)$, equation (2.3). Moreover, in the non-conserved case, (2.26) and, a.e. in $(0, T)$, equation (2.5) hold, while, in the conserved case, (2.27) and, a.e. in $(0, T)$, equations (2.6)–(2.7) hold. Moreover, there hold the initial condition (2.8) and, a.e. in $(0, T)$, the weak identification property

$$\vartheta \in \partial_{V, V'} J(u). \quad (2.30)$$

Finally, any V' -solutions satisfies, for all $t \in [0, T]$, the energy equality

$$\begin{aligned} \int_0^t (\|\nabla u\|^2 + \|\chi_t\|^2) &= J^*(\vartheta_0) - J^*(\vartheta(t)) + \langle \vartheta_0, 1 \rangle - \langle \vartheta(t), 1 \rangle \\ &+ \int_{\Omega} \left(\chi_0 + \frac{1}{2} |\nabla \chi_0|^2 + \widehat{b}(\chi_0) - \frac{1}{2} \chi_0^2 \right) \\ &- \int_{\Omega} \left(\chi(t) + \frac{1}{2} |\nabla \chi(t)|^2 + \widehat{b}(\chi(t)) - \frac{1}{2} \chi(t)^2 \right), \end{aligned} \quad (2.31)$$

in the non-conserved case. In the conserved case, the same holds provided that the term $\|\chi_t\|^2$ on the left hand side is replaced by $\|\nabla w\|^2$.

We now turn to discussing regularization properties of solutions. The first result regards the function u .

THEOREM 2.4. *Let (2.9), (2.18) and (2.11) hold. Then, the V' -solution either to the non-conserved or to the conserved problem satisfies*

$$\|u\|_{L^\infty(1,\infty;V)} + \|u\|_{L^\infty((2,\infty)\times\Omega)} \leq Q(\mathbb{E}_0), \quad (2.32)$$

$$\|\chi\|_{L^\infty(1,\infty;H^2(\Omega))} + \|b(\chi)\|_{L^\infty(1,\infty;H)} \leq Q(\mathbb{E}_0). \quad (2.33)$$

Moreover, in the non-conserved case we also have the “separation property”

$$\|b(\chi)\|_{L^\infty((3,\infty)\times\Omega)} \leq Q(\mathbb{E}_0). \quad (2.34)$$

Here and below, Q is a computable non-negative function, monotone in each of its arguments, whose expression is independent of initial data and of time.

REMARK 2.5. It is worth noting that the above regularization properties are in fact *instantaneous*. Indeed, with minor modification in the proofs, one could easily see that (2.32)–(2.33) and (2.34) hold starting from any $\tau > 0$ (and not only from $\tau = 1$ or 2 or 3). Of course, then the functions Q on the right hand sides would also monotonically depend on τ^{-1} and possibly explode for $\tau \searrow 0$. The same considerations hold also for what is proved in Theorem 2.7 below.

In case the initial temperature, beyond satisfying (2.18), is an L^1 -function, we can say something more precise on regularity:

THEOREM 2.6 (($V' \cap L^1$)-solutions). *Let (2.9), (2.18) and (2.11) hold. Let also*

$$\vartheta_0 \in L^1(\Omega). \quad (2.35)$$

Then, the triplet (ϑ, u, χ) given by Theorem 2.3 additionally satisfies, for all $T > 0$,

$$\vartheta \in C^0([0, T]; L^1(\Omega)) \quad (2.36)$$

and the strong identification property (2.4).

It is easy to show that, if $\vartheta_0 \in L^p(\Omega)$ for $p > 1$, then $\vartheta(t)$ remains in $L^p(\Omega)$ for $t > 0$. What is more interesting is that, if $p > 3$, then ϑ is asymptotically uniformly bounded:

THEOREM 2.7. *Let (2.9), (2.18) and (2.11) hold and let also*

$$\vartheta_0 \in L^{3+\varepsilon}(\Omega) \quad \text{for some } \varepsilon > 0. \quad (2.37)$$

Then, the $(V' \cap L^1)$ -solution either to the non-conserved or to the conserved problem satisfies the additional bound

$$\|\vartheta\|_{L^\infty((2,\infty)\times\Omega)} \leq Q(\mathbb{E}_0, \|\vartheta_0\|_{3+\varepsilon}). \quad (2.38)$$

REMARK 2.8. Relation (2.37) suggests that 3 should play the role of a critical exponent for equation (2.3) in space dimension 3. Actually, it is easy to check that the related “very-fast diffusion” equation

$$\vartheta_t + \Delta \vartheta^{-1} = 0 \quad (2.39)$$

over the whole space $(0, +\infty) \times \mathbb{R}^3$ admits the similarity solution (see, e.g., [33])

$$\vartheta(t, x) = \frac{2(T-t)_+^{1/2}}{|x|}, \quad (2.40)$$

which belongs to $L_{\text{loc}}^p(\mathbb{R}^3)$ for all $p < 3$ and all $t \geq 0$ and does not exhibit any instantaneous regularizing effect (of course, it satisfies a delayed regularization property since it extinguishes in a finite time; however, this effect is not expected to hold in the case of a finite domain when we have conservation of mass). In any case, we do not know what happens in the critical case of an initial datum ϑ_0 belonging to $L^3(\Omega)$.

3. Proofs

All proofs will be in principle given only for the *conserved* case which is, actually, more difficult. The properties holding only for the non-conserved case (as well any significant differences in the proofs) will be remarked on occurrence.

3.1. Proof of Theorem 2.3

We start by proving existence for weak initial data. Given ϑ_0 satisfying (2.18), we then set, for $n \in \mathbb{N}$,

$$\vartheta_{0,n} + n^{-1}A\vartheta_{0,n} = \vartheta_0. \quad (3.1)$$

The properties of this approximation deserve to be stated in a lemma.

LEMMA 3.1. *Let (2.18) hold and let $\vartheta_{0,n}$ be defined by (3.1). Then, $\vartheta_{0,n} \in V$ for all $n \in \mathbb{N}$. Moreover,*

$$\vartheta_{0,n} \rightarrow \vartheta_0 \text{ strongly in } V' \text{ and } J^*(\vartheta_{0,n}) \leq J^*(\vartheta_0) \quad \forall n \in \mathbb{N}. \quad (3.2)$$

Moreover, if also (2.35) holds, then we also have

$$\vartheta_{0,n} \rightarrow \vartheta_0 \text{ strongly in } L^1(\Omega) \quad (3.3)$$

and

$$\log \vartheta_{0,n} \in L^1(\Omega), \quad \int_{\Omega} -1 - \log \vartheta_{0,n} \leq \int_{\Omega} -1 - \log \vartheta_0, \quad \forall n \in \mathbb{N}. \quad (3.4)$$

Proof. Being $\vartheta_0 \in V'$, it is clear that $\vartheta_{0,n} \in V$ for all n . Moreover, the V' -strong convergence in (3.2) can be proved by standard Hilbert techniques. In order to complete the proof of (3.2), we introduce, for any given $n \in \mathbb{N}$, the strictly positive sequence $\vartheta_{0,n,k} := \max\{\vartheta_{0,n}, \frac{1}{k}\}$, for $k \in \mathbb{N}$. Correspondingly, we set $u_{0,n,k} := -\frac{1}{\vartheta_{0,n,k}}$. Note that, by construction, $u_{0,n,k} \in V$ and the map $\vartheta_{0,n} \mapsto \vartheta_{0,n,k}$ are monotone. As a consequence, we can write

$$\begin{aligned} 0 &\geq \left\langle -\frac{1}{n}A\vartheta_{0,n}, u_{0,n,k} \right\rangle = \langle \vartheta_{0,n} - \vartheta_0, u_{0,n,k} \rangle = \langle \vartheta_{0,n,k} - \vartheta_0, u_{0,n,k} \rangle \\ &\quad + \langle \vartheta_{0,n} - \vartheta_{0,n,k}, u_{0,n,k} \rangle. \end{aligned} \quad (3.5)$$

