Normalized functionals of first passage Brownian motion and a curious connection with the maximal relative height of fluctuating interfaces

Michael J Kearney

Senate House, University of Surrey, Guildford, Surrey, GU2 7XH, UK

Richard J Martin

Department of Mathematics, Imperial College London,

South Kensington, London, SW7 2AZ, UK

Abstract

A study is made of the normalized functionals $\mathcal{M} \equiv M/T^{1/2}$ and $\mathcal{A} \equiv A/T^{3/2}$ associated with one-dimensional first passage Brownian motion with positive initial condition, where M is the maximum value attained and A is the area swept out up to the random time T at which the process first reaches zero. Both $\mathcal{M}$ and $\mathcal{A}$ involve two strongly correlated random variables associated with a given Brownian path. Through their study, fresh insights are provided into the fundamental nature of such first passage processes and the underlying correlations. The probability density and the moments of $\mathcal{M}$ and $\mathcal{A}$ are calculated exactly and the theoretical results are shown to be in good agreement with those derived from simulations. Intriguingly, there is a precise equivalence in law between the variable $\mathcal{A}$ and the maximal

relative height of the fluctuating interface in the one-dimensional Edwards-Wilkinson model with free boundary conditions. This observation leads to some interesting and still partially unresolved questions.

1. Introduction

First-passage processes related to Brownian motion are commonly encountered in the physical, life and social sciences and their characterization is therefore of practical interest [1, 2]. If one considers a one-dimensional Brownian motion,

$$y(t) = y_0 + \sigma W(t) \quad (1)$$

where $y_0 > 0$ is the initial condition and $W(t)$ is the Wiener process with $W(0) \equiv 0$, then we define the first passage process to be the evolution of (1) up to the random time $T = \inf\{t : y(t) = 0\}$ at which the path $y(t)$ first reaches zero. Hereafter the choice $\sigma = 1$ is made for convenience. The probability density of the first passage time is well-known [3; Sect. 1, 2.0.2],

$$P(T) = \frac{y_0}{\sqrt{2\pi T^3}} \exp\left\{-\frac{y_0^2}{2T}\right\} \quad (2)$$

and the process is recurrent, i.e. the first passage time is finite with probability one.

The moments $\langle T^k \rangle$, on the other hand, diverge for all positive integer k because the likelihood of finding paths of long duration only decays slowly as $T \rightarrow \infty$.

Two important random variables associated with (technically we say they are functionals of) the first passage process are the maximum $M = \max_{0 \leq t \leq T} y(t)$ of the path till the first passage time and the area $A = \int_0^T y(t) dt$ swept out by the path till the first

passage time. The probability densities of these two variables are given by [3; Sect. 1, 2.1.2; Sect. 1, 2.8.2], see also [4];

$$P(M) = \frac{y_0}{M^2}; \quad M \geq y_0$$

$$P(A) = \frac{2^{1/3}}{3^{2/3} \Gamma(\frac{1}{3})} \frac{y_0}{A^{4/3}} \exp \left\{ -\frac{2y_0^3}{9A} \right\}. \quad (3)$$

Since both M and A are strongly correlated with T , as expected the moments $\langle M^k \rangle$ and $\langle A^k \rangle$ also diverge for all positive integer k . Both variables have been studied within the physics literature and find application within the broader study of Brownian functionals *per se* [5]. An early classic problem was to find the distribution of the maximum value of Brownian motion up to some fixed time [6]. Subsequent studies have addressed the same question in relation to constrained variants of Brownian motion, such as excursions and meanders [7, 8], together with consideration of the actual time to reach the maximum [9], and this has been further extended to more complex Brownian functionals which relate in some way to the maximum value attained. For a recent account see [10]; areas of related application include the study of kinetically growing Gaussian interfaces and extreme and near extreme events in time series analysis [11-16], as well as search optimisation problems [17, 18]. Regarding the area variable, early work also focussed on variants of Brownian motion up to some fixed time [19-22]; these studies are comprehensively reviewed in [23]. The first passage area variable arises naturally in the study of burst processes in solar physics, animal movement patterns, DNA breathing dynamics and models of

evolution [24-27]. There is also a direct link with various types of queueing phenomena such as traffic jams [28, 29].

The general scaling properties of the Wiener process suggest that $M \sim T^{1/2}$ and $A \sim T^{3/2}$ as $T \rightarrow \infty$. It is therefore natural in the context of characterizing the first passage process to consider the scaled or normalized random variables

$$\mathcal{M} \equiv \frac{M}{T^{1/2}}; \quad \mathcal{A} \equiv \frac{A}{T^{3/2}} = \frac{1}{T^{3/2}} \int_0^T y(t) dt. \quad (4)$$

The variables $\mathcal{M}$ and $\mathcal{A}$ both involve two correlated random variables associated with a given path, making their analysis somewhat tricky. Some aspects of the underlying mathematical structure have been explored before in the context of studying the normalized functional,

$$\mathcal{A}_a^{(m)} = \frac{1}{T_a^{1+m/2}} \int_0^{T_a} |W(t)|^m \text{sgn}(W(t)) dt \quad (5)$$

where $T_a = \inf\{t : W(t) = a\}$ is the first passage or hitting time for the Wiener process to reach a fixed boundary $a > 0$ [30]. In that work it was shown that the expectation of this random variable is independent of the boundary position a , but is strictly positive (negative) depending on whether $m > 1$ ($m < 1$) and precisely zero when $m = 1$. In what follows we calculate in relation to (4) the exact probability densities $f_{\mathcal{M}}(x)$ and $f_{\mathcal{A}}(x)$, and the corresponding moments $\langle \mathcal{M}^k \rangle$ and $\langle \mathcal{A}^k \rangle$, for all positive integer k , noting in advance that:

1. The densities $f_{\mathcal{M}}(x)$ and $f_{\mathcal{A}}(x)$ and distributions $\Pr(\mathcal{M} \leq x) \equiv \int_0^x f_{\mathcal{M}}(x') dx'$ and $\Pr(\mathcal{A} \leq x) \equiv \int_0^x f_{\mathcal{A}}(x') dx'$ are independent of the initial condition y_0 .
2. The moments $\langle \mathcal{M}^k \rangle$ and $\langle \mathcal{A}^k \rangle$ are all finite.

Amongst many interesting features of the results it turns out that there is a curious, indeed rather remarkable, equivalence in law (distribution) between the variable $\mathcal{A}$ and the maximal relative height of the fluctuating interface in the one-dimensional Edwards-Wilkinson model [31] with free boundary conditions [32, 33]. We explore this connection in some depth from a path perspective which highlights certain interesting and still unresolved questions. A calculation is also made of the more general expectation $y_0^{2\alpha-\beta} \langle M^\beta / T^\alpha \rangle$ for all $\alpha > 0$ and $\beta < 2\alpha + 1$; this provides additional insight into the relationship between M and T beyond that which can be ascertained from studying $\langle \mathcal{M}^k \rangle$ alone.

To verify the analysis, the theoretical results are compared with results derived from numerical simulations, consisting of many independent sample paths with $y_0 = 1$ and, in each case, the pair of correlated values $\{M, T\}$ and $\{A, T\}$ being recorded. The simulations are easy to implement as discussed in [34]. As an early illustration, in Figure 1 sample outcomes are plotted showing the value of $\mathcal{M} = M / T^{1/2}$ as a function of T . Evidently $M \geq y_0$ by definition and so all the data points lie above the asymptote $y_0 / T^{1/2}$. As a simple check on the accuracy of the simulations one can show using (2) that;

$$y_0 \left\langle \frac{1}{T^{1/2}} \right\rangle = y_0 \int_0^\infty \frac{1}{T^{1/2}} P(T) dT = \sqrt{\frac{2}{\pi}} = 0.7978... \quad (6)$$

and based on the simulated data with $y_0 = 1$ one finds that $\langle 1/T^{1/2} \rangle \approx 0.80$.

