The effect of bottom boundary condition type on the behavior of adhesive contact of spherical probe on an elastic film

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Abstract

This study presents an investigation on the behavior of adhesive contact between a rigid sphere and an elastic film which is either perfectly bonded (Case I) or in frictionless contact (Case II) with a rigid substrate. By using linear fracture mechanics, we formulate an convenient semi-analytical approach to develop relations between the applied force, penetration depth and contact radius. Finite element analysis (FEA) is used to verify the relationships. Our results reveal that the interfacial boundary conditions between the film and substrate have distinct effects on the adhesive contact behavior between the sphere and the film. The aim of the present study is to provide an instructive inspiration for controlling adhesion strength of the thin film subject to adhesive contact.

Keywords: adhesive contact, JKR theory, boundary condition, fracture mechanics, pull-off force, finite element analysis (FEA)

1. Introduction

Surface adhesion plays a paramount role in the performance and durability of miniaturized devices (or components), e.g. microelectromechanical (MEMS) systems, nanoelectromechanical (NEMS) systems, hard-disk devices, and atomic force microscopy (AFM) in which surface forces dominate other bulk forces. To date, there is a significant number of literatures on calssical analytical models accounting for adhesive contact, such as JKR (Johnson-Kendall-Roberts), DMT (Deraguin–Miller–Toporov) and MD (Maugis–Dugdale) models which are developed to investigate the effect of adhesive force on the contact behavior under different assumptions. The JKR model considers adhesive forces are caused by short-range surface interactions and thus the forces are confined within the contact region [1]. In contrast, the DMT model ascribes adhesion to long-ranged forces and the adhesive forces exist in an annulus encircling the contact area without impacting the surface profiles [2]. Although the pull-off forces predicted by JKR model and DMT model equal $1.5\pi\Delta\gamma R$ and $2\pi\Delta\gamma R$ respectively, this divergence is interpreted by the alleged Tabor parameter μ [3] given as

$$\mu = [R\Delta\gamma^2/E^*z_0^3]^{1/3} \quad (1)$$

where $E^* = [(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2]^{-1}$ is the effective Young's modulus, (E_1 , ν_1 and E_2 , ν_2 denote the elastic modulus and Possion's ratio of the two contacting spheres, respectively), and z_0 represents the atomic equilibrium distance. The Tabor parameter could be regarded as the ratio of the elastic surface displacement at the moment of detachment to the effective range of surface force characterized by the atomic equilibrium distance z_0 [4]. In general, the DMT model holds for a small and stiffer sphere ($\mu < 0.1$) whilst the JKR model is preferable for a large and compliant sphere ($\mu > 5$) [5]. Maguis [6] used Dugdale-Barenblatt model to approximate the interaction forces, and presented the transition between DTM and JKR extremes (i.e. MD model).

In the studies mentioned above, the contact region is considered small compared with the dimensions of the contacting bodies, and therefore the latter ones could be treated as semi-infinite body. However, this assumption is not always satisfied in factual circumstance where the dimension

in the vertical direction (the direction parallel to the load direction) is always comparable to the contacting size. In this sense, treating the contacting body as thin film would be more practical and appropriate. For example, the MEMS component [7] would be a coated thin film and the biological cells might spread like a thin layer bonded on petri dish. In both cases, the contact radius would coincide with the film thickness in the order of magnitude, the former even larger than the latter. Regarding this, the substrate effect becomes significant, which could lead to a divergence from the traditional JKR model. To date, there is a large number of studies which extended the conventional JKR theory to adhesive contact of an axisymmetric probe on an elastic layer [8–14]. However regarding the bottom boundary condition, the thin film is assumed to be perfectly bonded to the substrate (referred as Case I if the substrate is rigid hereafter) in the above studies, while the adhesive contact of the thin film frictionlessly bonded to the substrate has not been thoroughly investigated, nor the comparison of results between these two boundary conditions has been made.

