Polytopes and the Existence of Approximate Equilibria in Discontinuous Games*

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Abstract

Radzik (1991) showed that, by strengthening the usual quasi-concavity assumption on players' payoff functions, upper semi-continuous two-player games on compact intervals of the real line have ε -equilibria for all $\varepsilon > 0$. Ziad (1997) then stated that the same conclusion holds for n -player games on compact, convex subsets of $\mathbb{R}^m$, $m \geq 1$, provided that the upper semi-continuity condition is strengthened. Both Radzik's and Ziad's proofs rely crucially on the lower hemi-continuity of the ε -best reply correspondence. We show that: (1) in contrast to what is stated by Ziad, his conditions fail to be sufficient for the lower hemi-continuity of the approximate best-reply correspondence, (2) the approximate best-reply correspondence is indeed lower hemi-continuous if players' action spaces are polytopes, and (3) with action spaces as polytopes, Ziad's theorem can be stated so that it properly generalizes Radzik's theorem.

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1 Introduction

Classic existence results (e.g., Nash (1950)) show that a normal-form game has a Nash equilibrium provided that two sets of conditions hold. First, each player's action space is a non-empty, compact, convex subset of a euclidean space. Second, each player's payoff function is jointly continuous and separately quasi-concave in his own action. The goal of this note is to show that, for the existence of approximate equilibria in normal-form games with polytopes as action spaces, the continuity requirement on players' payoff functions can be weakened from joint continuity to joint upper semi-continuity by strengthening the quasi-concavity requirement.

Our results build upon those of Radzik (1991) and Ziad (1997). In order to explain our results and how they relate to theirs, we briefly recall some game-theoretical and mathematical notions (Definitions 1 and 2, respectively).

- Definition 1** 1. A normal form game G consists of a finite set of players $N = \{1, \dots, n\}$, and, for all players $i \in N$, a pure strategy set X_i , which is a non-empty compact subset of $\mathbb{R}^m$, and a bounded payoff function $u_i : X \rightarrow \mathbb{R}$, where $X = \times_{i \in N} X_i$.
2. The symbol $-i$ denotes "all players but i ." In particular, $X_{-i} = \times_{j \neq i} X_j$. The value function of player i is $V_i : X_{-i} \rightarrow \mathbb{R}$ defined by $V_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$.
3. Given a game G and $\varepsilon \geq 0$, an ε -equilibrium of G is $x^* \in X$ such that $u_i(x^*) \geq u_i(x_i, x_{-i}^*) - \varepsilon$ for all $i \in N$ and $x_i \in X_i$. A Nash equilibrium of G is an ε -equilibrium for $\varepsilon = 0$.
4. For all $\varepsilon > 0$ and $i \in N$, the player i 's ε -best-reply correspondence is $BR_i^\varepsilon : X_{-i} \rightrightarrows X_i$ defined by $BR_i^\varepsilon(x_{-i}) = \text{cl}(\{x_i \in X_i : u_i(x_i, x_{-i}) > \sup_{\hat{x}_i \in X_i} u_i(\hat{x}_i, x_{-i}) - \varepsilon\})$.

$\varepsilon\}$) for all $x_{-i} \in X_{-i}$.¹ The ε -best-reply correspondence is $BR_\varepsilon : X \rightrightarrows X$ defined by $BR_\varepsilon(x) = BR_1^\varepsilon(x_{-1}) \times \cdots \times BR_n^\varepsilon(x_{-n})$ for all $x \in X$.

Definition 2 Let X and Y be compact and convex subsets of a euclidean space.