To make progress in the study of the variables $\mathcal{M}$ and $\mathcal{A}$ one needs information about the respective joint probability densities $P(M, T)$ and $P(A, T)$. In the former case an explicit expression for the joint probability density $P(M, T)$ is known [34],

$$P(M, T) = \pi \frac{\partial}{\partial M} \frac{1}{M^2} \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi y_0}{M}\right) e^{-\frac{n^2 \pi^2}{2M^2} T}. \quad (7)$$

From (7) one can, in principle, derive the marginal densities by integrating over the redundant variable, e.g. $P(T) = \int_{y_0}^{\infty} P(M, T) dM$ or $P(M) = \int_0^{\infty} P(M, T) dT$, although as will become clear later algebraic manipulations in relation to (7) are not always trivial to execute. An equivalent expression for the probability density $P(A, T)$ is not known, although its double Laplace transform has been well characterised, see e.g. [34, 35] and references therein. This is sufficient to enable the key results to be established, although the mathematical steps involved are somewhat more involved as a consequence.

The paper is arranged as follows. In Section 2, we consider the derivation of the probability density and moments of the variable $\mathcal{M}$, making detailed comparison

with the results of simulations. In Section 3, we repeat the exercise in the context of the variable $\mathcal{A}$, wherein the connection with the maximal relative height of the fluctuating interface in the Edwards-Wilkinson model is made clear. In Section 4, this connection is explored from a different perspective based on the transformation properties and implied equivalences of distinct sets of Brownian paths; this leads to some interesting and still unresolved questions. Finally, in Section 5, conclusions are drawn and avenues for future research identified.

2. The probability density and moments of the normalized maximum

Although not new, it is useful in what follows to sketch the derivation of the joint probability density $P(M, T) \equiv \frac{\partial}{\partial m} \Pr(M \leq m, T) \Big|_{m=M}$ given by (7). Further details may be found in [34]. The key observation is that the Laplace transform of the time specific probability distribution $\Pr(M \leq m, T)$,

$$\tilde{Q}(m, p, y_0) = \int_0^\infty e^{-pT} \Pr(M \leq m, T) dT \quad (8)$$

satisfies a backward Fokker-Plank equation when considered as a function of the initial condition;

$$\left[\frac{1}{2} \frac{\partial^2}{\partial y_0^2} - p \right] \tilde{Q}(m, p, y_0) = 0. \quad (9)$$

The relevant boundary conditions are $\tilde{Q}(m, p, y_0 = 0) = 1$ and $\tilde{Q}(m, p, y_0 = m) = 0$, whereupon the solution is [3; Sect. 1, 2.1.4]

$$\tilde{Q}(m, p, y_0) = \frac{\sinh(\sqrt{2p}(m - y_0))}{\sinh(\sqrt{2p}m)}. \quad (10)$$

The inversion is straightforward using the method of residues and leads to (7). By considering the limit $p \rightarrow 0$ it follows from (8) and (10) that $\Pr(M \leq m) = 1 - y_0/m$, whereupon differentiating with respect to m leads to the first result in (3).

The probability density $f_{\mathcal{M}}(x)$ may be expressed in terms of the joint probability density $P(M, T)$ as follows,

$$f_{\mathcal{M}}(x) \equiv \int_{y_0}^{\infty} \int_0^{\infty} P(M, T) \delta\left(x - \frac{M}{T^{1/2}}\right) dT dM = \int_{y_0}^{\infty} \frac{2M^2}{x^3} P\left(M, \frac{M^2}{x^2}\right) dM. \quad (11)$$

Given the form of (7) this result requires care to manipulate (i.e. to control the sequence of operations involving summation, differentiation and integration). The first step is to carry out the differentiation in (7) explicitly, then substitute $T = M^2/x^2$ and finally invoke the change of variables $\theta = y_0/M$ to derive;

$$f_{\mathcal{M}}(x) = \int_0^1 \left[-\frac{4\pi}{x^3} \sum_{n=1}^{\infty} n \frac{\sin(n\pi\theta)}{\theta} e^{-\frac{n^2\pi^2}{2x^2}} - \frac{2\pi^2}{x^3} \sum_{n=1}^{\infty} n^2 \cos(n\pi\theta) e^{-\frac{n^2\pi^2}{2x^2}} \right. \\ \left. + \frac{2\pi^3}{x^5} \sum_{n=1}^{\infty} n^3 \frac{\sin(n\pi\theta)}{\theta} e^{-\frac{n^2\pi^2}{2x^2}} \right] d\theta. \quad (12)$$

It is now permissible to integrate term by term, whereupon the terms involving $\cos(n\pi\theta)$ vanish and one obtains,

$$f_{\mathcal{M}}(x) = -\frac{4\pi}{x^3} \sum_{n=1}^{\infty} n \text{Si}(n\pi) e^{-\frac{n^2\pi^2}{2x^2}} + \frac{2\pi^3}{x^5} \sum_{n=1}^{\infty} n^3 \text{Si}(n\pi) e^{-\frac{n^2\pi^2}{2x^2}} \quad (13)$$

or in a more compact form,

$$f_{\mathcal{M}}(x) = 2\pi \frac{\partial}{\partial x} \frac{1}{x^2} \sum_{n=1}^{\infty} n \text{Si}(n\pi) e^{-\frac{n^2\pi^2}{2x^2}} \quad (14)$$

where $\text{Si}(z)$ is the sine integral $\text{Si}(z) = \int_0^z \theta^{-1} \sin \theta d\theta$. This is our first main result and, as shown in Figure 2, the theoretical density profile agrees well with the results derived from simulations. When carrying out the simulations we have fixed $y_0 = 1$, but the form of $f_{\mathcal{M}}(x)$ is independent of the choice made. For the purpose of comparison, truncating the summation in the theoretical result to the first ten terms is more than sufficient. The density profile shows that the typical values of $\mathcal{M}$ are concentrated in the range $0.7 < \mathcal{M} < 3$ (which is evident also in Figure 1) and this

highlights the degree of correlation between M and $T^{1/2}$. From (14) one can also show that the density is correctly normalized,

$$\int_0^\infty f_{\mathcal{M}}(x) dx = \lim_{x \rightarrow \infty} \frac{2\pi}{x^2} \sum_{n=1}^\infty n \text{Si}(n\pi) e^{-\frac{n^2 \pi^2}{2x^2}} \sim \lim_{x \rightarrow \infty} \frac{\pi^2}{x^2} \int_1^\infty u e^{-\frac{u^2 \pi^2}{2x^2}} du = 1 \quad (15)$$

where the summation has been replaced by an integral, noting that the ‘dominant’ terms in the summation have $n \approx x/\pi$ and $\lim_{x \rightarrow \infty} \text{Si}(x) = \pi/2$. Regarding the asymptotes one has that

$$f_{\mathcal{M}}(x) \sim \begin{cases} \frac{2\pi^3}{x^5} \text{Si}(\pi) e^{-\pi^2/2x^2}; & x \rightarrow 0 \\ \sqrt{\frac{8}{\pi}} e^{-x^2/2}; & x \rightarrow \infty. \end{cases} \quad (16)$$

The $x \rightarrow 0$ behaviour is obtained directly from the $n = 1$ term of (13). The $x \rightarrow \infty$ behaviour is more difficult to derive and has, in fact, been deduced from a later result for the asymptotic behaviour of the moments $\langle \mathcal{M}^k \rangle$ as $k \rightarrow \infty$. Numerical comparison with (13) supports the case that the $x \rightarrow \infty$ asymptote obtained in this way is correct.