In the present study, we aim to extend the classical JKR theory to the adhesive contact between a rigid sphere and an elastic film in frictionless contact with a rigid substrate (referred as Case II hereafter), and compare the results with that of Case I. In this paper, we firstly reintroduce the elastic solution of an axisymmetric mixed boundary problem in Sec. 2 where an elastic layer is either perfectly bonded to or in frictionless contact with a rigid substrate. The results are obtained by solving the Fredholm integral equations of the second kind. In Sec. 3, by analogy to linear fracture mechanics, the contact edge is treated as an external crack of the first kind. The normal stress underneath the indenter is given from which we formulate the expression for stress intensity factor K_I of the first kind and energy release rate G . Equating G to the work of adhesion $\Delta\gamma$ enables us to establish a mutual relationship between force, penetration and contact area. In Sec. 4 we present a meaningful discussion on the factors affecting characteristics of adhesive contact, e.g. pull-off force and pull-off radius, and we also compare the converse effects of these factors on the adhesive contact behaviors of thin film in case I and II. Sec. 5 presents a mathematical analysis on the stability of equilibrium obtained by letting $G = \Delta\gamma$. Finite element analysis is implemented to validate the obtained results in Sec. 6. Finally conclusions are drawn from this study.

2. Formulation of elastic film indented by a rigid sphere

Consider the axisymmetric frictionless contact problem of a rigid sphere on an elastic layer overlaying on a rigid substrate as illustrated in figure 1. The layer is infinite with a thickness h , Young's modulus E and Poisson's ratio ν . The cylindrical coordinate (r, φ, z) is used in figure 1, where z axis coincides with the generatrix of the sphere, r axis is perpendicular to z axis and φ denotes the angular distance between a reference line and r axis. The origin of the cylindrical coordinate is selected at the intersection point where the generatrix goes through the upper surface of the film. Based on the above boundary conditions, one has $\sigma_z = 0$ ($r > a, z = 0$), $\tau_{rz} = 0$ ($0 \leq r < \infty, z = 0$) and $w = \delta - \Psi(r)$ ($0 \leq r \leq a, z = 0$), where σ_z and τ_{rz} are the normal and tangential stress components respectively, a represents the contact radius and $\Psi(r)$ denotes the shape of the indenter. Regarding the interface between the film and the rigid substrate, if it is perfectly bonded (case I), one has as $w = u = 0$ ($0 \leq r < \infty, z = h$), and if the interface is frictionless (case II), one then has $w = \tau_{rz} = 0$ ($0 \leq r < \infty, z = h$). Since the deformation of the layer subjected to the normal force P is axisymmetric, the radial displacement u and vertical displacement w are independent of the hoop coordinate φ and they satisfy the field equations of the linear theory of elasticity [15] for homogeneous, isotropic and linear elastic materials, given as

$$(1 - 2\nu)\nabla^2 \mathbf{u} + \nabla(\nabla \cdot \mathbf{u}) = 0 \quad (2)$$

where $\mathbf{u} = (u, 0, w)$ is the displacement vector, and ∇ represents gradient operator, i.e. $\nabla = (\mathbf{i}_r \partial / \partial r + r^{-1} \mathbf{j}_\phi \partial / \partial \phi + \mathbf{k}_z \partial / \partial z)$.

Figure 1. Contact between a rigid sphere and an elastic film sitting on a rigid substrate.

The case of the axisymmetric contact problem shown in figure 1 could be solved by means of Papkovitch–Neuber solution for the expression of the components of displacement vector:

$$2Gu_r = -\frac{\partial}{\partial r} [\Phi_0(r, z) + z\Phi_1(r, z)] \quad (3a)$$

$$2Gu_z = -\frac{\partial}{\partial z} \Phi_0(r, z) - z \frac{\partial}{\partial z} \Phi_1(r, z) + (3 - 4\nu)\Phi_1(r, z) \quad (3b)$$

and stress vector:

$$\sigma_z = 2(1 - \nu) \frac{\partial}{\partial z} \Phi_1(r, z) - \frac{\partial^2}{\partial z^2} \Phi_0(r, z) - z \frac{\partial^2}{\partial z^2} \Phi_1(r, z) \quad (4a)$$