1. A polytope is the convex hull of finitely many points.
2. For a function $f : X \rightarrow \mathbb{R}$, an upper set is of the form $\{x \in X : f(x) \geq a\}$ for some $a \in \mathbb{R}$.
3. A function is upper semi-continuous if all its upper sets are closed, quasi-concave if all its upper sets are convex, and polyhedral quasi-concave if all its upper sets are polytopes.
4. For a function $f : X \times Y \rightarrow \mathbb{R}$, these notions apply jointly to the argument (x, y) of $f(x, y)$. Separate upper semi-continuity, quasi-concavity and polyhedral quasi-concavity in x , say, apply to $f(x, y)$ as a function of x .
5. A correspondence $\beta : X \rightrightarrows Y$ is closed-valued if $\beta(x)$ is closed for all $x \in X$. It is lower hemi-continuous if the set $\{x \in X : \beta(x) \subseteq F\}$ is closed for each closed subset F of Y . A closed-valued correspondence is upper hemi-continuous if its graph is closed.
6. A point $x \in X$ is a fixed point of β if $x \in \beta(x)$. A function $f : X \rightarrow Y$ is a continuous selection of β if f is continuous and $f(x) \in \beta(x)$ for all $x \in X$.

Recall, as well, that if $\beta : X \rightrightarrows X$ is such that $\beta(x)$ is nonempty, compact and convex for all $x \in X$ and X is finite-dimensional and compact, then: (1) If β is upper hemi-continuous, by Kakutani's fixed point theorem there is a fixed point x of β . (2) If β is lower hemi-continuous, then Michael's selection theorem implies that there is a continuous selection f of β .

Radzik (1991) considers payoff functions that are jointly upper semi-continuous and separately quasi-concave in each player's own action. Radzik considers only two players, each with an interval as strategy set. He shows that these assumptions

¹For all subsets A of a topological space Y , $\text{cl}(A)$ denotes the closure of A .

are not enough to guarantee the existence of an ε -equilibrium for all $\varepsilon > 0$. A solution to this existence problem can be obtained by strengthening upper semi-continuity to continuity since this would imply the existence of a Nash equilibrium using Kakutani's theorem. Radzik obtained an alternative solution by strengthening separate quasi-concavity to the following notion of *piecewise quasi-concavity*: For player 1, the compact interval X_2 that is the strategy set of player 2 is suitably covered by a finite number of intervals $[b_i, b_{i+1}]$ so that player 1's payoff function u_1 is jointly quasi-concave on $X_1 \times [b_i, b_{i+1}]$ for all i (piecewise quasi-concavity of player 2's payoff function is defined analogously).² His main result shows that if each player's payoff function is jointly upper semi-continuous and piecewise quasi-concave, then the game has an ε -equilibrium for all $\varepsilon > 0$.

Radzik proves his result by establishing that the ε -best reply correspondence is lower hemi-continuous, with nonempty, compact and convex values. Michael's selection theorem then implies that ε -best reply correspondence has a continuous selection, to which we can apply Brouwer's fixed point theorem to obtain an ε -equilibrium.

Ziad (1997) also considers the existence of approximate equilibrium in normal-form games. His setting is more general than that of Radzik (1991) since it allows for n players (not just two) and nonempty, convex, compact subsets of euclidean spaces as strategy spaces (not just compact intervals on the real line). He also extends the notion of piecewise quasi-concavity to this more general setting as follows: The finite cover of X_{-i} by closed intervals is generalized to a finite cover by convex compact sets X_{-i}^l , so that u_i is jointly quasi-concave on $X_i \times X_{-i}^l$. For ease of reference, this notion is presented in the following definition.

Definition 3 *Player i 's payoff function is piecewise quasi-concave if there is a finite cover of X_{-i} by compact convex subsets X_{-i}^l such that u_i is quasi-concave on $X_i \times X_{-i}^l$ for all l .³*

²Radzik has named this property *strong quasi-concavity*. While the property is a strengthening of separate quasi-concavity in each player's own action, the term strong quasi-concavity suggests that it is a strengthening of joint quasi-concavity, which it is not. For this reason, we refer to it as piecewise quasi-concavity.

³When $n = 2$ and players' action spaces are intervals in $\mathbb{R}$, this definition collapses into Radzik's

He then claims, as his main result, that the game has an ε -equilibrium for all $\varepsilon > 0$ if each player's payoff function u_i is upper semi-continuous, piecewise quasi-concave and if it satisfies an additional property called i -upper semi-continuity. Definition 4 presents its definition, which will be discussed below.