Turning now to the moments, attempts at calculations based on (13) or (14) are plagued with mathematical challenges. Even going back to (7) one must still carry out the summation explicitly at some juncture which in general is hard to do. As an illustration, for the first moment one can write down based on (7),

$$\begin{aligned}
\langle \mathcal{M} \rangle &\equiv \left\langle \frac{M}{T^{1/2}} \right\rangle = \int_{y_0}^{\infty} M \int_0^{\infty} \frac{1}{T^{1/2}} P(M, T) dT dM \\
&= \sqrt{2\pi} \int_1^{\infty} u \frac{\partial}{\partial u} \frac{1}{u} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{u}\right) du
\end{aligned} \tag{17}$$

where the integration over T has been carried out followed by the substitution $u = M / y_0$. It is important to note, however, that the summation in (17) is now no longer formally convergent. It therefore needs to be regularised (assigned a value) through a technique such as Abel or Cesàro summation which gives,

$$\sum_{n=1}^{\infty} \sin(n\varphi) = \lim_{\varepsilon \rightarrow 0^+} \text{Im} \sum_{n=1}^{\infty} e^{n(i\varphi - \varepsilon)} = \frac{1}{2} \cot\left(\frac{\varphi}{2}\right); \quad 0 < \varphi < \pi. \tag{18}$$

After another change of variables (17) then reduces to an expression which is easy to evaluate numerically,

$$\langle \mathcal{M} \rangle = \sqrt{\frac{\pi}{2}} \int_0^{\pi/2} \frac{\theta - \sin \theta \cos \theta}{\theta \sin^2 \theta} d\theta = 1.4962... \tag{19}$$

which compares well with the result derived from simulations $\langle \mathcal{M} \rangle \approx 1.49$. It will be shown shortly using a different approach that (19), despite questions over the use of (18), generates the correct answer. However, the extension of the line of reasoning that led to (17) to higher order moments is more difficult; for example, at second order the integration over T diverges at every term.

A more robust approach to calculating the moments, and one which works to all orders, is to recognise from (8) and (10) that,

$$\int_0^\infty e^{-pT} P(M, T) dT = \left. \frac{\partial \tilde{Q}}{\partial m} \right|_{m=M} = \sqrt{2p} \frac{\sinh(\sqrt{2p} y_0)}{\sinh^2(\sqrt{2p} M)}. \quad (20)$$

By exploiting the identity $\int_0^\infty p^{a-1} e^{-pT} dp \equiv T^{-a} \Gamma(a)$ for $a > 0$ one then has;

$$\begin{aligned} \int_0^\infty \frac{1}{T^{k/2}} P(M, T) dT &= \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty \left[\int_0^\infty p^{(k-2)/2} e^{-pT} dp \right] P(M, T) dT \\ &= \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty p^{(k-2)/2} \left. \frac{\partial \tilde{Q}}{\partial m} \right|_{m=M} dp. \end{aligned} \quad (21)$$

This means that,

$$\begin{aligned} \langle \mathcal{M}^k \rangle &\equiv \int_{y_0}^\infty M^k \int_0^\infty \frac{1}{T^{k/2}} P(M, T) dT dM \\ &= \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty p^{(k-2)/2} \int_{y_0}^\infty M^k \left. \frac{\partial \tilde{Q}}{\partial m} \right|_{m=M} dM dp \end{aligned} \quad (22)$$

which, using (20), simplifies to

$$\begin{aligned}
\langle \mathcal{M}^k \rangle &= \frac{1}{\Gamma(\frac{k}{2})2^{(k-2)/2}} \int_0^\infty \frac{\sinh u}{u} \int_u^\infty \frac{t^k}{\sinh^2 t} dt du \\
&= \frac{1}{\Gamma(\frac{k}{2})2^{(k-2)/2}} \int_0^\infty \frac{t^k}{\sinh^2 t} \int_0^t \frac{\sinh u}{u} du dt.
\end{aligned} \tag{23}$$

Expanding the inner integrand as a power series in u , integrating term by term and then using the result [36],

$$\int_0^\infty \frac{t^a}{\sinh^2 t} dt = \frac{\Gamma(a+1)\zeta(a)}{2^{a-1}}; \quad a > 1 \tag{24}$$

where $\zeta(z)$ is the Riemann zeta function, one finally obtains,

$$\langle \mathcal{M}^k \rangle = \frac{1}{\Gamma(\frac{k}{2})2^{3(k-2)/2}} \sum_{j=1}^\infty \frac{\Gamma(2j+k)\zeta(2j+k-1)}{4^j(2j-1)\Gamma(2j)}. \tag{25}$$

This is our second main result. It holds true for all $k > 0$, not just integer k , and in the limit $k \rightarrow 0$ has the correct behaviour $\langle \mathcal{M}^k \rangle \rightarrow 1$ (this may be shown using the fact that $\zeta(1+k) \sim 1/k$ and $\Gamma(k/2) \sim 2/k$ as $k \rightarrow 0$). Setting $k=1$ in (25) gives,

$$\langle \mathcal{M} \rangle = \sqrt{\frac{8}{\pi}} \sum_{j=1}^\infty \frac{2j\zeta(2j)}{4^j(2j-1)} = 1.4962... \tag{26}$$

in perfect agreement with (19) and, incidentally, demonstrating an interesting identity.

Setting $k=2$ in (25) gives,

$$\langle \mathcal{M}^2 \rangle = \sum_{j=1}^{\infty} \frac{(2j+1)(2j)\zeta(2j+1)}{4^j(2j-1)} = 2.4245\dots \quad (27)$$

Based on simulations one has $\langle \mathcal{M}^2 \rangle \approx 2.42$ so again the agreement is good. Higher order (as well as non-integer) moments similarly agree. Regarding the asymptotic behaviour as $k \rightarrow \infty$, one can show using the second integral representation in (23), noting that $\int_0^t u^{-1} \sinh u \, du \sim e^t / 2t$ as $t \rightarrow \infty$,

$$\langle \mathcal{M}^k \rangle \sim \frac{2}{\Gamma(\frac{k}{2})2^{k/2}} \int_0^{\infty} 4t^k e^{-2t} \left[\frac{e^t}{2t} \right] dt \sim 2^{3/2} k^{k/2} e^{-k/2}. \quad (28)$$

From the behaviour of the moments as $k \rightarrow \infty$ one can deduce the asymptotic behaviour of the probability density $f_{\mathcal{M}}(x)$ as $x \rightarrow \infty$, i.e. that given in (16).

Although not central to our discussion here, by adapting the approach discussed above a calculation can also be made of the more general expectation $y_0^{2\alpha-\beta} \langle M^\beta / T^\alpha \rangle$ for all $\alpha > 0$ and $\beta < 2\alpha + 1$;

$$y_0^{2\alpha-\beta} \left\langle \frac{M^\beta}{T^\alpha} \right\rangle = \frac{1}{\Gamma(\alpha)8^{\alpha-1}} \sum_{j=1}^{\infty} \frac{\Gamma(2j+2\alpha)\zeta(2j+2\alpha-1)}{4^j(2j+2\alpha-\beta-1)\Gamma(2j)}. \quad (29)$$

By setting $\alpha = k/2$ and $\beta = k$ one recovers (25). Further details of the calculation are presented in Appendix A alongside comparison with simulations; Appendix A also highlights an interesting zeta-function identity which follows directly from (29).

One can also use (29) in other ways to explore the relationship between M and T .

For example, the standard linear correlation coefficient between the variables M and $T^{1/2}$ is undefined (each variable has infinite variance). However, using (29) in conjunction with (2) and (3) one can show that the linear correlation coefficient between, say, the reciprocal variables M^{-1} and $T^{-1/2}$ is given by

$$R(M^{-1}, T^{-1/2}) \equiv \frac{\langle M^{-1} T^{-1/2} \rangle - \langle M^{-1} \rangle \langle T^{-1/2} \rangle}{\sqrt{\text{Var}(M^{-1}) \text{Var}(T^{-1/2})}} = 0.8856... \quad (30)$$

which compares well with the result $R(M^{-1}, T^{-1/2}) \approx 0.89$ obtained from simulations.

3. The probability density and moments of the normalized area

As in the previous section, it is helpful to begin the analysis by summarising what is known about the joint probability density $P(A, T)$, whose double Laplace transform may be written as;

$$\tilde{P}(s, p, y_0) \equiv \int_0^\infty \int_0^\infty P(A, T) e^{-sA} e^{-pT} dA dT = \langle e^{-sA - pT} \rangle. \quad (31)$$

As discussed in [34, 35], one may show that $\tilde{P}(s, p, y_0)$ satisfies the following backward Fokker-Planck equation;

$$\frac{1}{2} \frac{\partial^2 \tilde{P}}{\partial y_0^2} - (s y_0 + p) \tilde{P} = 0. \quad (32)$$

The boundary conditions are $\tilde{P}(s, p, y_0 = 0) = 1$ and $\tilde{P}(s, p, y_0 \rightarrow \infty) = 0$, and the solution is given by [35],

$$\tilde{P}(s, p, y_0) = \frac{\text{Ai}\left(\frac{p + s y_0}{2^{-1/3} s^{2/3}}\right)}{\text{Ai}\left(\frac{p}{2^{-1/3} s^{2/3}}\right)} \quad (33)$$

where $\text{Ai}(z)$ is the standard Airy function. There is no elementary double inversion of this expression to yield $P(A, T)$. However, as a check, the marginal densities $P(T)$ and $P(A)$ given by (2) and (3) may be obtained by considering the limits $s \rightarrow 0$ and $p = 0$ respectively and then carrying out the remaining inversion.