$$\tau_{rz} = \frac{\partial}{\partial r} [(1 - 2\nu)\Phi_1(r, z) - \frac{\partial}{\partial z} \Phi_0(r, z) - z \frac{\partial}{\partial z} \Phi_1(r, z)] \quad (4b)$$

where Φ_i ($i = 0, 1$) is harmonic function known as the Boussinesq–Papkovitch potential functions [16] and G denotes shear modulus, i.e. $G = E/2(1 + \nu)$. Since the solutions of stresses and displacements under the mixed boundary conditions have already been developed by previous studies [13, 17–20], its detailed derivation procedure is not repeated here. Herein, we directly formulate the dependence of the normal stress σ_z beneath the sphere on the contact radius a as

$$\sigma_z|_{z=0} = -\frac{\delta E}{\pi(1-\nu^2)a} \left[\frac{\omega(1)}{\sqrt{1-\varrho^2}} - \int_{\tau=\varrho}^1 \frac{\omega'(\tau)d\tau}{\sqrt{\tau^2-\varrho^2}} \right] (\varrho = \frac{r}{a}) \quad (5)$$

where the unknown function $\omega(\tau)$ could be written as

$$\omega(\tau) = \omega_f(\tau) + \frac{a^2}{\delta R} \omega_s(\tau) \quad (6)$$

The $\omega_f(\tau)$ is the solution for a flat tip, which is given as

$$\omega_f(\xi) + \frac{1}{\pi} \int_{\tau=0}^1 \omega_f(\tau) [K(\tau + \xi) + K(\tau - \xi)] d\tau = 1 \quad (7a)$$

and $\omega_s(\tau)$ satisfies

$$\omega_s(\xi) + \frac{1}{\pi} \int_{\tau=0}^1 \omega_s(\tau) [K(\tau + \xi) + K(\tau - \xi)] d\tau = -\xi^2 \quad (7b)$$

where the kernel function $K(u)$ is dependent on the type of boundary condition between the film and the rigid substrate, and is given by

$$K(u) = \begin{cases} \frac{a}{h} \int_{\alpha=0}^{\infty} \frac{(3-4\nu)e^{-\alpha sh\alpha} - \alpha(1+\alpha) - 4(1-\nu)^2}{\alpha^2 + 4(1-\nu)^2 + (3-4\nu)sh^2\alpha} \cos\left(\frac{\alpha au}{h}\right) d\alpha & \text{(case I)} \\ \frac{a}{h} \int_{\alpha=0}^{\infty} \frac{\alpha + e^{-\alpha sh\alpha}}{\alpha + chash\alpha} \cos\left(\frac{\alpha au}{h}\right) d\alpha & \text{(case II)} \end{cases} \quad (8a)$$

$$(8b)$$

Equation 7 could be solved by numerical method for a given value of a/h and ν .

3. Extention of JKR model to thin film by linear fracture mechanics

Consider the normal stress $\sigma_z(\varrho, 0)$ ($\varrho = r/a$) at the proximity of the contact edge. If one lets $r = a(1 - \varepsilon)$ ($\varepsilon > 0$), the pressure could be rewritten as

$$\sigma_z|_{z=0}(\varrho) \approx -\frac{\delta E}{\pi(1-\nu^2)a} \left[\frac{\omega(1)}{\sqrt{2\varepsilon}} - \int_{\tau=1-\varepsilon}^1 \frac{\omega'(\tau)d\tau}{\sqrt{\tau^2-1+2\varepsilon}} \right] (\varepsilon = 1 - \varrho) \quad (9)$$