Definition 4 *Player i 's payoff function is i -upper semi-continuous if*

$$\limsup_k V_i(\alpha_k z_{-i}^k + (1 - \alpha_k)x_{-i}) \leq \limsup_k V_i(z_{-i}^k) \quad (1)$$

for all $x_{-i} \in X_{-i}$, $\{z_{-i}^k\}_{k=1}^\infty$ converging to x_{-i} and $\{\alpha_k\}_{k=1}^\infty \subseteq (0, 1]$ converging to zero.

Our contributions to the problem of existence of approximate equilibrium in normal-form games are the following: First, to note the following mistake in Ziad's paper: while he claims that the ε -best reply correspondence is upper hemi-continuous, his argument in the proof of the theorem attempts, in effect, to show that such a correspondence is lower hemi-continuous. Consequently, Ziad's argument would need Michael's selection theorem rather than Kakutani's fixed point theorem (as was the case in Radzik (1991)).

Second, we show that, under Ziad's assumptions, the ε -best reply correspondence may however fail to be lower hemi-continuous and to have a continuous selection. This implies that Ziad's argument does not prove the existence result he claims.

Third, we show that in Ziad's framework the ε -best reply correspondence is indeed lower hemi-continuous if players' action spaces are polytopes. This implies that Ziad's existence theorem is valid for games with polytopes as action spaces.

Fourth, we show that this modified version of Ziad's result (with polytopes as action spaces) implies Radzik's result. Recall that those two results differ because Ziad's result requires an additional condition on players' payoff functions (i -upper semi-continuity, Definition 4 above) while it allows for more than two players and for more general action spaces. We show that under Radzik's assumptions, players' payoff functions are i -upper semi-continuous, which immediately implies that Radzik's result follows from that of Ziad.

notion of piecewise quasi-concavity.

Fifth, we show that i -upper semi-continuity is implied by the following requirement on players' value functions:

Definition 5 *Player i 's value function is piecewise polyhedral quasi-concave if there is a finite cover of X_{-i} by compact convex subsets X_{-i}^l such that V_i is polyhedral quasi-concave on X_{-i}^l for all l .*

That is, we show that if player i 's value function is piecewise polyhedral quasi-concave, then his value function is i -upper semi-continuous.⁴ Furthermore, we show that, in Radzik's setting each player's value function is piecewise polyhedral quasi-concave. Thus, the sufficiency of the piecewise polyhedral quasi-concavity of V_i for the i -upper semi-continuity of u_i allows us to state an existence result for approximate equilibria of n -person games that simultaneously dispenses with the lower semi-continuity of u_i altogether and generalizes Radzik's theorem.

Proposition 6 *Let G be an n -player game such that for all $i \in N$, (1) X_i is a polytope, (2) u_i is upper semi-continuous and piecewise quasi-concave (Definitions 2 and 3), and (3) V_i is piecewise polyhedral quasi-concave (Definition 5). Then, G has an ε -equilibrium for all $\varepsilon > 0$.*

In conclusion, in order to obtain the existence of an approximate equilibrium, the continuity of players' payoff function can be weakened to upper semi-continuity. This is the case when players' action spaces are polytopes and when players' payoff and value functions satisfy a form of quasi-concavity stronger than separate quasi-concavity in each player's own action. The usefulness of Proposition 6 therefore lies in games whose players' payoff functions do not satisfy any form of lower semi-continuity, but which are, in compensation, upper semi-continuous and satisfy a relatively strong

⁴Clearly, i -upper semi-continuity is implied by the continuity of V_i and so can be thought of as a weak form of continuity. Note, however, that i -upper semi-continuity is neither implied nor does it imply upper semi-continuity. Indeed, player 2's payoff function in Example 11 below is upper semi-continuous but not i -upper semi-continuous. Conversely, if $X_1 = X_2 = [0, 1]$ and u_1 is defined by $u_1(x_1, x_2) = 1$ if $x_1 \leq 1/2$ and $u_1(x_1, x_2) = 2$ if $x_1 > 1/2$, then u_1 is i -upper semi-continuous but not upper semi-continuous.

form of quasi-concavity. In contrast, if players' payoff functions are discontinuous and separately quasi-concave in each player's own action but satisfy no stronger quasi-concave property, then one has to rely on the results of Dasgupta and Maskin (1986) and Reny (1999) to establish the existence of equilibria.