The probability density $f_{\mathcal{A}}(x)$ may be expressed in terms of the joint probability density $P(A, T)$ as follows,

$$f_{\mathcal{A}}(x) \equiv \int_0^\infty \int_0^\infty P(A, T) \delta\left(x - \frac{A}{T^{3/2}}\right) dT dA. \quad (34)$$

This result is of limited use as it stands; however, progress can be made by

considering the Laplace transform $\tilde{f}_{\mathcal{A}}(u) \equiv \int_0^\infty e^{-ux} f_{\mathcal{A}}(x) dx$,

$$\tilde{f}_{\mathcal{A}}(u) = \int_0^\infty \int_0^\infty e^{-uA/T^{3/2}} P(A, T) dA dT = \int_0^\infty \left\langle e^{-uA/T^{3/2}} \right\rangle_T P(T) dT \quad (35)$$

where, in the final step, the notation $\langle \dots \rangle_T$ implies expectation taken at a *fixed* value of the first passage time T , i.e.,

$$\langle \dots \rangle_T \equiv \frac{\int_0^\infty \dots P(A, T) dA}{P(T)}. \quad (36)$$

To evaluate the quantity $\langle e^{-uA/T^{3/2}} \rangle_T$ appearing in (35), we start by rewriting (31) so as to first calculate the quantity $\langle e^{-sA} \rangle_T$. This procedure was carried through in [35] and we summarise here the key steps in the derivation for ease of reference. From (31) and (36) one has that,

$$\int_0^\infty e^{-pT} \langle e^{-sA} \rangle_T P(T) dT = \tilde{P}(s, p, y_0). \quad (37)$$

The inversion of (33) with respect to the Laplace variable p is straightforward to accomplish by the method of residues and leads, after using (2), to;

$$\langle e^{-sA} \rangle_T = \sqrt{2\pi} \left(\frac{s^{2/3} T^{3/2}}{2^{1/3} y_0} \right) e^{y_0^2 / 2T} \sum_{n=1}^\infty \frac{\text{Ai}(2^{1/3} y_0 s^{1/3} - \alpha_n)}{\text{Ai}'(-\alpha_n)} e^{-\alpha_n s^{2/3} T / 2^{1/3}} \quad (38)$$

where $-\alpha_n$, with $\alpha_1 = 2.338\dots$, $\alpha_2 = 4.087\dots$, etc., are the zeros of the Airy function [37] and $\text{Ai}'(-\alpha_n)$ is the first derivative evaluated at the n -th zero. A change of

variables to $s = u/T^{3/2}$, recognising that the expectation in (38) is carried out with the variable T fixed, then implies,

$$\begin{aligned} \left\langle e^{-uA/T^{3/2}} \right\rangle_T &\equiv \left\langle e^{-u\mathcal{A}} \right\rangle_T = \sqrt{2\pi} \left(\frac{u^{2/3} T^{1/2}}{2^{1/3} y_0} \right) e^{y_0^2/2T} \\ &\times \sum_{n=1}^{\infty} \frac{\text{Ai}\left(\frac{2^{1/3} y_0 u^{1/3}}{T^{1/2}} - \alpha_n\right)}{\text{Ai}'(-\alpha_n)} e^{-\alpha_n u^{2/3}/2^{1/3}}. \end{aligned} \quad (39)$$

Using this we can evaluate (35) after a change of variables $t = y_0/T^{1/2}$ as follows;

$$\begin{aligned} \tilde{f}_{\mathcal{A}}(u) &= 2^{2/3} u^{2/3} \sum_{n=1}^{\infty} \frac{e^{-\alpha_n u^{2/3}/2^{1/3}}}{\text{Ai}'(-\alpha_n)} \int_0^{\infty} \frac{\text{Ai}(2^{1/3} u^{1/3} t - \alpha_n)}{t} dt \\ &= 2^{-1/3} u^{2/3} \sum_{n=1}^{\infty} C(\alpha_n) e^{-\alpha_n u^{2/3}/2^{1/3}} \end{aligned} \quad (40)$$

where $C(\alpha_n)$ is given by,

$$C(\alpha_n) = \frac{2}{\text{Ai}'(-\alpha_n)} \int_{-\alpha_n}^{\infty} \frac{\text{Ai}(t)}{t + \alpha_n} dt = \left[\frac{\int_{-\alpha_n}^{\infty} \text{Ai}(t) dt}{\text{Ai}'(-\alpha_n)} \right]^2. \quad (41)$$

The non-trivial last step in (41) is established in Appendix B.

To invert (40) we need the inverse Laplace transform $\mathcal{L}_u^{-1}(u^{2/3} e^{-\gamma u^{2/3}})$. To this end we recognise the fact that the function $g_{2/3}(x)$ defined by,

$$\int_0^\infty e^{-ux} g_{2/3}(x) dx = e^{-u^{2/3}} \quad (42)$$

is the one-sided Lévy distribution of index $2/3$ [19, 38];

$$g_{2/3}(x) = \frac{2^{4/3}}{3^{3/2} \sqrt{\pi}} \frac{1}{x^{7/3}} e^{-4/27x^2} U\left(\frac{1}{6}, \frac{4}{3}, \frac{4}{27x^2}\right) \quad (43)$$

where $U(a, b, z)$ is the confluent hypergeometric function [37]. It follows after elementary operations that we have our third main result,

$$f_{\mathcal{A}}(x) = \frac{2^{-1/3}}{\sqrt{3\pi} x^{7/3}} \sum_{n=1}^{\infty} \alpha_n C(\alpha_n) \left[U\left(\frac{1}{6}, \frac{4}{3}, \frac{b_n}{x^2}\right) + 2U\left(-\frac{5}{6}, \frac{4}{3}, \frac{b_n}{x^2}\right) \right] e^{-b_n/x^2} \quad (44)$$

where $b_n = 2\alpha_n^3/27$. As mentioned in the Introduction, this is *precisely* the result obtained in [33] for the probability density of maximal relative height of the fluctuating interface in the one-dimensional Edwards-Wilkinson model with free boundary conditions (we have used the notation of [33] to make this clear). We return to this equivalence in the next section. As an alternative representation of $f_{\mathcal{A}}(x)$, and one which highlights the underpinning role played by the Airy function, the function $g_{2/3}(x)$ can also be written as [38],

$$g_{2/3}(x) = 6\zeta^{7/4} e^{-(2/3)\zeta^{3/2}} \left[\text{Ai}(\zeta) - \frac{\text{Ai}'(\zeta)}{\sqrt{\zeta}} \right]; \quad \zeta \equiv (3x)^{-4/3}. \quad (45)$$

It then follows that an equivalent way to write (44) is,

$$f_{\mathcal{A}}(x) = \frac{2^{2/3}}{3^{2/3} x^{5/3}} \sum_{n=1}^{\infty} C(\alpha_n) e^{-\alpha_n^3 / 27 x^2} F\left(\frac{\alpha_n^2}{2^{2/3} 3^{4/3} x^{4/3}}\right) \quad (46)$$

where the function $F(z)$ is given by,

$$F(z) \equiv \text{Ai}(z)[4z^2 - 2\sqrt{z}] - \text{Ai}'(z)[4z^{3/2} - 1]. \quad (47)$$

In Figure 3 we show the probability density $f_{\mathcal{A}}(x)$ as determined from (46) and compare it against that deduced from numerical simulations. The agreement is very good. As when studying the normalized maximum, for the purposes of the simulations we have fixed $y_0 = 1$ (noting again that the form of $f_{\mathcal{A}}(x)$ is independent of the choice made) and we have truncated the summation in (46) to the first ten terms. The asymptotes of $f_{\mathcal{A}}(x)$ are given by,

$$f_{\mathcal{A}}(x) \sim \begin{cases} \frac{2\sqrt{2}}{27\sqrt{\pi}} C(\alpha_1) \alpha_1^{7/2} x^{-4} e^{-2\alpha_1^3 / 27 x^2}; & x \rightarrow 0 \\ \frac{2\sqrt{6}}{\sqrt{\pi}} e^{-3x^2 / 2}; & x \rightarrow \infty. \end{cases} \quad (48)$$

The $x \rightarrow 0$ behaviour is obtained directly from the $n = 1$ term of (46); the $x \rightarrow \infty$ behaviour is given by the analysis presented in [33] and refined in [23, 39].