Now, since $\vartheta_{0,n,k} = -\frac{1}{u_{0,n,k}}$ almost everywhere in Ω , we have $\vartheta_{0,n,k} \in \partial_{V,V'} J(u_{0,n,k})$. Equivalently, $u_{0,n,k} \in \partial_{V',V} J^*(\vartheta_{0,n,k})$ (where the subdifferential acts now in the duality between V' and V) for any k . Hence,

$$\langle \vartheta_{0,n,k} - \vartheta_0, u_{0,n,k} \rangle \geq J^*(\vartheta_{0,n,k}) - J^*(\vartheta_0), \quad (3.6)$$

by definition of subdifferential. On the other hand, since $\vartheta_{0,n,k} \equiv \vartheta_{0,n}$ in $\Omega \cap \{\vartheta_{0,n} \geq 1/k\}$,

$$\langle \vartheta_{0,n} - \vartheta_{0,n,k}, u_{0,n,k} \rangle = \int_{\Omega \cap \{\vartheta_{0,n} \leq 1/k\}} (-k) \cdot (\vartheta_{0,n} - \vartheta_{0,n,k}) \geq 0. \quad (3.7)$$

Thus, collecting the above computations we have

$$J^*(\vartheta_{0,n,k}) \leq J^*(\vartheta_0). \quad (3.8)$$

Finally, since for any $n \in \mathbb{N}$, we have that $\vartheta_{0,n,k} \xrightarrow{k \nearrow +\infty} \vartheta_{0,n}$ strongly in $L^p(\Omega)$ for any $p \in [1, 6)$ (hence, a fortiori, in V'), we have the following chain of inequalities:

$$J^*(\vartheta_{0,n}) \leq \liminf_{k \nearrow +\infty} J^*(\vartheta_{0,n,k}) \leq \limsup_{k \nearrow +\infty} J^*(\vartheta_{0,n,k}) \leq J^*(\vartheta_0), \quad \text{for any } n \in \mathbb{N}, \quad (3.9)$$

i.e., (3.2) holds.

Now, we assume that also (2.35) holds, namely we assume that $\vartheta_0 \in V \cap L^1(\Omega)$. Note that, thanks to Remark 2.2, in this new regularity framework, (3.4) is nothing else than $J^*(\vartheta_{0,n}) \leq J^*(\vartheta_0)$, which as been proved above.

Thus, we only need to prove the L^1 -convergence. To this end, we have to be a bit more careful. First, we define the Banach space $X := V' \cap L^1(\Omega)$, endowed with the norm

$$\|\cdot\|_X := \|\cdot\|_{V'} + \|\cdot\|_{L^1(\Omega)},$$

and introduce the unbounded linear operator $\mathcal{A}$ on X defined as $\mathcal{A}v := \Delta v$ with domain

$$D(\mathcal{A}) := \{v \in V : \Delta v \in L^1(\Omega)\},$$

where Δ is the usual distributional Laplace operator. Then, we have that $\mathcal{A}$ is an accretive operator on the space X . Indeed, by [4, Prop. II.3.1], this corresponds to checking that, if $\lambda > 0$ and

$$x_i + \lambda \mathcal{A}x_i = f_i, \quad i = 1, 2, \quad (3.10)$$

for $f_i \in X$, then

$$\|x_1 - x_2\|_X \leq \|f_1 - f_2\|_X. \quad (3.11)$$

Actually, the analogue of (3.11) w.r.t. the V' -norm can be obtained by testing the difference

$$x_1 - x_2 + \lambda \mathcal{A}(x_1 - x_2) = f_1 - f_2 \quad (3.12)$$

by $(I + A)^{-1}(x_1 - x_2)$ where I is the identity mapping of H (and, hence, $I + A : V \rightarrow V'$ is the Riesz isomorphism). On the other hand, the L^1 -analogue of (3.11) is obtained by testing (3.12) by $\text{sign}(x_1 - x_2)$ and applying the Brezis-Strauss theorem [9, Lemma 2]. Moreover, we have that $D(\mathcal{A})$ is dense in $V' \cap L^1(\Omega)$. To see this, let us take $z \in V' \cap L^1(\Omega)$. Then, setting

$$z_n := \rho_n * z = \int_{\Omega} \rho_n(x - y) z(y) \, dy, \quad (3.13)$$

where ρ_n is the standard mollifier, it is clear that z_n is smooth (hence, in particular, it belongs to $D(A)$). Moreover, the convergence $z_n \rightarrow z$ in $L^1(\Omega)$ follows from standard properties of convolutions, while the convergence $z_n \rightarrow z$ in V' follows from the density of H in V' and from the fact that the mapping $z \mapsto z_n$ is a contraction w.r.t. the V' -norm (this may be verified for $z \in H$ by using Fubini's theorem and then extended to V' by density). These facts permit to apply [4, Prop. 3.2 (e)], which gives exactly the convergence property (3.3), which concludes the proof. $\square$

Thus, taking $\vartheta_{0,n}$ as an initial datum for equation (2.3) (while the initial datum χ_0 is kept fixed), existence of a corresponding solution (ϑ, u, χ) is guaranteed by Theorem 2.1. Our aim will be now that of removing the approximation of the initial datum letting $n \nearrow +\infty$. With this aim, we start by recalling a couple of basic a-priori estimates. The procedure is detailed only in the conserved case, the differences occurring in the non-conserved case being pointed out at the end. For the meanwhile, we will not emphasize the dependence on n in the notation.

Energy estimate. We test (2.3) by $1 + u$, (2.6) by w and (2.7) by χ_t . This formal procedure will be justified at the end, when we prove (2.31). We obtain

$$\frac{d}{dt} \mathcal{E} + \|\nabla u\|^2 + \|\nabla w\|^2 = 0, \quad (3.14)$$

where $\mathcal{E}$ was defined in (2.20). Using also the properties of A , we immediately get

$$\|\mathcal{E}\|_{L^\infty(0,T)} + \|\nabla u\|_{L^2(0,T;H)} + \|\nabla w\|_{L^2(0,T;H)} + \|\chi_t\|_{L^2(0,T;V')} \leq c\mathbb{E}_0. \quad (3.15)$$

Here and below, the letters c and κ will denote generic positive constants, independent of initial data and of time, whose value possibly varies on occurrence, κ being used in estimates from below. In particular, the above estimate is uniform with respect to T .

A generalized Poincaré inequality. To estimate the full V -norm of u (and not just the H -norm of its gradient), we need a proper form of Poincaré's inequality (cf., e.g., [18, Lemma 5.1] for a similar tool), which we prove just for the sake of completeness:

LEMMA 3.2. Assume Ω is a bounded open subset of $\mathbb{R}^d$. Suppose $v \in W^{1,1}(\Omega)$ and $v \geq 0$ a.e. in Ω . Then, setting $K := \int_{\Omega} (\log v)^+$, the following estimate holds:

$$\|v\|_{L^1(\Omega)} \leq |\Omega|e^{C_1 K} + \frac{C_2}{|\Omega|} \|\nabla v\|_{L^1(\Omega)}, \quad (3.16)$$

the constants C_1 and C_2 depending only on Ω .

Proof. First of all, we recall that for any function $z \in W^{1,1}(\Omega)$ such that $|E_0| > 0$ (with $E_0 := \{x \in \Omega : z(x) = 0\}$) the following Poincaré type inequality (see [20, Lemma 5.1, pag. 89]) holds:

$$\|z\|_{L^1(\Omega)} \leq \frac{C}{|E_0|} \|\nabla z\|_{L^1(\Omega)}, \quad (3.17)$$

where the constant C can be explicitly computed and depends only on Ω . Now, let v be a function in the hypothesis of the Lemma. Set $K := \int_{\Omega} (\log v)^+$ and note that, thanks to the Chebychev inequality, we have, for any fixed $N > 0$,

$$|\{x \in \Omega : (\log v)^+ > N\}| \leq \frac{K}{N}, \quad (3.18)$$

and consequently

$$|\{x \in \Omega : v > N\}| \leq \frac{K}{\log N}. \quad (3.19)$$

Thanks to (3.18) and (3.19), we can fix $\bar{N} = \bar{N}(K, \Omega) = e^{\frac{2K}{|\Omega|}}$ in such a way that

$$|\{x \in \Omega : v \leq \bar{N}\}| = |\Omega| - |\{x \in \Omega : v > \bar{N}\}| \geq \frac{|\Omega|}{2}. \quad (3.20)$$

As a consequence, the inequality (3.17), with $z = (v - \bar{N})^+$ and (3.20), entails

$$\begin{aligned} \|v\|_{L^1(\Omega)} &\leq \bar{N}|\Omega| + \|(v - \bar{N})^+\|_{L^1(\Omega)} \leq \bar{N}|\Omega| + \frac{C}{|\{v \leq \bar{N}\}|} \|\nabla(v - \bar{N})^+\|_{L^1(\Omega)} \\ &\leq |\Omega|e^{\frac{2K}{|\Omega|}} + 2C|\Omega|\|\nabla v\|_{L^1(\Omega)}, \end{aligned} \quad (3.21)$$

which is (3.16) with $C_1 = 2/|\Omega|$ and $C_2 = 2C$, C being the constant in (3.17). $\square$

Consequences of the energy estimate.