It can be seen that as $\varepsilon \rightarrow 0^+$, the second term in the bracket tends towards zero whilst the first term tends towards infinity as $\varepsilon^{-1/2}$. In fact, the square root singularity of $\sigma_z(\varrho, 0)$ is the characteristic of linear fracture mechanics. Since linear fracture mechanics has already succeeded in representing an alternative method to derive JKR model where the contacting body is semi-infinite [19], it inspires us to imitate the same derivation when the semi-infinite space is curtailed into a film. If one treats the contact edge as the cutting-edge of an external crack of the first kind, the dependence of σ_z on the stress intensity of the first kind is given by

$$\sigma_z(1 - \varepsilon) = \frac{K_I}{\sqrt{2\pi a \varepsilon}} \quad (10)$$

and comparing the right sides of equations (9) and (10), one has

$$K_I = -\frac{\omega(1)}{\sqrt{\pi a}} \frac{E\delta}{(1-\nu^2)} \quad (11)$$

The summation of the normal stress σ_z numerically equals the applied force P . In another word, P could be obtained by integrating equation (5) along the radius r until it reaches the contact radius a as

$$P = 2\pi a^2 \int_{\varrho=0}^1 \varrho \sigma_z(\varrho, 0) d\varrho \quad (12)$$

Let us denote P_1 the apparent load which, if $\omega(1) = 0$, would give the same contact radius as that under the applied load P with $\omega(1) \neq 0$ [19]. Using $\omega(1) = 0$ and substituting equation (5) into equation (12) results in

$$P_1 = -\frac{2Ea^3\omega_s(1)}{R(1-\nu^2)\omega_f(1)} \int_{\tau=0}^1 \omega(\tau)d\tau \quad (13)$$

Hereupon, we introduce the following dimensionless parameters:

$$\kappa\left(\frac{a}{h}, \nu\right) = \int_0^1 \omega(\tau)d\tau; \quad \left(\frac{a}{h}, \nu\right) = -\frac{\omega_f(1)}{\omega_s(1)}; \quad \psi\left(\frac{a}{h}, \nu\right) = \kappa\left(\frac{a}{h}, \nu\right)/\chi\left(\frac{a}{h}, \nu\right) \quad (14)$$

Based on the above non-dimensional parameters, equation (13) could be rewritten as

$$P_1 = \frac{2Ea^3}{R(1-\nu^2)} \psi\left(\frac{a}{h}, \nu\right) \quad (15)$$

Combining equations (5) and (12), one then has

$$P_1 - P = -\frac{4E\delta\omega(1)a}{(1-\nu^2)} = K_I\sqrt{4\pi a^3} \quad (16)$$

According to linear fracture mechanics, K_I is related to the energy release rate G by

$$G = \frac{(1-\nu^2)K_I^2}{2E} \quad (17)$$

Substituting equation (16) into equation (17) leads to

$$G = \frac{1-\nu^2}{E} \frac{(P_1-P)^2}{8\pi a^3} \quad (18)$$

Equilibrium requires $G = \Delta\gamma$, from which one has

$$P = \frac{2Ea^3}{R(1-\nu^2)} \psi\left(\frac{a}{h}, \nu\right) - \sqrt{\frac{8\pi a^3 \Delta\gamma E}{1-\nu^2}} \quad (19)$$

Let $\tau = 1$ in equation (6), and then one has

$$\omega(1) = \omega_f(1) + \frac{a^2}{\delta R} \omega_s(1) \quad (20)$$

Substituting equation (20) into equation (11) results in

$$K_I = -\frac{E}{\sqrt{\pi a(1-\nu^2)}} [\delta \omega_f(1) + \frac{a^2}{R} \omega_s(1)] \quad (21)$$

Inserting equation (21) into equation (17), one has

$$G = \frac{E}{2\pi a(1-\nu^2)} [\delta \omega_f(1) + \frac{a^2}{R} \omega_s(1)]^2 \quad (22)$$

Again, the equilibrium requirement $G = \Delta\gamma$ leads to

$$\delta = \frac{1}{\omega_f(1)} \sqrt{\frac{2\pi a \Delta\gamma (1-\nu^2)}{E}} + \frac{1}{\chi\left(\frac{a}{h}, \nu\right)} \frac{a^2}{R} \quad (23)$$

Equations (19) and (23) present the dependence of load and penetration on the contact radius respectively. It should be pointed out that equation (14) represents a perfectly bonded interface in case I, and for the corresponding dimensionless parameters of case II, Poisson's ratio ν is absent.