We will shortly outline a general method for calculating all the integer moments. Before that, and rather instructively, the first moment can be calculated as follows,

$$\langle \mathcal{A} \rangle = \int_0^{\infty} \langle \mathcal{A} \rangle_T P(T) dT \quad (49)$$

where $\langle \mathcal{A} \rangle_T$ represents the average of $\mathcal{A}$ taken at a fixed value of T . As shown in [35], an expression for $\langle \mathcal{A} \rangle_T$ is already known whereupon we can simply write down (given that the value of T is fixed),

$$\langle \mathcal{A} \rangle_T \equiv \frac{\langle A \rangle_T}{T^{3/2}} = \frac{y_0}{2T^{1/2}} + \sqrt{\frac{\pi}{8}} e^{y_0^2/2T} \operatorname{erfc}\left(\frac{y_0}{\sqrt{2T}}\right). \quad (50)$$

In Figure 4 we show the variation of $\mathcal{A}$ with T derived from simulations (again carried out with $y_0 = 1$) and compare the numerical averages so obtained with the theoretical result (50), with good agreement. From (49) and (50) one can calculate $\langle \mathcal{A} \rangle$ and after a change of variables $t = y_0/T^{1/2}$ one has,

$$\langle \mathcal{A} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[t e^{-t^2/2} + \sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{t}{\sqrt{2}}\right) \right] dt = \sqrt{\frac{2}{\pi}}. \quad (51)$$

This theoretical result $\langle \mathcal{A} \rangle = \sqrt{2/\pi} = 0.7978\dots$ accords with the result from the simulations $\langle \mathcal{A} \rangle \approx 0.80$. We note from (50) that for large first passage times (i.e. $T \gg y_0^2$) so $\lim_{T \rightarrow \infty} \langle \mathcal{A} \rangle_T = \sqrt{\pi/8} = 0.6266\dots$; the fact that the unconditional

expectation $\langle \mathcal{A} \rangle$ exceeds this is due to the contribution of paths for which the first passage time is “small”, as is evident from Figure 4.

By comparing (51) with (6) we see that $\langle \mathcal{A} \rangle = y_0 \langle 1/T^{1/2} \rangle$. If one integrates both sides of (1) with respect to time over the first passage time, then divides through by $T^{3/2}$ and averages, it must therefore be the case that $\langle T^{-3/2} \int_0^T W(t) dt \rangle = 0$, where $T = \inf\{t : W(t) = -y_0\}$ is the first passage or hitting time for the Wiener process to reach any given negative boundary $-y_0 < 0$. The inversion property of the Wiener process implies that this expectation must be also be zero for any given positive boundary as well. A natural assumption would be that the zero value can be inferred from the transformation properties of the Wiener process alone, but this is not the case and the result is non-trivial. In fact, it is a special case of the general theorem proved in [30] and discussed in the Introduction where the expectation of the normalized functional (5) is only zero for the case $m = 1$. The analysis here establishes this particular case in a way which is independent of that presented in [30].

To calculate the higher order moments we utilize an approach based on the fact that the correlator $\langle A^k e^{-pT} \rangle$ may be determined recursively from the definition (31) by applying the operator $(-d/ds)^k$ term-by-term to (32) and then setting $s = 0$. One finds that,

$$\left[\frac{1}{2} \frac{d^2}{dy_0^2} - p \right] \langle A^k e^{-pT} \rangle = -k y_0 \langle A^{k-1} e^{-pT} \rangle. \quad (52)$$

When $k = 0$, the right hand side of (52) vanishes and the solution is simply the Laplace transform of the first passage time probability density (2);

$$\langle e^{-pT} \rangle \equiv \int_0^\infty e^{-pT} P(T) dT = e^{-(2p)^{1/2} y_0}. \quad (53)$$

When $k > 0$, the boundary conditions for (52) are: (i) as $y_0 \rightarrow 0$ so $T \rightarrow 0$ and $A \rightarrow 0$ almost surely such that $\langle A^k e^{-pT} \rangle \rightarrow 0$; (ii) as $y_0 \rightarrow \infty$ so $T \rightarrow \infty$ and $A \rightarrow \infty$ such that $\langle A^k e^{-pT} \rangle \rightarrow 0$. One can now derive the general result for $k > 0$,

$$\langle A^k e^{-pT} \rangle = e^{-(2p)^{1/2} y_0} \frac{k!}{4^k} \sum_{j=1}^{2k} a(j, k) \frac{2^j y_0^j}{j! (2p)^{(3k-j)/2}} \quad (54)$$

where $a(1,1) = a(2,1) = 1$ and, by exploiting the properties of telescoping series, the integer coefficients $a(j, k)$ may be shown to satisfy the recursive relation,

$$a(j, k) = \sum_{\ell=j-2}^{2k-2} (\ell+1) a(\ell, k-1); \quad 3 \leq j \leq 2k \quad (55)$$

$$a(1, k) = a(2, k) = a(3, k).$$

The first few of these coefficients are listed in Table 1 for ease of reference. It follows from (55) that $a(2k, k) = (2k-1)!!$, but in general there is no simple product-form solution for $a(j, k)$. This can be seen from Table 1, where it is clear that the coefficients are not necessarily trivial in terms of their prime factors (e.g.

$a(6,6) = 2 \times 5 \times 58417$). The sequence $a(1,k) = 1, 5, 60, 1105 \dots$ occurs in many disparate (and seemingly unconnected) problems [40, 41]; given (33), we see that the deep-rooted mathematical structure relates to the properties of the Airy function.

$j \backslash k$	1	2	3	4	5	6
1	1	5	60	1105	27120	828250
2	1	5	60	1105	27120	828250
3		5	60	1105	27120	828250
4		3	50	985	24910	774010
5			35	805	21595	692650
6			15	565	17175	584170
7				315	12250	459620
8				105	7420	330050
9					3465	209825
10					945	111825
11						45045
12						10395

Table 1: The coefficients $a(j,k)$ for $1 \leq j \leq 12$ and $1 \leq k \leq 6$.

The integer moments $\langle \mathcal{A}^k \rangle$ can be now derived from the correlator $\langle A^k e^{-pT} \rangle$ by

exploiting the identity $\int_0^\infty p^{a-1} e^{-pT} dp \equiv T^{-a} \Gamma(a)$;

$$\langle \mathcal{A}^k \rangle \equiv \frac{1}{\Gamma(\frac{3k}{2})} \int_0^\infty p^{\frac{3k-2}{2}} \langle A^k e^{-pT} \rangle dp. \quad (56)$$

Using (54) it then follows that one can obtain a closed form expression which is our final main result,

$$\langle \mathcal{A}^k \rangle = \frac{k!}{2^{\frac{7k}{2}} \Gamma(\frac{3k}{2})} \sum_{j=1}^{2k} \frac{2^{j+1} a(j, k)}{j}. \quad (57)$$

For completeness, the first four moments determined from (57) and using the coefficients from Table 1 are given by,

$$\langle \mathcal{A} \rangle = \sqrt{\frac{2}{\pi}} \quad \langle \mathcal{A}^2 \rangle = \frac{17}{24} \quad \langle \mathcal{A}^3 \rangle = \sqrt{\frac{2}{\pi}} \frac{123}{140} \quad \langle \mathcal{A}^4 \rangle = \frac{2963}{3840}. \quad (58)$$

These are identical to the results presented in [33] where they were obtained in a totally different context (that of characterizing interface fluctuations in the Edwards-Wilkinson model) and using very different techniques; see also [23, 39]. The first moment agrees with (51) and, as reported earlier, accords with the result based on simulations. The second moment $\langle \mathcal{A}^2 \rangle = 17 / 24 = 0.7083 \dots$ also agrees very well with the results of the simulations $\langle \mathcal{A}^2 \rangle \approx 0.71$. Regarding the asymptotic behaviour as $k \rightarrow \infty$, one can show on the basis of the $x \rightarrow \infty$ behaviour of (48) that $\langle \mathcal{A}^k \rangle \sim 2\sqrt{2}(k/3)^{k/2} e^{-k/2}$, as given in [23, 39].