Using the above lemma, (3.15) additionally gives

$$\|u\|_{L^2(t, t+1; V)} \leq Q(\mathbb{E}_0) \quad \forall t \geq 0. \quad (3.22)$$

Applying standard techniques to system (2.6)–(2.7), we also have

$$\|X\|_{L^2(t, t+1; H^2(\Omega))} \leq Q(\mathbb{E}_0). \quad (3.23)$$

Moreover, testing (2.7) by $\chi - \chi_\Omega$ and proceeding, e.g., as in the Appendix of [23], it is not difficult to arrive at

$$\|b(\chi)\|_{L^2(t, t+1; H)} \leq Q(\mathbb{E}_0), \quad (3.24)$$

whence a comparison of terms in (2.7) and estimate (3.15) also give

$$\|w\|_{L^2(t, t+1; V)} \leq Q(\mathbb{E}_0). \quad (3.25)$$

Passage to the limit. We will now let $n \nearrow +\infty$, still referring to the conserved case. With this aim, we rename as $(\vartheta_n, u_n, \chi_n)$ the solution to the n -approximation. By (3.15) and (3.22)–(3.25), we then have, for any $T > 0$,

$$u_n \rightarrow u \quad \text{weakly in } L^2(0, T; V), \quad (3.26)$$

$$\chi_n \rightarrow \chi \quad \text{weakly in } H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad (3.27)$$

$$w_n \rightarrow w \quad \text{weakly in } L^2(0, T; V), \quad (3.28)$$

$$b(\chi_n) \rightarrow b \quad \text{weakly in } H^1(0, T; H), \quad (3.29)$$

for suitable limit functions u, χ, b . Then, the Aubin-Lions compactness Lemma and the usual monotonicity argument [4, Prop. 1.1, p. 42] permit to see that $b = b(\chi)$ a.e. in $(0, T) \times \Omega$. Moreover, the above relation suffice to pass to the limit in system (2.6)–(2.7).

Taking the limit in (2.3) and in (2.4) is a bit more involved. Actually, (3.15) and a comparison of terms in (2.3) give

$$\vartheta_{n,t} \rightarrow \vartheta_t \quad \text{weakly in } L^2(0, T; V'). \quad (3.30)$$

Then, integrating in time and using the V' -convergence in (3.2), we obtain more precisely

$$\vartheta_n \rightarrow \vartheta \quad \text{weakly in } H^1(0, T; V'). \quad (3.31)$$

This is sufficient to take the limit of equation (2.3), but not of (2.4). Actually, to identify ϑ in terms of u , we have to work a little bit more. Namely, we have to integrate (2.3) with respect to time both at the n -level and in the limit and then test, respectively, by u_n and by u . Notice that, even at the limit level, the use of u as a test function is guaranteed by the fact that $u \in L^2(0, T; V)$ and all terms in (2.3) lie at least in $L^2(0, T; V')$.

Then, at the n -level, we obtain

$$\int_0^T \langle \vartheta_n, u_n \rangle + \frac{1}{2} \|\nabla(1 * u_n)(t)\|^2 = \int_0^T \langle \vartheta_{0,n} + \chi_{0,n} - \chi_n, u_n \rangle, \quad (3.32)$$

where $*$ denotes convolution in time. Taking the supremum limit in the above relation and comparing the result with the limit equation, we obtain

$$\limsup_{n \nearrow +\infty} \int_0^T \langle \vartheta_n, u_n \rangle \leq \int_0^T \langle \vartheta, u \rangle. \quad (3.33)$$

Since

$$\vartheta_n = -\frac{1}{u_n} \in \partial_{V,V'} J(u_n) \quad \text{a.e. in } (0, +\infty), \quad (3.34)$$

(notice that we used here property (2.16)), relations (3.26) and (3.31) and the standard monotonicity argument [4, Prop. 1.1, p. 42], applied here in the duality pairing between V and V' , permit to obtain (2.30), which concludes the proof of existence in the conserved case.

Differences occurring in the non-conserved case.

At the level of estimates, the only relevant difference is in the energy relation, which is now obtained testing (2.3) by $1 + u$ and (2.5) by χ_t . This gives

$$\frac{d}{dt} \mathcal{E} + \|\nabla u\|^2 + \|\chi_t\|^2 = 0. \quad (3.35)$$

Thus, we have the H -norm of χ_t rather than the V' -norm on the left hand side (and consequently we obtain (2.27) in place of (2.26)). Estimates (3.22)–(3.24) hold without variations while (3.25) makes no longer sense. Notice that (3.24) can now be obtained testing directly (2.5) by $b(\chi)$. The passage to the limit is analogous.

Proof of (2.31).

We first observe that (2.30) is equivalent to

$$u \in \partial_{V',V} J^*(\vartheta), \quad (3.36)$$

almost everywhere in $(0, T)$. Then, the standard integration by parts formula [7, p. 73], applied in the duality between V' and V , gives

$$J^*(\vartheta) \in AC([0, T]), \quad \langle \vartheta_t, u \rangle = \frac{d}{dt} J^*(\vartheta). \quad (3.37)$$

Thanks to this formula, in the non-conserved case for any V' -solution, we are allowed to test (2.3) by u and (2.5) by χ_t . Integrating over $(0, t)$ for arbitrary $t > 0$, we obtain exactly (2.31). In the conserved case, instead, we have to test (2.3) by u , (2.6) by w and (2.7) by χ_t . Note that this is still possible for any V' -solutions. Indeed, thanks to (2.27) and the properties of A , we have that $w \in L^2(0, T; V)$. Thus, w can be used as a test function in (2.6) (which is a relation in $L^2(0, T; V')$), and χ_t can be used as a test function in (2.7) (which is a relation in $L^2(0, T; V)$ thanks to the above discussion). However, we have to notice that, while

$$AX + b(\chi) \in L^2(0, T; V), \quad (3.38)$$

it is not expected to be true that, separately, $AX \in L^2(0, T; V)$ and $b(\chi) \in L^2(0, T; V)$. Nevertheless, as shown, e.g., in [28, Lemma 4.1], property (3.38) is sufficient to prove that

$$\langle \chi_t, AX + b(\chi) \rangle = \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} |\nabla \chi|^2 + \widehat{b}(\chi) \right) \quad (3.39)$$

almost everywhere in $(0, T)$. Thus, we still have (2.31), of course with $\|\nabla w\|^2$ in place as $\|X_t\|^2$.

Proof of uniqueness.

It works exactly as in the standard case (so, we just sketch it for the conserved model). Namely, we can take a couple of solutions $(\vartheta_1, u_1, X_1), (\vartheta_2, u_2, X_2)$ starting from the same initial datum, write the system for both solutions and take the difference. Then, setting $(\vartheta, u, X) := (\vartheta_1, u_1, X_1) - (\vartheta_2, u_2, X_2)$, we integrate (the difference of) (2.3) in time and test it by u . Moreover, we test (the difference of) (2.6) by $A^{-1}X$ (note that X has zero-mean value, so A^{-1} is well defined) and the difference of (2.7) by X . Collecting everything and noting that two couples of terms cancel, we obtain

$$\langle \vartheta, u \rangle + \frac{1}{2} \frac{d}{dt} (\|\nabla(1 * u)\|^2 + \|X\|_{V'}^2) + \|\nabla X\|^2 \leq \|X\|^2 \leq \frac{1}{2} \|\nabla X\|^2 + c \|X\|_{V'}^2, \quad (3.40)$$

Noting that, by monotonicity, $\langle \vartheta, u \rangle = \langle \vartheta_1 - \vartheta_2, u_1 - u_2 \rangle \geq 0$, the thesis follows then from Gronwall's lemma.