4. Relations between applied force, penetration depth and contact radius

If one nondimensionalizes the force and length in the current situation by following the same way in the conventional JKR theory [19], as

$$P^* = \frac{P}{3\pi \Delta\gamma R}; \quad a^* = \frac{a}{(9\pi \Delta\gamma R^2/4E^*)^{1/3}}; \quad h^* = \frac{h}{(9\pi \Delta\gamma R^2/4E^*)^{1/3}}; \quad \delta^* = \frac{\delta}{(3\pi^2 \Delta\gamma^2 R/16E^{*2})^{1/3}} \quad (24)$$

where $E^* = E/(1-\nu^2)$ denotes the effective Young's modulus. Equations (19) and (23) can be non-dimensionalized to

$$P^* = \frac{3}{2} a^{*3} \psi\left(\frac{a^*}{h^*}, \nu\right) - \sqrt{2} a^{*3/2} \quad (25a) \quad \text{and} \quad \delta^* = \frac{3a^{*2}}{\chi\left(\frac{a^*}{h^*}, \nu\right)} - \frac{\sqrt{8}}{\omega_f\left(\frac{a^*}{h^*}, \nu\right)} a^{*1/2} \quad (25b)$$

respectively. Again, the Poisson's ratio is absent in equation (25) if the bottom interface is frictionless (Case II). It is worth our attention that, equation (25) presents a semi-analytical solution for elastic film subject to adhesive contact, and it is very convenient for practical applications. Moreover, equation (25) reveals that both h^* and ν govern the relations between the dimensionless parameters P^* , δ^* and a^* in case I (in case II, the sole governing parameter is ν).

4.1 The relationship between dimensionless force and contact radius

Figure 2 illustrates the effect of dimensionless parameter h^* on the distribution of a^*-P^* relation curves, and for comparison purposes, the conventional JKR counterpart is also plotted. For simplicity, figure 2(a) only represents the cluster of $P^*-\delta^*$ curves corresponding to $\nu = 0.4$ in case I, and the results could be generalized with no difficulty to different values of Poisson's ratio. An overall view of figure 2 suggests that the distribution of the a^*-P^* curves is dependent on the parameter h^* . The conventional JKR model is an asymptotical solution for h^* tending towards infinity, which is consistent with the expectation that thin film will evolve into JKR theory if its thickness approaches infinity. In case I, all the a^*-P^* curves lie on the right side of JKR solution, whereas all the a^*-P^* curves are located on the left side of JKR solution in case II. For a given value of ν in case I, higher force is needed to obtain the same contact radius for thinner film while converse trend is observed in case II.

Figure 2. Equilibrium relation between the contact radius a and the applied force P for (a) case I and (b) case II.

Figure 3 shows the dependence of a^* on P^* corresponding to different ν for a given h^* . It can be seen that as h^* increases, the divergence between the two cases and JKR model decreases, and the discrepancy is almost indistinguishable at $h^* = 5$. In another word, the conventional JKR model could be a good approximation for thin film in terms of the a^*-P^* relation if the dimensionless height h^* is larger than 5, regardless of the type of bottom interface. For a given of h^* , a greater force is required to retain the same contact radius for a film with higher Poisson's ratio in case I.

Figure 3. The relation between the dimensionless force and contact radius for (a) $h^* = 0.5$ (b) $h^* = 1$ (c) $h^* = 2$ and (d) $h^* = 5$

The effects of the dimensionless parameter h^* and Poisson's ratio ν on the pull-off force (the minimum force required to detach contacting bodies) are illustrated in figure 4(a). In case I, for a given value of ν the pull-off force increases monotonically and asymptotically approaches the counterpart in JKR model (i.e. $P_{JKR} = 1.5\pi\Delta\gamma R$) with an increase in h^* . On the other hand, for a given value of h^* the pull-off force increases with ν decreasing from 0.5 to 0.1. In case II, the pull-off force is solely dependent on h^* as predicted, and its variation trend is opposite to its counterpart of case I, and it also tends asymptotically towards $1.5\pi\Delta\gamma R$. The critical contact radius ($a_{pull-off}$) at full-off instant is also affected by h^* and ν as shown in figure 4(b), and it can be seen that the effect of these two parameters on $a_{pull-off}$ is the same as that of pull-off force.