4. Fluctuating interfaces, path transformations and open questions

We turn now to discussing the unexpected equivalence in law $\mathcal{A} \stackrel{d}{=} h_m$, which was highlighted in the previous section, between the distribution of the normalized area variable $\mathcal{A}$ in first passage Brownian motion and that of the maximal relative height

h_m of the fluctuating interface in the Edwards-Wilkinson model [33]. The interface height profile $H(x, t)$ in this model satisfies the following stochastic differential equation in appropriately scaled units;

$$\frac{\partial H(x, t)}{\partial t} = \frac{\partial^2 H(x, t)}{\partial x^2} + \eta(x, t) \quad (59)$$

$$\langle \eta(x, t) \eta(x', t') \rangle = 2\delta(x - x')\delta(t - t')$$

defined over a one-dimensional substrate of unit length with free boundary conditions.

Here, $\eta(x, t)$ is a Gaussian white noise forcing term with zero mean. Various

definitions of relative interface height are possible, see e.g. [42]; in this instance the

choice made is $h(x, t) = H(x, t) - \overline{H(x, t)}$, where $\overline{H(x, t)} = \int_0^1 H(x, t) dx$ is the spatially

averaged height at time t for a given realization. The maximal relative height of the

interface at time t is then given by $h_m = \max_{0 \leq x \leq 1} h(x, t)$. For long enough times the

relative interface profile $h(x, t)$ becomes stationary in a statistical sense and, as shown

in [33], its spatial dependence may be represented in terms of a so-called Brownian

double meander on the unit interval (for later comparative purposes we use the

variable τ instead of x),

$$B_{dm}(\tau) = W(\tau) - \min_{0 \leq u \leq 1} W(u); \quad \tau \in [0, 1] \quad (60)$$

who's area integral $\mathcal{B}_{dm} = \int_0^1 B_{dm}(\tau) d\tau$ is equivalent to h_m ; see Figure 5a. The area constraint $\int_0^1 h(x, t) dx = 0$, which follows directly from the definition of $h(x, t)$, plays a key role in establishing this equivalence. Using path integrals and other techniques, the analysis in [33] and subsequently [23] shows that the probability density of $\mathcal{B}_{dm}$ and hence h_m is given by (44). Thus the results in the present paper actually prove in a roundabout way that $\mathcal{A} \stackrel{d}{=} \mathcal{B}_{dm}$, with (57) providing a new expression for the moments of $\mathcal{B}_{dm}$ and hence h_m . What one would ideally like is to understand this particular equivalence in law in a more direct or fundamental way.

To that end, a natural next step is to define another process $\hat{y}(\tau)$ in relation to (1) by reversing each path $y(t)$ in time and scaling with respect to the path specific first passage time $\hat{y}(\tau) = T^{-1/2} y((1 - \tau)T)$, a so-called random Brownian scaling, see e.g. [43, 44]. The properties of the Wiener process are such that the local characteristics of the paths are preserved by this transformation. The transformed paths are defined on the unit interval $\tau \in [0, 1]$, with every path starting at $\hat{y}(\tau = 0) = 0$ and constrained to be positive thereafter; see Figure 5b. By construction, the end-point $\hat{y}(1)$ takes the random value $T^{-1/2} y_0$ and therefore, based on (2), has density $f(x) = \sqrt{2/\pi} e^{-x^2/2}$; evidently the process $\hat{y}(\tau)$ is independent of y_0 . It is important to note that, despite appearances, $\hat{y}(\tau)$ is quite distinct from the standard Brownian meander process on the interval $[0, 1]$, for which the end-point has density $f(x) = x e^{-x^2/2}$ [7, 8]. The area functional $\mathcal{A} = \int_0^1 \hat{y}(\tau) d\tau$ is readily seen to be the same as the normalized functional

$\mathcal{A} = A/T^{3/2}$ attributed to the original path $y(t)$. Thus the task of directly showing

that $\mathcal{A} \stackrel{d}{=} \mathcal{B}_{dm}$ is the same as establishing an equivalence in law between $\hat{y}(\tau)$ and

$B_{dm}(\tau)$, in the specific sense that $\int_0^1 \hat{y}(\tau) d\tau \stackrel{d}{=} \int_0^1 B_{dm}(\tau) d\tau$.

We pose this as an open challenge, since we have been unable to construct an argument which we feel is mathematically *rigorous*. One possible strategy is to demonstrate a pairing between paths of the two processes $\hat{y}(\tau)$ and $B_{dm}(\tau)$ which, when suitably transformed by cutting, translating, inverting or time reversing (see e.g. [45]), demonstrably have the same area. A helpful but not sufficient condition for such a pairing to exist is the fact that for the double meander the absolute value of the difference between the start and end points $|B_{dm}(1) - B_{dm}(0)|$ is characteristic of $|W(1)|$ and therefore has density $f(x) = \sqrt{2/\pi} e^{-x^2/2}$; this is the same as the density of the end-point $\hat{y}(1)$. If such a pairing could be demonstrated then it might also provide insights into the statistics of different interface models or even the extension to the case where there is a drift term present [46, 47]. More fundamentally, it seems clear that the process $\hat{y}(\tau)$ is worthy of further study in its own right, and one should anticipate the sort of mathematical subtleties as are found in other random Brownian scaling problems [43, 44]. We note that in the present work, not only have we characterized the area $\mathcal{A} = \int_0^1 \hat{y}(\tau) d\tau$, we have also characterized the maximum $\mathcal{M} = \max_{0 \leq \tau \leq 1} \hat{y}(\tau)$, since this too is the same as the corresponding normalized functional $\mathcal{M} = M/T^{1/2}$ attributed to the original path $y(t)$.

5. Conclusions

In this paper we have considered, in relation to the first passage process $y(t)$ defined by (1), the normalized variables $\mathcal{M} \equiv M/T^{1/2}$ and $\mathcal{A} = A/T^{3/2}$, and shown how to calculate the probability densities $f_{\mathcal{M}}(x)$, given by (14), and $f_{\mathcal{A}}(x)$, given by (44). In addition, we have provided results for the integer moments $\langle \mathcal{M}^k \rangle$, given by (25), and $\langle \mathcal{A}^k \rangle$, given by (57), as well as an expression for the more general expectation $y_0^{2\alpha-\beta} \langle M^\beta / T^\alpha \rangle$, given by (29). The variables $\mathcal{M}$ and $\mathcal{A}$ correspond directly to the maximum and swept-out area associated with the randomly scaled process $\hat{y}(\tau) = T^{-1/2} y((1-\tau)T)$ which is defined on the interval $\tau \in [0,1]$. All the theoretical results are in excellent agreement with results derived from simulations.

This analysis provides fresh insights into the first passage process and sheds further light on the relationship between the underlying strongly correlated variables. By construction, the densities and moments of the variables $\mathcal{M}$ and $\mathcal{A}$ are independent of the initial condition y_0 and thus they say something fundamental about the geometric structure of Brownian motion itself. This is also evident in the fact that the paths defined by $\hat{y}(\tau)$ are independent of y_0 . This means, for example, that given knowledge of, say, T , but no knowledge of the initial condition y_0 , one has a means through the relevant probability densities $f_{\mathcal{M}}(x)$ and $f_{\mathcal{A}}(x)$ of estimating both M and A . The moments $\langle \mathcal{M}^k \rangle$ and $\langle \mathcal{A}^k \rangle$ capture the contribution of all paths on an equal basis rather than simply focussing on paths of long duration.