3.2. Proof of Theorem 2.4

We start by deducing an additional a-priori estimate. As before, we present it just in the conserved case, the variations in the non-conserved case being given at the end.

Second estimate – local version.

We test (2.3) by $tu_t = t\vartheta_t/\vartheta^2$ and add the result to (2.6) multiplied by tw_t . Then, we add also the time derivative of (2.7) multiplied by tX_t . We obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{t}{2} \|\nabla u\|^2 + \frac{t}{2} \|\nabla w\|^2 \right) + t \int_{\Omega} \frac{\vartheta_t^2}{\vartheta^2} + t \|\nabla X_t\|^2 + t \int_{\Omega} b'(X) X_t^2 \\ & \leq \frac{1}{2} (\|\nabla u\|^2 + \|\nabla w\|^2) + t \|X_t\|^2. \end{aligned} \quad (3.41)$$

Then, noting that

$$t \|X_t\|^2 \leq \frac{t}{2} \|\nabla X_t\|^2 + ct \|\nabla w\|^2, \quad (3.42)$$

integrating (3.41) between 0 and $t \in (0, 1]$ and taking advantage of (3.15), we obtain

$$t \|\nabla u(t)\|^2 + t \|\nabla w(t)\|^2 + t \|X_t(t)\|_{V'}^2 + \int_0^t s \|\nabla X_t(s)\|^2 ds \leq c \mathbb{E}_0. \quad (3.43)$$

Second estimate—global version.

We test (2.3) by $u_t = \vartheta_t/\vartheta^2$ and add the result to (2.6) multiplied by w_t . Then, we add also the time derivative of (2.7) multiplied by X_t . Proceeding as above, we obtain

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\nabla w\|^2 \right) + \int_{\Omega} \frac{\vartheta_t^2}{\vartheta^2} + \frac{1}{2} \|\nabla X_t\|^2 + \int_{\Omega} b'(X) X_t^2 \leq c \|\nabla w\|^2. \quad (3.44)$$

Integrating between 1 and $t \geq 1$ and recalling (3.15) and (3.43), we infer

$$\|\nabla u\|_{L^\infty(1,\infty;H)} + \|\nabla w\|_{L^\infty(1,\infty;H)} + \|\chi_t\|_{L^\infty(1,\infty;V')} + \int_1^\infty \|\nabla \chi_t(t)\|^2 dt \leq Q(\mathbb{E}_0). \quad (3.45)$$

Using again the logarithmic Poincaré inequality (Lemma 3.16), we also have

$$\|u\|_{L^\infty(1,\infty;V)} \leq Q(\mathbb{E}_0), \quad (3.46)$$

i.e., the first (2.32).

In the non-conserved case, the procedure is similar. In place of (3.45), we rather obtain

$$\|\nabla u\|_{L^\infty(1,\infty;H)} + \|\chi_t\|_{L^\infty(1,\infty;H)} + \int_1^\infty \|\nabla \chi_t(t)\|^2 dt \leq Q(\mathbb{E}_0). \quad (3.47)$$

As a further consequence, we can look at equation (2.7) in the conserved case and (2.5) in the non-conserved case, respectively. Thanks to estimates (3.45)–(3.46) for u and w (respectively, to estimate (3.47) for u), applying standard regularity results for elliptic equations with monotone nonlinearities, we then obtain (2.33).

Asymptotic uniform regularity of u .

Our aim is now to show the second (2.32). The key step is represented by the following lemma:

LEMMA 3.3. *Let u be a solution of the problem*

$$\vartheta_t + Au = f, \quad \vartheta = -1/u \quad (3.48)$$

over the time interval $(S, S+2)$, where we additionally assume that

$$\|u\|_{L^3(S,S+2;L^{3/2}(\Omega))} \leq M, \quad \|f\|_{L^2(S,S+2;L^{3+\varepsilon}(\Omega))} \leq F, \quad (3.49)$$

for some (given) constants $M > 0$, $F > 0$ and some $\varepsilon > 0$. Moreover, let us assume that

$$u(S) \in L^1(\Omega). \quad (3.50)$$

Then, we have

$$\|u\|_{L^\infty((S+1,S+2)\times\Omega)} \leq Q(F, M, \|u(S)\|_1). \quad (3.51)$$

Proof. We test (3.48) by $-|u|^{p+1}$, where $p \geq 1$ will be specified later (although $-|u|^{p+1}$ needs not necessarily be an admissible test function, the procedure could be easily justified by truncation arguments, we omit the details). This gives

$$\frac{1}{p} \frac{d}{dt} \|u\|_p^p + \frac{4(p+1)}{(p+2)^2} \|\nabla |u|^{\frac{p+2}{2}}\|^2 \leq \int_\Omega |f||u|^{p+1}. \quad (3.52)$$

We then set $r := \frac{3+\varepsilon}{2+\varepsilon}$ to be the conjugate exponent of $3 + \varepsilon$. Then, multiplying by p , we can estimate the right hand side as

$$p \int_{\Omega} |f||u|^{p+1} \leq p \|f\|_{3+\varepsilon} \| |u|^{p+1} \|_r = p \|f\|_{3+\varepsilon} \|u\|_{r(p+1)}^{p+1}. \quad (3.53)$$

Then, in order to recover the full V -norm from the gradient term, we add

$$\| |u|^{\frac{p+2}{2}} \|_1^2 = \|u\|_{\frac{p+2}{2}}^{p+2} \quad (3.54)$$

to both hands sides of (3.52). Integrating (3.52) over (τ, t) , for t a generic point in $(\tau, S+2)$ and choosing, for the first iteration, $p = 1$ and $\tau = S$, we obtain

$$\begin{aligned} & \|u\|_{L^\infty(\tau, S+2; L^p(\Omega))}^p + \|u\|_{L^{p+2}(\tau, S+2; L^{3p+6}(\Omega))}^{p+2} \leq c \|u(\tau)\|_{L^p(\Omega)}^p \\ & + cp \|f\|_{L^2(S, S+2; L^{3+\varepsilon}(\Omega))} \|u\|_{L^{2(p+1)}(\tau, S+2; L^{r(p+1)}(\Omega))}^{p+1} + c \int_{\tau}^{S+2} \|u(s)\|_{\frac{p+2}{2}}^{p+2} ds \\ & \leq cpF \|u\|_{L^{2(p+1)}(\tau, S+2; L^{r(p+1)}(\Omega))}^{p+1} + Q(M, \|u(S)\|_p), \end{aligned} \quad (3.55)$$

where in the last inequality, we took advantage of (3.49) using that $p = 1$ and $\tau = S$.

Being non-restrictive to assume that $u \geq 1$ almost everywhere (otherwise, we can replace u with $\max\{u, 1\}$), we can then define

$$J_p^p := \|u\|_{L^\infty(\tau_p, S+2; L^p(\Omega))}^p + \|u\|_{L^{p+2}(\tau_p, S+2; L^{3p+6}(\Omega))}^p, \quad (3.56)$$

where, for now, we take $\tau_p = \tau = 0$. Then, by interpolation, we obtain

$$\|u\|_{L^{2(q+1)}(\tau_p, S+2; L^{r(q+1)}(\Omega))} \leq \|u\|_{L^\infty(\tau_p, S+2; L^p(\Omega))}^\alpha \|u\|_{L^{p+2}(\tau_p, S+2; L^{3p+6}(\Omega))}^{1-\alpha}, \quad (3.57)$$

for some $\alpha \in (0, 1)$.