Finally, we note that Proposition 6 is related to Theorem 2 of Gale, Klee, and Rockafellar (1968), which, in particular, states that, for all closed convex subsets D of a finite-dimensional Euclidean space, every convex function on D is upper semi-continuous if and only if the intersection of D with any polytope is a polytope. This result implies that every concave function on a polytope is lower semi-continuous.⁵ This, in turn, implies the following existence result: If G is an n -player game such that for all $i \in N$, (1) X_i is a polytope, (2) u_i is upper semi-continuous and (3) there exists a finite cover of X_{-i} by polytopes X_{-i}^l such that u_i is concave on $X_i \times X_{-i}^l$ for all l , then G has a Nash equilibrium. In fact, by their Theorem 2, u_i would be lower semi-continuous on $X_i \times X_{-i}^l$ for all l , and so on $X = X_i \times X_{-i}$. Together with the upper semi-continuity of u_i , this implies that u_i is actually continuous and the existence of equilibrium follows from Kakutani's fixed point theorem. Thus, in comparison, Proposition 6 allows for compact convex partitions of X_{-i} and it requires only quasi-concavity of u_i on $X_i \times X_{-i}^l$ but it only guarantees the existence of approximate equilibrium and it requires V_i to be polyhedral quasi-concave on X_{-i}^l . More importantly, Proposition 6 allows for discontinuous payoff functions whereas the above existence result applies only to payoff functions that are, in effect, continuous.

⁵Indeed, if D is a polytope, then D is convex and closed and its intersection with a polytope is also a polytope. Furthermore, if $f : D \rightarrow \mathbb{R}$ is concave, then $-f$ is convex. Thus, Theorem 2 of Gale, Klee, and Rockafellar (1968) implies that $-f$ is upper semi-continuous and so f is lower semi-continuous.

2 Characterizations of Polytopes and Ziad's Theorem

We start by stating that Ziad's theorem is valid when players' action spaces are polytopes.

Theorem 7 (Ziad) *Let G be an n -player game such that for all $i \in N$, (1) X_i is a polytope and (2) u_i is upper semi-continuous, i -upper semi-continuous and piecewise quasi-concave. Then for each $\varepsilon > 0$, each player's ε -best reply correspondence is lower hemi-continuous. Consequently, G has an ε -equilibrium for all $\varepsilon > 0$.*

The reason why polytopes are important for Ziad's theorem is given in Lemma 9, which is a simple consequence of the characterization of polytopes that we provide in Proposition 8. The importance of Lemma 9 is that it allows us to establish the lower hemi-continuity of the ε -best reply correspondence (which, as explained in the introduction, implies the existence of an ε -equilibrium together with Michael's selection theorem and Brouwer's fixed point theorem).

We start by providing two characterizations of polytopes.

Proposition 8 *Let $P \subseteq \mathbb{R}^n$. Then, the following conditions are equivalent:*

1. P is a polytope;
2. P is compact, convex and satisfies the following property: for all $x \in P$, there exists $r > 0$ such that for all $\tilde{x} \in P$, $\tilde{x} \neq x$,

$$x + r \frac{\tilde{x} - x}{\|\tilde{x} - x\|} \in P;$$

3. P is compact, convex and all its extreme points are isolated.

The last condition in property 2 says that the ball of radius r intersected with the cone of all interior directions from x is a subset of P .⁶

⁶I am thankful to an associate editor for this interpretation and for the present proof of Proposition 8.

Proof of Proposition 8. We will prove that 1 implies 2, 2 implies 3 and 3 implies 1. For convenience, let $E(C)$ denote the set of extreme points of a convex set C .

Property 2 implies 3 because if there were a sequence $\{x_k\}_{k=1}^\infty$ of extreme points of P converging to x , its elements would eventually be in an $r/2$ -ball around x , but then its elements would not be extreme: defining $x' = x + 2(x_k - x)$, then $x_k = (x' + x)/2$, a contradiction since x' belongs to P . Indeed, $x + r(x_k - x)/\|x_k - x\| \in P$ by property 2 and $x' = \theta(x + r(x_k - x)/\|x_k - x\|) + (1 - \theta)x$ with $\theta = 2\|x_k - x\|/r \in (0, 1)$.