Figure 4. The effect of dimensionless parameter h^* and Poisson's ratio ν on the critical (a) force and (b) radius at the pull-off instant.

4.2 The relationship between dimensionless force and penetration

The dependence of the dimensionless force P^* on the reduced penetration δ^* could be obtained by eliminating a^* in equation (25), and it is illustrated in figure 5. Negative and positive force represent attraction and repulsion between the rigid sphere and the elastic film respectively. For simplicity, figure 5(a) only represents the cluster of $P^*-\delta^*$ curves corresponding to $\nu = 0.4$ in case I, and the results could be generalized with no difficulty to different values of Poisson's ratio. Figure 5(b) illustrates $P^*-\delta^*$ curves corresponding to case II. It can be seen that the distribution of the $P^*-\delta^*$ curve of both cases is dependent on the value of h^* , and there is no overlapping between two curves with different h^* values. Particularly, in case I, the curve of smaller h^* value is surrounded by the next curve of larger h^* value, whereas converse distribution trend is observed in case II as shown in figure 5(b).

Figure 5. Relationship between the non-dimensionalized force and penetration for (a) case I and (b) case II.

It can be seen that at a given value of ν , the stiffness $dP/d\delta$ is highest in case I, lowest in case II and JKR model is between case I and II as seen in figure 5. The reason is that when the upper part of film is subject to contact, the film will expand in lateral direction. In case I, the bottom part of film is bonded to the rigid substrate which is nondeformable, which will in turn suppress the lateral expansion of the upper part film. In general, this suppression will enhance the stiffness of the indenter-film-substrate system. Since this suppression will be decreased as the thickness of film increases, it also explains why a film with larger thickness corresponds to lower stiffness, and all the stiffness of case I is larger than that of JKR theory. In case II, this suppression never holds since the interface between the film and substrate is frictionless. In this sense, thicker film consisting of more material suppresses the lateral expansion more significantly and thus film of larger thickness results higher stiffness, which will increase to the limit of JKR theory where the height of film tends towards infinity.

Figure 6. Relationship between the dimensionless force and penetration for (a) $h^* = 0.5$, (b) $h^* = 1.0$, (c) $h^* = 2.0$ and (d) $h^* = 20$.

Figure 6 illustrates the dependence of the dimensionless force on the dimensionless penetration depth for four different values of h^* . It can be seen that at a given value of h^* , the stiffness $dP/d\delta$ is highest in case I ($\nu = 0.1, 0.2, 0.3, 0.4$ and 0.5), lowest in case II and between case I and II for the JKR model. The reason is that when the upper part of film is subject to contact, the film will expand in lateral direction. In case I, the film is perfectly bonded to the rigid substrate which is nondeformable, the lateral expansion will be suppressed which in turn enhances the stiffness of the indenter-film-substrate system. However, in case II, the lateral expansion is never restrained due to the frictionless interface, which exhibits a lower stiffness in general. Since the Poisson's ratio is characterized by this lateral expansion, it makes sense that higher Poisson's ratio corresponds higher stiffness in case I as shown in figure 6. Again as h^* increases, the two cases will converge to the JKR model, and at $h^* = 20$, almost no distinguishable difference will be observed among the three cases.