Then, raising to the power p and using the Young inequality with exponents $P = 1/\alpha$ e $Q = 1/(1 - \alpha)$, we get

$$\begin{aligned} & \|u\|_{L^{2(q+1)}(\tau_p, S+2; L^{r(q+1)}(\Omega))}^p \leq \alpha \|u\|_{L^\infty(\tau_p, S+2; L^p(\Omega))}^P \\ & + (1 - \alpha) \|u\|_{L^{p+2}(\tau_p, S+2; L^{3p+6}(\Omega))}^p, \end{aligned} \quad (3.58)$$

which implies, upon dividing by $\max\{\alpha, 1 - \alpha\}$ (that is different from 0 and 1)

$$J_p^p \geq \|u\|_{L^{2(q+1)}(\tau_p, S+2; L^{r(q+1)}(\Omega))}^p, \quad (3.59)$$

where the index q and the interpolation exponent ρ are given by the system

$$\begin{cases} \frac{1-\rho}{p+2} = \frac{1}{2(q+1)}, \\ \frac{\rho}{p} + \frac{1-\rho}{3(p+2)} = \frac{1}{r(q+1)}. \end{cases} \quad (3.60)$$

Dividing the second equation in (3.60) by the first one, we actually have

$$\left(\frac{\rho}{p} + \frac{1-\rho}{3(p+2)}\right) \frac{p+2}{1-\rho} = \frac{2}{r}, \quad (3.61)$$

whence

$$\frac{\rho}{p} \frac{p+2}{1-\rho} = \frac{2}{r} - \frac{1}{3} =: K_\varepsilon, \quad (3.62)$$

and it is easy to compute

$$K_\varepsilon = \frac{9+7\varepsilon}{9+3\varepsilon}. \quad (3.63)$$

From (3.62) and the first equation in (3.60), we then have

$$\rho = \frac{\frac{p}{p+2} K_\varepsilon}{1 + \frac{p}{p+2} K_\varepsilon} \in (0, 1) \quad \forall p \geq 1. \quad (3.64)$$

Being

$$1 - \rho = \frac{1}{1 + \frac{p}{p+2} K_\varepsilon}, \quad (3.65)$$

we then obtain from the first (3.60)

$$\begin{aligned} q &= \frac{1}{2} \frac{p+2}{1-\rho} - 1 = \frac{1}{2} \left(1 + \frac{p}{p+2} K_\varepsilon\right) (p+2) - 1 \\ &= \frac{K_\varepsilon + 1}{2} p = \frac{9+5\varepsilon}{9+3\varepsilon} p =: Hp, \end{aligned} \quad (3.66)$$

where, obviously, $H = H(\varepsilon) > 1$ whenever $\varepsilon > 0$.

Given that $p_0 = 1$, let us set, by induction, $p_{i+1} = Hp_i = H^{i+1}$. Then, let $i \geq 0$ and let us rewrite (3.55) by taking $p = p_{i+1}$ and $\tau = \tau_{i+1}$ (the latter will be chosen below). Setting also, for brevity, $J_i := J_{p_i}$, we then obtain, thanks also to (3.59),

$$J_{i+1}^{p_{i+1}} \leq c \|u(\tau_{i+1})\|_{p_{i+1}}^{p_{i+1}} + c p_{i+1} F J_i^{p_{i+1}+1} + c \int_{\tau_{i+1}}^2 \|u(s)\|_{\frac{p_{i+1}+2}{2}}^{p_{i+1}+2} ds. \quad (3.67)$$

Now, let (for instance), for $i \geq 1$,

$$\sigma_i = \frac{6}{\pi^2 i^2}, \quad \text{so that} \quad \sum_{i=1}^{\infty} \sigma_i = 1. \quad (3.68)$$

Then, we observe that, given τ_i , we can choose $\tau_{i+1} \in (\tau_i, \tau_i + \sigma_{i+1})$ such that

$$\begin{aligned} \|u(\tau_{i+1})\|_{p_{i+1}}^{p_i} &\leq \frac{1}{\sigma_{i+1}} \int_{\tau_i}^{\tau_i + \sigma_{i+1}} \|u(t)\|_{p_{i+1}}^{p_i} dt \\ &\leq c i^2 \int_{\tau_i}^{\tau_i + \sigma_{i+1}} \|u(t)\|_{3(p_i+2)}^{p_i} dt \leq c i^2 J_i^{p_i}, \end{aligned} \quad (3.69)$$

where we used that $p_{i+1} \leq 3(p_i + 2)$.

Analogously, we have that

$$c \int_{\tau_{i+1}}^{S+2} \|u(s)\|_{\frac{p_{i+1}+2}{2}}^{p_{i+1}+2} ds \leq 2c \|u\|_{L^\infty(\tau_i, S+2; L^{p_i}(\Omega))}^{p_{i+1}+2} \leq c J_i^{p_{i+1}+2}. \quad (3.70)$$

Collecting (3.69)-(3.70), (3.67) gives

$$J_{i+1}^{p_{i+1}} \leq c i^{2H} J_i^{p_{i+1}} + c p_{i+1} F J_i^{p_{i+1}+1} + c J_i^{p_{i+1}+2}. \quad (3.71)$$

Thus, we finally obtain

$$J_{i+1}^{H^{i+1}} \leq c(i^{2H} + H^{i+1}F + 1)J_i^{H^{i+1}+2}. \quad (3.72)$$

Thus, setting $\eta_i := (H^i + 2)/H^i$, we have

$$J_{i+1} \leq B_i^{H^{-(i+1)}} J_i^{\eta_{i+1}}, \quad \text{where } B_i := c(i^{2H} + H^{i+1}F + 1) \quad (3.73)$$

whence a simple induction argument (see, e.g., [29]) permits to obtain (3.51). $\square$

Conclusion of the proof.

To obtain the uniform boundedness of u , it is now sufficient to notice that, by the first (2.32), $u(t) \in L^6(\Omega)$ for all $t \geq 1$. Then, (3.50) holds. Moreover, (3.49) are a consequence of (3.45) (or, in the non-conserved case, (3.47)), which gives the required regularity for $f = -\chi_t$, and of (3.46), which gives the required regularity for u . The second (2.32) is then a consequence of the lemma (applied with the choice of $S = 2$). It is also worth noting that, at the level of V' -solution, the identification property in (3.48) needs not hold in the strong (pointwise) form (but just in the “weak” sense (2.30)). However, one can apply Lemma 3.3 at the n -regularized level and then pass to the limit noting that the procedure yields n -uniform estimates.

To complete the proof, we have to show (2.34) in the non-conserved case. Of course, such a property is significant only in the case when I , the domain of b , does not coincide with the real line, i.e., we are in presence of a singular potential (like the logarithmic one (1.1)). Otherwise, (2.34) is (also in the conserved case, of course) an immediate consequence of (2.33).

That said, let us prove the upper bound, the lower one working in a similar way. Being, by the second (2.32), $|u(t, x)| \leq U$ for some $U = Q(\mathbb{E}_0)$ and a.e. $(t, x) \in (2, \infty) \times \Omega$, we can then apply the comparison principle to (2.5). This gives that χ is bounded from above by the solution χ^+ to the forward Cauchy problem

$$\chi_t^+ + b(\chi^+) - \chi^+ = U, \quad \chi^+(2) = I^+, \quad \text{where } I^+ := \sup I. \quad (3.74)$$

Actually, by the last (2.9), $\lim_{r \nearrow \sup I} b(r) - r - U = +\infty$. Thus, $\chi^+(t) \leq I^+ - \delta$ for all $t \geq 3$ and some $\delta > 0$. Moreover, this bound is uniform in time. Then, (2.34) is a consequence of the comparison principle. This concludes the proof of the theorem.