Thus the first moment of the normalized area $\langle \mathcal{A} \rangle = 0.7978\dots$ is different from the limiting value $\lim_{T \rightarrow \infty} \langle \mathcal{A} \rangle_T = 0.6266\dots$

The stand-out result is the equivalence in law between the variable $\mathcal{A}$ and the maximal relative height of the fluctuating interface h_m in the one-dimensional Edwards-Wilkinson model with free boundary conditions. An interesting by-product of this finding is a new expression for the moments of h_m , given by (57). More fundamentally, the equivalence $\mathcal{A} \stackrel{d}{=} h_m$ raises interesting and not yet fully resolved questions as to the relationship between the randomly scaled process $\hat{y}(\tau)$ and the Brownian double meander process $B_{dm}(\tau)$. A deeper examination of this relationship may provide new perspectives on related interface models.

Appendix A: Further results relating to the first passage maximum

Generalizing the arguments in Section 2 that led to (23) it follows for all $\alpha > 0$ and $\beta < 2\alpha + 1$ that,

$$\begin{aligned} y_0^{2\alpha-\beta} \left\langle \frac{M^\beta}{T^\alpha} \right\rangle &= \frac{1}{\Gamma(\alpha)} \int_0^\infty p^{\alpha-1} \int_{y_0}^\infty M^\beta \frac{\partial \tilde{Q}}{\partial m} \Big|_{m=M} dM dp \\ &= \frac{1}{\Gamma(\alpha) 2^{\alpha-1}} \int_0^\infty u^{2\alpha-\beta-1} \sinh u \int_u^\infty \frac{t^\beta}{\sinh^2 t} dt du. \end{aligned} \tag{A1}$$

Rearranging the order of integration gives,

$$y_0^{2\alpha-\beta} \left\langle \frac{M^\beta}{T^\alpha} \right\rangle = \frac{1}{\Gamma(\alpha)2^{\alpha-1}} \int_0^\infty \frac{t^\beta}{\sinh^2 t} \int_0^t u^{2\alpha-\beta-1} \sinh u \, du \, dt \quad (\text{A2})$$

whereupon expanding the inner integrand as a power series in u , integrating term by term and then using the result (24) one finally obtains,

$$y_0^{2\alpha-\beta} \left\langle \frac{M^\beta}{T^\alpha} \right\rangle = \frac{1}{\Gamma(\alpha)8^{\alpha-1}} \sum_{j=1}^\infty \frac{\Gamma(2j+2\alpha)\zeta(2j+2\alpha-1)}{4^j(2j+2\alpha-\beta-1)\Gamma(2j)}. \quad (\text{A3})$$

Other quantities are now accessible besides those considered in Section 2, which correspond to making the specific choice $\alpha = k/2$ and $\beta = k$.

To illustrate this, setting $\alpha = 1/2$ and $\beta = 1/2$ in (A3) gives,

$$y_0^{1/2} \left\langle \frac{M^{1/2}}{T^{1/2}} \right\rangle = \sqrt{\frac{8}{\pi}} \sum_{j=1}^\infty \frac{2j\zeta(2j)}{4^j(2j-\frac{1}{2})} = 1.0348\dots \quad (\text{A4})$$

Based on simulations with $y_0 = 1$ one finds that $\langle M^{1/2}/T^{1/2} \rangle \approx 1.03$. If instead the choice $\alpha = 1$ and $\beta = 1$ is made one obtains,

$$y_0 \left\langle \frac{M}{T} \right\rangle = \sum_{j=1}^\infty \frac{(2j+1)\zeta(2j+1)}{4^j} = 1.3862\dots \quad (\text{A5})$$

and based on simulations with $y_0 = 1$ one has $\langle M/T \rangle \approx 1.38$. As a final example, setting $\alpha = 2$ and $\beta = 2$ leads to,

$$y_0^2 \left\langle \frac{M^2}{T^2} \right\rangle = \frac{1}{8} \sum_{j=1}^{\infty} \frac{\Gamma(2j+4) \zeta(2j+3)}{4^j (2j+1) \Gamma(2j)} = 4.2071... \quad (\text{A6})$$

and simulations with $y_0 = 1$ reveal that $\langle M^2/T^2 \rangle \approx 4.20$.

As an adjunct, setting $\alpha = 1$ and $\beta = 1$ in (A1) gives,

$$\begin{aligned} y_0 \left\langle \frac{M}{T} \right\rangle &= \int_0^{\infty} \sinh u \left[\int_u^{\infty} \frac{t}{\sinh^2 t} dt \right] du \\ &= \int_0^{\infty} \sinh u [u \coth u - u - \log(1 - e^{-2u})] du. \end{aligned} \quad (\text{A7})$$

This may be simplified thus,

$$\begin{aligned} y_0 \left\langle \frac{M}{T} \right\rangle &= \int_0^{\infty} [ue^{-u} - \sinh u \log(1 - e^{-2u})] du \\ &= 1 - \int_0^{\infty} \sinh u \log(1 - e^{-2u}) du. \end{aligned} \quad (\text{A8})$$

Expanding the logarithm and evaluating the integral term by term then gives,

$$y_0 \left\langle \frac{M}{T} \right\rangle = \frac{3}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j(j+1)(2j+1)} = 2 \log 2 = 1.3862... \quad (\text{A9})$$

where the summation may be evaluated through the use of partial fractions and the fact that $\sum_{j=1}^{\infty} (-1)^{j+1} / j = \log 2$. This agrees with (A5) and through comparison provides an interesting identity;

$$\sum_{j=1}^{\infty} \frac{(2j+1)\zeta(2j+1)}{4^j} = 2 \log 2. \quad (\text{A10})$$

In a similar vein, setting $\beta = 0$ in (A1) one has,

$$\begin{aligned} y_0^{2\alpha} \left\langle \frac{1}{T^\alpha} \right\rangle &= \frac{1}{\Gamma(\alpha)2^{\alpha-1}} \int_0^\infty u^{2\alpha-1} \sinh u \left[\int_u^\infty \frac{dt}{\sinh^2 t} \right] du \\ &= \frac{1}{\Gamma(\alpha)2^{\alpha-1}} \int_0^\infty u^{2\alpha-1} \sinh u [\coth u - 1] du = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)2^{\alpha-1}}. \end{aligned} \quad (\text{A11})$$

This result can also be derived directly from (2). Comparison of (A11) and (A3) with $\beta = 0$ leads to a fascinating zeta-function identity valid for all $\alpha > 0$,

$$\sum_{j=1}^{\infty} \frac{\Gamma(2j+2\alpha-1)\zeta(2j+2\alpha-1)}{4^j \Gamma(2j)} = 4^{\alpha-1} \Gamma(2\alpha) \quad (\text{A12})$$

from which many intriguing results can be derived. For example, setting $\alpha = 1/2$ and $\alpha = 1$ gives respectively,

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{4^j} = \frac{1}{2}, \quad \sum_{j=1}^{\infty} \frac{2j\zeta(2j+1)}{4^j} = 1. \quad (\text{A13})$$

Through comparison with (A10) and after minor manipulations of the second identity in (A13) one therefore has

$$\sum_{j=1}^{\infty} \frac{\zeta(2j+1)}{4^j} = 2\log 2 - 1 \quad (\text{A14})$$

and combined with the first result in (A13) this means that

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{2^j} = \log 2. \quad (\text{A15})$$

This result is known via alternative means, see e.g. [48], but other results obtainable from (A12) for larger or even non-rational values of α are likely to be new.

Appendix B: Derivation of an Airy function identity

We wish to establish the identity,

$$\text{Ai}'(-\alpha_n) \int_{-\alpha_n}^{\infty} \frac{\text{Ai}(t)}{t + \alpha_n} dt = \frac{1}{2} \left[\int_{-\alpha_n}^{\infty} \text{Ai}(t) dt \right]^2 \quad (\text{B1})$$

where $-\alpha_n$ is a given zero of the Airy function. The starting point is to make use of the following result,

$$\int_0^{\infty} \text{Ai}(\omega + t) \text{Ai}(\omega + s) d\omega = \frac{\text{Ai}(t) \text{Ai}'(s) - \text{Ai}'(t) \text{Ai}(s)}{t - s} \quad (\text{B2})$$

which may be proved by multiplying through by $t - s = (\omega + t) - (\omega + s)$ and integrating by parts using the defining equation $\text{Ai}''(z) = z\text{Ai}(z)$. The right hand side of (B2) is known as the Airy kernel in the context of random matrix theory [49].