5. Analysis of stability of equilibrium

Although equilibrium is achieved by letting $G = \Delta\gamma$, the stability of equilibrium is still unknown yet for both cases. Figure 7 illustrates the relationship between applied force and penetration depth for a certain value of ν and h^* . Since the stability of equilibrium at fixed load and fixed grips are given by $(\partial G/\partial a^*)_{P^*} > 0$ and $(\partial G/\partial a^*)_{\delta^*} > 0$ respectively, we focus on the fixed load point (valley point C where $dP^*/da^* = 0$) and fixed grips point (leftmost point D where $d\delta^*/da^* = 0$) on the equilibrium curve. In equation (18), if one substitutes P_1 with equation (15), and then replaces P and a by their dimensionless counterparts in equation (24), the energy release rate would be given by

$$G = \left(\frac{9}{8}\psi^2 a^{*2} - \frac{3}{2}\psi P^* + \frac{P^{*2}}{2a^{*3}}\right)\Delta\gamma \quad (26a)$$

Partial derivative of G with respect to a^* results in

$$\left.\frac{\partial G}{\partial a^*}\right|_{P^*} = \left(\frac{9}{4}\psi\psi' \frac{a^{*3}}{h^*} + \frac{27}{8}\psi^2 a^{*2} - \frac{3}{2}\frac{\psi' P^*}{h^*} - \frac{3P^{*2}}{2a^{*4}}\right)\Delta\gamma \quad (26b)$$

where ψ' denotes the derivative of ψ with respect to a^*/h^* . Substituting equation (25a) into equation (26b), and noting $dP^*/da^* = 0$ at point C, one could find equation (26b) equals zero.

Figure 7. Equilibrium relation between force P and penetration depth δ .

In a similar way, G could be expressed in terms of dimensionless parameters as

$$G = \Delta \gamma \left[\frac{\omega_f(1)}{2\sqrt{2}} \frac{\delta^*}{\sqrt{a^*}} + \frac{3\omega_s(1)}{2\sqrt{2}} a^{*3/2} \right]^2 \quad (27)$$

If one partially differentiates equation (27) with respect to a^* , and notices that $d\delta^*/da^* = 0$ at point D, one will find that $(\partial G/\partial a^*)_{\delta^*}$ vanishes at Point D. The branches of stable and unstable equilibrium is denoted by solid line and dashdotted line respectively. By analogy to the situation in conventional JKR model [19], at fixed force, we can move along the equilibrium curve from top to bottom until the Point C, and under fixed grips condition, we can move along the equilibrium curve from right to left until the Point D. Since the above analysis always holds independent of the exact value of ν and h^* , in whichever case, this conclusion is universal.

In the conventional JKR theory, under zero force, the corresponding contact radius a_0^{JKR} and the penetration δ_0^{JKR} are

$$a_0^{JKR} = \left(\frac{9\pi \Delta \gamma R^2}{2E^*} \right)^{1/3} \quad \text{and} \quad \delta_0^{JKR} = \left(\frac{3\pi^2 \Delta \gamma^2 R}{4E^{*2}} \right)^{1/3} \quad (28)$$

respectively. Let us denote contact radius and the penetration corresponding to zero force by a_0 and δ_0 respectively in thin film case, and they could be obtained by the vertical intercept in figure 2 and the abscissa intercept in figure 5 respectively. It can be seen that these two values are dependent on h^* in both cases. A further analysis by letting P^* equal zero in equation (25) would derive the dependence of a_0 and δ_0 on h^* , as illustrated in figures 8(a) and (b) respectively. It can be seen that the dependence of a_0 and δ_0 on h^* resembles that of pull-off force as shown in figure 4(a).

Figure 8. The effect of dimensionless h^* and Poisson's ratio on the zero force (a) radius and (b) penetration depth.

6. Finite element analysis

To validate the above derivation, finite element method (FEM) was used to simulate adhesive

contact between a rigid sphere and a linearly elastic film. Finite element simulations were performed using commercial ANSYS 14.5 package.

Figure 9. Schematic of FE model of (a) an overview and (b) its amplification at contact region.

The thin film consists of 4949 four-node quadrilateral asisymmetrical (Plane 182) with a total numbe of 5030 nodes as shown in figure 9(a). The nodes at the left boundaries of the mesh were constrained against horizontal direction. The finite element mesh was designed to provide a high density of nodes at the contact region, to capture the details of the displacement and stress filed as illustrated in figure 9(b). The distance between two adjacent surface nodes of the refined mesh is euqal to $0.005R$, where R denotes the radius of the rigid sphere. For the nodes at the bottom of the film, their vertical displacements are restrained, and the suppression of their horizontal displacements is retained and freed to realize the boundary conditions in case I and II respectively.