3.3. Proof of Theorem 2.6

We start by giving the proof in the non-conserved case, the variations occurring in the conserved case being outlined at the end. Then, we know that any V' -solution (ϑ, u, χ) satisfies the energy equality (2.31). Analogously, if $(\vartheta_n, u_n, \chi_n)$ is the approximating solution constructed in the existence proof, the analogue of (2.31) reads

$$\begin{aligned} \int_0^t (\|\nabla u_n\|^2 + \|\chi_{n,t}\|^2) &= J^*(\vartheta_{0,n}) - J^*(\vartheta_n(t)) + \langle \vartheta_{0,n}, 1 \rangle - \langle \vartheta_n(t), 1 \rangle \\ &+ \int_{\Omega} \left(\chi_0 + \frac{1}{2} |\nabla \chi_0|^2 + \widehat{b}(\chi_0) - \frac{1}{2} \chi_0^2 \right) \\ &- \int_{\Omega} \left(\chi_n(t) + \frac{1}{2} |\nabla \chi_n(t)|^2 + \widehat{b}(\chi_n(t)) - \frac{1}{2} \chi_n(t)^2 \right). \end{aligned} \quad (3.75)$$

Our task is now to compute the supremum limit of (3.75) and compare it with (2.31). Then, we first observe that, by (3.27), the Aubin-Lions lemma, and lower semicontinuity of $\widehat{b}$,

$$\begin{aligned} \int_{\Omega} \left(\chi(t) + \frac{1}{2} |\nabla \chi(t)|^2 + \widehat{b}(\chi(t)) - \frac{1}{2} \chi(t)^2 \right) &\leq \liminf_{n \nearrow \infty} \\ &\times \int_{\Omega} \left(\chi_n(t) + \frac{1}{2} |\nabla \chi_n(t)|^2 + \widehat{b}(\chi_n(t)) - \frac{1}{2} \chi_n(t)^2 \right). \end{aligned} \quad (3.76)$$

Analogously, using convexity and lower semicontinuity of the functional J^* w.r.t. the V' -norm, (3.2), and the fact that, by (3.31), $\vartheta_n(t)$ tends to $\vartheta(t)$ weakly in V' for all $t \in [0, T]$, we obtain

$$J^*(\vartheta(t)) - \langle \vartheta_0, 1 \rangle + \langle \vartheta(t), 1 \rangle \leq \liminf_{n \nearrow \infty} \left(J^*(\vartheta_n(t)) - \langle \vartheta_{0,n}, 1 \rangle + \langle \vartheta_n(t), 1 \rangle \right). \quad (3.77)$$

So, it remains to prove that

$$\limsup_{n \nearrow \infty} J^*(\vartheta_{0,n}) \leq J^*(\vartheta_0). \quad (3.78)$$

The proof of this fact is actually a bit more involved. We prepare a Lemma

LEMMA 3.4. *Let $v \in V' \cap L^1(\Omega)$ such that $j^*(v) \in L^1(\Omega)$, j and j^* being given by (2.12). Then,*

$$J^*(v) = \int_{\Omega} j^*(v). \quad (3.79)$$

Proof. Let z belong to the domain of J , namely let $z \in V$ with $J(z) < +\infty$. Then, for a.e. $x \in \Omega$, by the definition of subdifferential in $\mathbb{R}$, we have

$$z(x)v(x) \leq j(z(x)) + j^*(v(x)). \quad (3.80)$$

Integrating over Ω , we would *formally* get

$$\int_{\Omega} zv \leq J(z) + \int_{\Omega} j^*(v). \quad (3.81)$$

However, the integral on the left hand side could make no sense since the function zv could not belong to $L^1(\Omega)$. Nevertheless, it is simple (see, e.g., [5, Lemma 2.2] and [4, Lemma 2.1]) to see that, in place of (3.81), there holds

$$\langle z, v \rangle \leq J(z) + \int_{\Omega} j^*(v). \quad (3.82)$$

Passing to the supremum w.r.t. z varying in the domain of J , we then get the $\leq$ sign in (3.79).

To prove the converse, we first let, for v as in the statement and $N \in (1, \infty)$,

$$z_N := -\max \{N^{-1}, \min\{v^{-1}, N\}\}. \quad (3.83)$$

Then, for $\epsilon \in (0, 1)$, we regularize by singular perturbation as in (3.1):

$$z_{\epsilon, N} + \epsilon A z_{\epsilon, N} = z_N. \quad (3.84)$$

Then,

$$\begin{aligned} J^*(v) &\geq \langle v, z_{\epsilon, N} \rangle - J(z_{\epsilon, N}) = \int_{\Omega} v z_{\epsilon, N} - J(z_{\epsilon, N}) = \int_{\Omega} v z_{\epsilon, N} + \int_{\Omega} \log(-z_{\epsilon, N}) \\ &\geq \int_{\Omega} v z_{\epsilon, N} + \int_{\Omega} \log(-z_N) \xrightarrow{\epsilon \searrow 0} \int_{\Omega} v z_N + \int_{\Omega} \log(-z_N) \\ &\xrightarrow{N \nearrow \infty} \int_{\Omega} (-1 + \log v^{-1}) = \int_{\Omega} j^*(v), \end{aligned} \quad (3.85)$$

where the first inequality follows from the definition of conjugate function, the second equality from the fact that $z_{\epsilon, N}$ is smooth and bounded, the third equality is trivial, the fourth inequality comes from (3.84), the convergence $\epsilon \searrow 0$ from standard properties of elliptic systems, and the convergence $N \nearrow \infty$ from Lebesgue's theorem. This proves the $\geq$ in (3.79) and the lemma. $\square$

Then, by definition of conjugate function (recall (2.12)),

$$\begin{aligned} J^*(\vartheta_{0, n}) &= \sup_{v \in V} \langle \vartheta_{0, n}, v \rangle - J(v) \leq \sup_{v \in H} \langle \vartheta_{0, n}, v \rangle - J(v) \\ &= \int_{\Omega} -1 - \log \vartheta_{0, n} \leq \int_{\Omega} -1 - \log \vartheta_0, \end{aligned} \quad (3.86)$$

where the last inequality follows from (3.4). In particular, we have that $-1 - \log \vartheta_0 \in L^1(\Omega)$. Thus, applying the above Lemma, we obtain

$$J^*(\vartheta_0) = \int_{\Omega} (-1 - \log \vartheta_0), \quad (3.87)$$

whence, computing the supremum limit of (3.86), (3.78) follows. Thus, we finally end up with

$$\limsup_{n \nearrow \infty} \int_0^t (\|\nabla u_n\|^2 + \|\chi_{n,t}\|^2) \leq \int_0^t (\|\nabla u\|^2 + \|\chi_t\|^2), \quad (3.88)$$

whence, recalling (3.26)-(3.29), we get in particular

$$\nabla u_n, \chi_{n,t} \rightarrow \nabla u, \chi_t \text{ strongly in } L^2(0, T; H). \quad (3.89)$$

As a final step of our procedure, we shall prove that $\{\vartheta_n\}$ is a Cauchy sequence in $C^0([0, T]; L^1(\Omega))$. Actually, writing equation (2.3) for a couple of indexes n and m and taking the difference, we obtain

$$\vartheta_{n,t} - \vartheta_{m,t} + A(u_n - u_m) = \chi_{n,t} - \chi_{m,t}. \quad (3.90)$$

Thus, testing by $\text{sign}(\vartheta_n - \vartheta_m)$, noticing that, by monotonicity, $\text{sign}(\vartheta_n - \vartheta_m) = \text{sign}(u_n - u_m)$, and applying the Brezis-Strauss theorem [9, Lemma 2], we arrive at

$$\frac{d}{dt} \|\vartheta_n - \vartheta_m\|_{L^1(\Omega)} \leq \|\chi_{n,t} - \chi_{m,t}\|_{L^1(\Omega)}, \quad (3.91)$$

whence, integrating in time and using the strong convergences (3.2) and (3.89), we end up with

$$\vartheta_n \rightarrow \vartheta \text{ strongly in } C^0([0, T]; L^1(\Omega)). \quad (3.92)$$

Thus, in particular, we have that, for all $t \in [0, T]$, $\vartheta(t) \in L^1(\Omega) \cap \partial_{V,V'} J(u(t))$, whence the pointwise identification (2.4) follows from (2.17). This concludes the proof in the non-conserved case.