Letting $s = -\alpha_n$ in (B2) we then have,

$$\int_0^\infty \text{Ai}(\omega + t)\text{Ai}(\omega - \alpha_n) d\omega = \frac{\text{Ai}(t)\text{Ai}'(-\alpha_n)}{t + \alpha_n}. \quad (\text{B3})$$

If we now integrate both sides of (B3) with respect to t from $-\alpha_n$ to ∞ we obtain,

$$\begin{aligned} \text{Ai}'(-\alpha_n) \int_{-\alpha_n}^\infty \frac{\text{Ai}(t)}{t + \alpha_n} dt &= \int_{-\alpha_n}^\infty \int_0^\infty \text{Ai}(\omega + t)\text{Ai}(\omega - \alpha_n) d\omega dt \\ &= \int_0^\infty \text{Ai}(\omega - \alpha_n) \left[\int_{-\alpha_n}^\infty \text{Ai}(\omega + t) dt \right] d\omega. \end{aligned} \quad (\text{B4})$$

Finally, if we now define $F(s) \equiv \int_s^\infty \text{Ai}(t) dt$ we obtain,

$$\begin{aligned} \text{Ai}'(-\alpha_n) \int_{-\alpha_n}^\infty \frac{\text{Ai}(t)}{t + \alpha_n} dt &= \int_0^\infty \text{Ai}(\omega - \alpha_n) F(\omega - \alpha_n) d\omega \\ &= - \int_{-\alpha_n}^\infty \left(\frac{dF}{ds} \right) F(s) ds = \frac{1}{2} [F(-\alpha_n)]^2 \end{aligned} \quad (\text{B5})$$

and the result is proved.

References

- [1] Redner S 2001 *A Guide to First-Passage Processes* (Cambridge: Cambridge University Press)
- [2] Bray A J, Majumdar S N and Schehr G 2013 *Adv. Phys.* **62** 225
- [3] Borodin A N and Salminen P 2002 *Handbook of Brownian Motion – Facts and Formulae*, 2nd ed. (Basel: Birkhauser)
- [4] Kearney M J and Majumdar S N 2005 *J. Phys. A: Math. Gen.* **38** 4097
- [5] Majumdar S N 2005 *Current Science* **89** 2076; arXiv:cond-mat/0510064
- [6] Erdős P and Kac M 1946 *Bull. Amer. Math. Soc.* **52** 292
- [7] Durrett R T, Iglehart D L and Miller D R 1977 *Annals of Probab.* **5** 117
- [8] Durrett R T and Iglehart D L 1977 *Ann. Probab.* **5** 130
- [9] Majumdar S N, Randon-Furling J, Kearney M J and Yor M 2008 *J. Phys. A: Math. Theor.* **41** 365005
- [10] Perret A, Comtet A, Majumdar S N and Schehr G 2015 *J. Stat. Phys.* **161** 1112
- [11] Raychaudhuri S, Cranston M, Przybyla C and Shapir Y 2001 *Phys. Rev. Lett.* **87** 136101
- [12] Györgyi G, Holdsworth P C W, Portelli B and Racz Z 2003 *Phys. Rev. E* **68** 056116
- [13] Sabhapandit S and Majumdar S N 2007 *Phys. Rev. Lett.* **98** 140201
- [14] Burkhardt T W, Györgyi G, Moloney N R and Racz Z 2007 *Phys. Rev. E* **76** 041119
- [15] Majumdar S N 2010 *Physica A* **389** 4299
- [16] Perret A, Comtet A, Majumdar S N and Schehr G 2013 *Phys. Rev. Lett.* **111** 240601

- [17] Odlyzko A M 1995 *Random Struct. Algor.* **6** 275
- [18] Chassaing P 1999 *Stoch. Proc. Appl.* **81** 129
- [19] Takacs L 1991 *Adv. Appl. Probab.* **23** 557
- [20] Takacs L 1993 *Annals Appl. Probab.* **3** 186
- [21] Takacs L 1995 *J. Appl. Probab.* **32** 375
- [22] Perman M and Wellner J A 1996 *Annals Appl. Probab.* **6** 1091
- [23] Janson S 2007 *Probab. Surv.* **4** 80
- [24] Watkins N W, Credington D, Sanchez R, Rosenberg S J and Chapman S C
2009 *Phys. Rev E* **79** 041124
- [25] Reynolds A M 2010 *J. Roy. Soc. Interface* **7** 1753
- [26] Bandyopadhyay M, Gupta S and Segal D 2011 *Phys. Rev. E* **83** 031905
- [27] Lavrentovich M O, Korolev K S and Nelson D R 2013 *Phys. Rev. E* **87**
102103
- [28] Kearney M J 2004 *J. Phys. A: Math. Gen.* **37** 8421
- [29] Kearney M J 2006 *Phys. Rev. E* **74** 061115
- [30] Elie R, Rosenbaum M and Yor M 2014 *Electron. J. Probab.* **19** 37
- [31] Edwards S F and Wilkinson D R 1982 *Proc. Roy. Soc. A: Math. Phys. Eng.
Sci.* **381**. 17
- [32] Majumdar S N and Comtet A 2004 *Phys. Rev. Lett.* **92** 225501
- [33] Majumdar S N and Comtet A 2005 *J. Stat. Phys.* **119** 777
- [34] Kearney M J and Majumdar S N 2014 *J. Phys. A: Math. Theor.* **47** 465001
- [35] Kearney M J, Pye A J and Martin R J 2014 *J. Phys. A: Math. Theor.* **47**
225002
- [36] Gradshteyn I S and Ryzhik I M 2000 *Tables of Integrals, Series, and Products*,
6th ed. (San Diego: Academic Press)

- [37] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions
(New York: Dover)
- [38] Kessler D A, Medalion S and Barkai E 2014 *J. Stat. Phys.* **156** 686
- [39] Janson S and Louchard G 2007 *Electron. J. Probab.* **12** 58
- [40] Kearney M J, Majumdar S N and Martin R J 2007 *J. Phys. A: Math. Theor.* **40**
F863
- [41] OEIS Foundation: *Online Encyclopedia of Integer Sequences*, www.oeis.org,
A062980
- [42] Rambeau J and Schehr G 2009 *J. Stat. Mech.: Theor. Expt.* P09004
- [43] Yor M 1995 *Prog. Prob.* **36** 243
- [44] Fujita T and Yor M 2007 *Prob. Math. Stat.* **27** 105
- [45] Bertoin J and Pitman J 1994 *Bull. Sci. Math.* **118** 147
- [46] Pruessner G 2004 *Phys. Rev. Lett.* **92** 246101.
- [47] Cheang S and Pruessner G 2011 *J. Phys. A: Math. Theor.* **44** 065003.
- [48] Borwein J M, Bradley D M and Crandall R E 2000 *J. Comp. Appl. Math.* **121**
247
- [49] Levin E and Lubinsky D S 2009 *Int. Eqs. Op. Theor.* **63** 427

Figure 1: Numerical data (triangles) showing the variation of $\mathcal{M}$ with T for $y_0 = 1$. The dashed line shows the asymptote $1/T^{1/2}$.

Figure 2: The probability density of $\mathcal{M}$ based on theory (solid line) and derived from numerical simulations (diamonds).

Figure 3: The probability density of $\mathcal{A}$ based on theory (solid line) and derived from numerical simulations (diamonds).

Figure 4: Numerical data (triangles) showing the variation of $\mathcal{A}$ with T for $y_0 = 1$. Also shown are the results for $\langle \mathcal{A} \rangle_T$ derived from the simulations (diamonds) compared to the theoretical expression (line).

Figure 5: (a) The double meander process $B_{dm}(\tau)$. (b) The process $\hat{y}(\tau)$ derived from the first passage process appropriately scaled and reversed in time.