The adhesive interaction was simulated by a USERINTER subroutine with a traction-separation relationship governed the Lennard–Jones (L–J) law [21]:

$$p(r) = \frac{8\Delta\gamma}{3z_0} \left\{ \left[\frac{z_0}{H(r)} \right]^3 - \left[\frac{z_0}{H(r)} \right]^9 \right\} \quad (29)$$

where $p(r)$ and $H(r)$ denote the load interaction force and separation between indenter and film surface respectively. Although we use L–J force law to simulate the real interfacial adhesion, the pull-off force is unlikely to be sensitive to the detailed form of the traction-separation law [22]. Moreover, the general results of the two extreme models (JKR and DMT model) were shown to be independent of the exact form the intermolecular interaction potential [23]. In the present section, the governing parameters E^* , z_0 , R and $\Delta\gamma$ in equation (1) are selected such that the corresponding Tabor parameter μ is larger than 5 to approximate the JKR extreme [5].

Figure 10. Finite element results for (a) $h^* = 1$ (b) $h^* = 0.5$ and (c) $h^* = 0.2$ in case I where the Poisson's ratio $\nu = 0.4$.

Figure 11. Finite element results for (a) $h^* = 1$ (b) $h^* = 0.5$ and (c) $h^* = 0.2$ in case II.

Figures 10 and 11 plot the FEM results of the non-dimensional P - δ relation for case I and II respectively, in comparison with the corresponding semi-analytical solutions given by equation (25). Good agreement is observed between semi-analytical solutions and FEM results for the pull-off force. The semi-analytical solutions, however, differ significantly from the FEM curves when the separation is large. This discrepancy is due to the fact that the surface traction of the Lennard-Jones law differs from that of JKR where interaction forces are considered by a delta function. It is worth our attention that this discrepancy is also consistent with the observations made by Greenwood [5] for semi-infinite space. It should be borne in mind that for simplicity, we only present the FEM results corresponding to three values of h^* , and the conclusion obtained above could be extended to other values of h^* and ν .

7. Conclusions

To the best of our knowledge, the present paper represents the first attempt to extend the conventional JKR theory to thin film in frictionless contact with rigid substrate, and compare its adhesive contact behavior with that of perfectly bonded boundary condition. In the study, we consider the free edge of the contact as an external crack, and derive the formula for energy release rate G inspired by the contact pressure at the proximity of contact edge. The equilibrium requires $G = \Delta\gamma$ from which we could ascertain the relations between applied force, penetration depth and contact radius. The results indicate that the type of the bottom boundary condition between the elastic film and the rigid substrate plays a significant role in affecting the pull-off force, i.e. the pull-force in case I is smaller than the counterpart in conventional JKR model, which is lower than that in case II. Moreover, in case I the pull-off force is dependent on h^* and ν with higher h^* and lower ν corresponding to higher pull-off force, while in case II the pull-off force is sloely determined by h^* with higher h^* resulting in lower pull-off force. We also investigate the stability of equilibrium in terms of the force-penetration curve, which is identical with that in conventional JKR theory. Finally finite element method is performed to validate the obtained semi-analytical results. In general, our results present a theoretical foundation and some physical insights that are potentially important for applications involving adhesive contact with thin film. For example, compared to the conventioal JKR theory where the space is semi-infinite, the thin film exhibits analogical adhesive behaviors but presenting more factors in controlling adhesion, e.g. the dimensionless height of film h^* , Poisson's ratio ν (only in case I) and the type of the bottom boundary condition. However, for a given type of bottom boundary condition, the pull off force cannot exceed the JKR limit regardless of the selection of h^* and ν , and thus the last factor inspires us to combine the two boundary conditions at the bottom to realize the continuous manipulation of adhesion strength beyond the JKR limit, which will be detailed in our future work.

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