Conserved case. To conclude the proof, we outline the differences occurring in the conserved case, which only regard the above L^1 -argument. Actually, the convergence of $\chi_{n,t}$ in (3.89) is now replaced by

$$\nabla w_n \rightarrow \nabla w, \text{ strongly in } L^2(0, T; H). \quad (3.93)$$

Of course, thanks to the properties of A , this also gives

$$\chi_{N,T} \rightarrow \chi_T \text{ strongly in } L^2(0, T; V'), \quad (3.94)$$

which, however, is not sufficient to proceed as before. On the other hand, we can rely on estimates (3.43) and (3.47) which tell us that

$$\|t^{1/2} \chi_{n,t}\|_{L^2(0,T;V)} + \|t^{1/2} \chi_{n,t}\|_{L^\infty(0,T;V')} \leq c. \quad (3.95)$$

Thus, by interpolation,

$$\|t^{1/2} \chi_{n,t}\|_{L^{\frac{4}{2-\epsilon}}(0,T;H^{1-\epsilon}(\Omega))} \leq c \quad (3.96)$$

for all $\epsilon \in (0, 1)$. Coming back to (3.91), we now have that, for all $t \in [0, T]$,

$$\begin{aligned} \|\chi_{n,t} - \chi_{m,t}\|_{L^1(\Omega)} &\leq c \|\chi_{n,t} - \chi_{m,t}\| \leq c \|\chi_{n,t} - \chi_{m,t}\|_{V'}^{\frac{1-\epsilon}{2-\epsilon}} \|\chi_{n,t} - \chi_{m,t}\|_{H^{1-\epsilon}(\Omega)}^{\frac{1}{2-\epsilon}} \\ &\leq c t^{-\frac{1}{2(2-\epsilon)}} \|\chi_{n,t} - \chi_{m,t}\|_{V'}^{\frac{1-\epsilon}{2-\epsilon}} \cdot t^{\frac{1}{2(2-\epsilon)}} \|\chi_{n,t} - \chi_{m,t}\|_{H^{1-\epsilon}(\Omega)}^{\frac{1}{2-\epsilon}}. \end{aligned} \quad (3.97)$$

Then, integrating over $(0, T)$, we arrive at

$$\begin{aligned} \|\chi_{n,t} - \chi_{m,t}\|_{L^1((0,T)\times\Omega)} &\leq c \left\| t^{-\frac{1}{2(2-\epsilon)}} \|\chi_{n,t} - \chi_{m,t}\|_{V'}^{\frac{1-\epsilon}{2-\epsilon}} \right. \\ &\quad \times \left. \left\| t^{\frac{1}{2(2-\epsilon)}} \|\chi_{n,t} - \chi_{m,t}\|_{H^{1-\epsilon}(\Omega)}^{\frac{1}{2-\epsilon}} \right\|_{L^4(0,T)} \right\|_{L^4(0,T)} \\ &\leq c \left(\int_0^T t^{-\frac{2}{3(2-\epsilon)}} \|\chi_{n,t} - \chi_{m,t}\|_{V'}^{\frac{4(1-\epsilon)}{3(2-\epsilon)}} dt \right)^{\frac{3}{4}} \\ &\leq c \left\| t^{-\frac{2}{3(2-\epsilon)}} \right\|_{L^2(0,T)}^{\frac{3}{4}} \left\| \|\chi_{n,t} - \chi_{m,t}\|_{V'}^{\frac{4(1-\epsilon)}{3(2-\epsilon)}} \right\|_{L^2(0,T)}^{\frac{3}{4}} \\ &\leq c \|\chi_{n,t} - \chi_{m,t}\|_{L^{\frac{8(1-\epsilon)}{3(2-\epsilon)}}(0,T;V')}^{\frac{1-\epsilon}{2-\epsilon}}, \end{aligned} \quad (3.98)$$

where the second inequality is a consequence of (3.96), the thirds follows from Hölder's inequality, and the fourth holds provided that we take ϵ so small that $4 < 3(2 - \epsilon)$. In particular, using (3.94) (note that $8(1 - \epsilon)/3(2 - \epsilon)$ is smaller than 2 for ϵ as above), we obtain that the right hand side tends to 0 for large m and n . At this point, the proof goes on like in the non-conserved case.

3.4. Proof of Theorem 2.7

To start, we need to prove some further a-priori estimates holding under the additional assumption (2.37). In particular, the key step will be that of showing that the $L^{3+\epsilon}$ regularity of ϑ is conserved uniformly in time. To show this, we will use L^p -techniques in equation (2.3) (actually, with $p = 3 + \epsilon$). However, doing this will require some care since, due to the low regularity of initial data, the “forcing term” χ_t needs not belong to $L^{3+\epsilon}(\Omega)$ for small values of the time variable. However, we will see that the $L^{3+\epsilon}$ -norm of $\chi_t(t)$ explodes, as $t \searrow 0$, in a way which is sufficiently slow for our purpose. As before, the proof is detailed just in the conserved case. That said, we start with the

Third estimate—local version. To start, we test (2.3) by $\vartheta^{2+\epsilon}$, to obtain

$$\frac{d}{dt} \|\vartheta\|_{3+\epsilon}^{3+\epsilon} \leq c \int_{\Omega} |\chi_t| \vartheta^{2+\epsilon} \leq c \|\chi_t\|_{3+\epsilon} \|\vartheta\|_{3+\epsilon}^{2+\epsilon}, \quad (3.99)$$

whence, clearly,

$$\frac{d}{dt} \|\vartheta\|_{3+\epsilon} \leq c \|\chi_t\|_{3+\epsilon}. \quad (3.100)$$

Thus, using that

$$H^{\frac{3+3\varepsilon}{6+2\varepsilon}}(\Omega) \subset L^{3+\varepsilon}(\Omega) \quad (3.101)$$

and that

$$\frac{3+3\varepsilon}{6+2\varepsilon} = \alpha \times 1 + (1-\alpha) \times (-1), \quad \text{with } \alpha = \frac{9+5\varepsilon}{12+4\varepsilon}, \quad (3.102)$$

we obtain

$$\begin{aligned} \frac{d}{dt} \|\vartheta\|_{3+\varepsilon} &\leq c \|X_t\|_V^{\frac{9+5\varepsilon}{12+4\varepsilon}} \|X_t\|_{V'}^{\frac{3-\varepsilon}{12+4\varepsilon}} \\ &\leq c \left(t^{\frac{9+5\varepsilon}{24+8\varepsilon}} \|X_t\|_V^{\frac{9+5\varepsilon}{12+4\varepsilon}} \right) \left(t^{\frac{3-\varepsilon}{24+8\varepsilon}} \|X_t\|_{V'}^{\frac{3-\varepsilon}{12+4\varepsilon}} \right) t^{-\frac{1}{2}} \\ &\leq Q(\mathbb{E}_0) \left(t^{\frac{9+5\varepsilon}{24+8\varepsilon}} \|X_t\|_V^{\frac{9+5\varepsilon}{12+4\varepsilon}} \right) t^{-\frac{1}{2}} \\ &\leq Q(\mathbb{E}_0) \left(t \|X_t\|_V^2 + t^{-\frac{12+4\varepsilon}{15+3\varepsilon}} \right), \end{aligned} \quad (3.103)$$

where (3.43) has also been exploited. Note that, for $\varepsilon \in (0, 1)$, the latter exponent lies in $(-1, 0)$. Notice also that in the non-conserved case, the exponents are even better since it is sufficient to interpolate between V and H (rather than between V and V'). Thus, integrating (3.103) between 0 and 1, and using once more (3.43), we infer

$$\|\vartheta\|_{L^\infty(0,1;L^{3+\varepsilon}(\Omega))} \leq Q(\mathbb{E}_0, \|\vartheta_0\|_{3+\varepsilon}). \quad (3.104)$$

Third estimate—global version. As before, we test (2.3) by $\vartheta^{2+\varepsilon}$. Taking now care also of the gradient term, we get

$$\frac{d}{dt} \|\vartheta\|_{3+\varepsilon}^{3+\varepsilon} + \kappa \|\nabla \vartheta^{\frac{1+\varepsilon}{2}}\|^2 \leq c \int_\Omega |\chi_t| \vartheta^{2+\varepsilon}. \quad (3.105)$$

Adding also the inequality (which is true thanks to (3.15))

$$\kappa \|\vartheta^{\frac{1+\varepsilon}{2}}\|_1^2 \leq c(1 + \|\vartheta\|_1^2) \leq Q(\mathbb{E}_0), \quad (3.106)$$

we then get

$$\begin{aligned} \frac{d}{dt} \|\vartheta\|_{3+\varepsilon}^{3+\varepsilon} + \kappa \|\vartheta^{\frac{1+\varepsilon}{2}}\|_6^2 &\leq c \|X_t\|_6 \|\vartheta^{\frac{3+\varepsilon}{2}}\|_{3/2} \|\vartheta^{\frac{1+\varepsilon}{2}}\|_6 + Q(\mathbb{E}_0) \\ &\leq c \|X_t\|_V^2 \|\vartheta^{\frac{3+\varepsilon}{2}}\|_{3/2}^2 + \frac{\kappa}{2} \|\vartheta^{\frac{1+\varepsilon}{2}}\|_6^2 + Q(\mathbb{E}_0) \\ &\leq c \|X_t\|_V^2 \|\vartheta\|_{\frac{9+3\varepsilon}{4}}^{3+\varepsilon} + \frac{\kappa}{2} \|\vartheta^{\frac{1+\varepsilon}{2}}\|_6^2 + Q(\mathbb{E}_0) \\ &\leq c \|X_t\|_V^2 \|\vartheta\|_{3+\varepsilon}^{3+\varepsilon} + \frac{\kappa}{2} \|\vartheta^{\frac{1+\varepsilon}{2}}\|_6^2 + Q(\mathbb{E}_0). \end{aligned} \quad (3.107)$$

Thus, setting $Y(t) := \|\vartheta(t)\|_{3+\varepsilon}^2$,

we obtain from (3.107)

$$Y' + \kappa \leq c \|X_t\|_V^2 Y + QY^{-\frac{1+\varepsilon}{2}}, \quad (3.108)$$

where we wrote Q in place of $Q(\mathbb{E}_0)$. Now, let us set

$$Z(t) := \max \left\{ Y(t), \left(\frac{Q}{\kappa} \right)^{\frac{2}{1+\varepsilon}} \right\}, \quad (3.109)$$

so that it is clear that Z satisfies,

$$Z'(t) \leq c \|X_t(t)\|_V^2 Z(t), \quad \text{for a.e. } t \geq 1, \quad \text{and } Z(1) \leq Q(\mathbb{E}_0, \|\vartheta_0\|_{3+\varepsilon}). \quad (3.110)$$

Hence, integrating between 1 and a generic $t \geq 1$ and recalling (3.45), we arrive at

$$\|\vartheta\|_{L^\infty(1,\infty;L^{3+\varepsilon}(\Omega))} \leq Q(\mathbb{E}_0, \|\vartheta_0\|_{3+\varepsilon}), \quad (3.111)$$

i.e., the global analogue of (3.104).

Asymptotic uniform regularity of ϑ . The key step is represented by the following counterpart of Lemma 3.3:

LEMMA 3.5. *Let u be a solution of the problem*

$$\vartheta_t + Au = f, \quad \vartheta = -1/u, \quad (3.112)$$

over the time interval $(S, S+2)$, where we additionally assume that

$$\|\vartheta\|_{L^\infty(S,S+2;L^1(\Omega))} \leq M, \quad \|f\|_{L^2(S,S+2;L^{3+\varepsilon}(\Omega))} \leq F, \quad (3.113)$$

for some (given) constants $M > 0$, $F > 0$ and some $\varepsilon > 0$. Moreover, let us assume that $\vartheta(S) \in L^{3+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then,

$$\|\vartheta\|_{L^\infty((S+1,S+2) \times \Omega)} \leq Q(F, M, \|\vartheta(S)\|_{3+\varepsilon}). \quad (3.114)$$

Proof. We test (3.48) by ϑ^{p-1} , where $p > 3$ will be specified later. This gives

$$\frac{1}{p} \frac{d}{dt} \|\vartheta\|_p^p + \frac{4(p-1)}{(p-2)^2} \|\nabla \vartheta^{\frac{p-2}{2}}\|^2 \leq \int_{\Omega} |f| \vartheta^{p-1}. \quad (3.115)$$

Setting as before $r := \frac{3+\varepsilon}{2+\varepsilon}$ and multiplying by p , we have

$$p \int_{\Omega} |f| \vartheta^{p-1} \leq p \|f\|_{3+\varepsilon} \|\vartheta^{p-1}\|_r = p \|f\|_{3+\varepsilon} \|\vartheta\|_{r(p-1)}^{p-1} \quad (3.116)$$

and, in order to recover the full V -norm from the gradient term, we add

$$\|\vartheta^{\frac{p-2}{2}}\|_1^2 = \|\vartheta\|_{\max\{\frac{p-2}{2}, 1\}}^{p-2}. \quad (3.117)$$

Integrating (3.115) over (τ, t) , for t , a generic point in $(\tau, S+2)$ and choosing, for the first iteration, $p = 3 + \varepsilon$ and $\tau = S$, we then have the analogue of (3.55):

$$\begin{aligned} & \|\vartheta\|_{L^\infty(\tau, S+2; L^p(\Omega))}^p + \|\vartheta\|_{L^{p-2}(\tau, S+2; L^{3p-6}(\Omega))}^{p+2} \\ & \leq c\|u(\tau)\|_{L^p(\Omega)}^p + cpF\|\vartheta\|_{L^{2(p-1)}(\tau, S+2; L^{r(p-1)}(\Omega))}^{p+1} + c \int_\tau^{S+2} \|\vartheta(s)\|_{\max\{\frac{p-2}{2}, 1\}}^{p-2} ds \\ & \leq cpF\|\vartheta\|_{L^{2(p-1)}(\tau, t; L^{r(p-1)}(\Omega))}^{p-1} + Q(M, \|u_0\|_1), \end{aligned} \quad (3.118)$$

Now, the iteration scheme goes through similarly as before. Actually, in place of (3.60), we get the system

$$\begin{cases} \frac{1-\rho}{p-2} = \frac{1}{2(q-1)}, \\ \frac{\rho}{p} + \frac{1-\rho}{3(p-2)} = \frac{1}{r(q-1)}, \end{cases} \quad (3.119)$$

whence one computes, exactly as before,

$$K_\varepsilon = \frac{9+7\varepsilon}{9+3\varepsilon}. \quad (3.120)$$

and, finally,

$$\begin{aligned} q &= \frac{1}{2} \frac{p-2}{1-\rho} + 1 = \frac{1}{2} \left(1 + \frac{p}{p-2} K_\varepsilon \right) (p-2) + 1 \\ &= \frac{K_\varepsilon + 1}{2} p = \frac{9+5\varepsilon}{9+3\varepsilon} p = Hp, \end{aligned} \quad (3.121)$$

which, exactly as before, is larger than one. Hence, the procedure continues as before, with small variations in the numerical values of the indices. Of course, the $L^{3+\varepsilon}$ regularity of the initial datum is used since we need to take $p = p_0 = 3 + \varepsilon$ at the first iteration (for smaller values of p , we get no summability gain from the gradient term). $\square$

Conclusion of proof. Thanks to estimate (3.111), ϑ satisfies the first (3.113) for any $S \geq 1$ (where M is the quantity on the right hand side of (3.111) which is independent of S). Analogously, we have $\vartheta(S) \in L^{3+\varepsilon}$ for (almost) all $S \geq 1$. Moreover, combining (3.15) and (3.45), we have the second of (3.113), still with F independent of S . We then conclude applying the above Lemma over the generic interval $(S, S+2)$, with $S \geq 1$.

REMARK 3.6. Of course, with (2.32) and (2.38) at our disposal, equation (2.3) is both non-singular and non-degenerate. Consequently, we can prove, with standard tools, further regularization properties of solutions. In the non-conserved case, thanks to (2.34), also the (possibly) singular character of b is lost. Thus, the smoothness of the solution is limited only by the differentiability properties of b . For instance, if $b \in C^\infty$, then also the solution is infinitely differentiable for strictly positive times.

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