Property 3 implies 1 because P is equal to the convex hull of its extreme points and $E(P)$ is finite. Indeed, $E(P)$ is closed (see Rockafellar (1970, Corollary 18.1.1)) and clearly bounded; so, if it were to be infinite, then it would have an accumulation point, a contradiction to property 3.

In order to establish that property 1 implies 2, note that polyhedra (and so polytopes) are intersections of finitely many half-spaces. Let $P = \bigcap_{i \in F} \{z \in \mathbb{R}^n : c_i \cdot z \leq d_i\}$ for some finite set F , vectors $c_i \in \mathbb{R}^n$ and scalars $d_i \in \mathbb{R}$. Let $T = \{i \in F : c_i \cdot x = d_i\}$ and $T^c = F \setminus T$. Take r to be the smallest distance of x from the hyperplanes $H_i = \{z \in \mathbb{R}^n : c_i \cdot z = d_i\}$ for all $i \in T^c$, unless $T^c = \emptyset$, in which case we can take $r = 1$. Let $\tilde{x} \in P$ and define $\bar{x} = x + r(\tilde{x} - x)/\|\tilde{x} - x\|$. We claim that $c_i \cdot \bar{x} \leq d_i$ holds for all $i \in F$, which clearly implies that $\bar{x} \in P$. For convenience, for all $i \in F$, let $S_i = \{z \in \mathbb{R}^n : c_i \cdot z \leq d_i\}$ and $d(x, H_i) = \inf_{z \in H_i} \|z - x\|$ denote the distance of x from H_i . In the case $i \in T$, we have that $c_i \cdot x = d_i$ and $c_i \cdot \tilde{x} \leq d_i$ and so $c_i \cdot \bar{x} = d_i + r(c_i \cdot \tilde{x} - d_i)/\|\tilde{x} - x\| \leq d_i + 0 = d_i$. In the case $i \in T^c$, note that $\|\bar{x} - x\| = r \leq d(x, H_i)$, which implies that $\bar{x} \in S_i$. In fact, if $\bar{x} \notin S_i$, then define $\theta = (d_i - c_i \cdot x)/(c_i \cdot \bar{x} - c_i \cdot x) \in (0, 1)$ (since $0 < d_i - c_i \cdot x < c_i \cdot \bar{x} - c_i \cdot x$) and note that $\theta\bar{x} + (1 - \theta)x \in H_i$. But then we obtain that $\|\theta\bar{x} + (1 - \theta)x - x\| = \theta r < r \leq d(x, H_i)$, a contradiction. ■

The following lemma is a simple consequence of Proposition 8 and is key to the proof of Theorem 7.

Lemma 9 *Let P be a polytope, $x \in P$, $\{x_k\}_{k=1}^\infty$ and $\{\alpha_k\}_{k=1}^\infty$ be such that $\lim_k x_k = x$, $0 < \alpha_k \leq 1$ and $\alpha_k x_k + (1 - \alpha_k)x \in P$ for all $k \in \mathbb{N}$. Then, there exist sequences*

$\{\hat{x}_k\}_{k=1}^\infty$ and $\{\theta_k\}_{k=1}^\infty$ such that $\lim_k \theta_k = 0$, $\hat{x}_k \in P$, $\theta_k \in (0, 1)$ and $x_k = \theta_k \hat{x}_k + (1 - \theta_k)x$ for all k sufficiently large. Furthermore, if $\{\alpha_k\}_{k=1}^\infty$ is bounded away from zero, then $\{\hat{x}_k\}_{k=1}^\infty$ and $\{\theta_k\}_{k=1}^\infty$ can be chosen so that $\lim_k \hat{x}_k = x$.

The following two special cases illustrate the usefulness of Lemma 9. The first case (to be considered in the proof of Theorem 7 below) is obtained by letting $\alpha_k = 1$ for all k . Thus, in this case, we have a sequence $\{x_k\}_{k=1}^\infty$ contained in the polytope P , converging to a point x also in P . Lemma 9 then asserts that, for all k sufficiently large, each x_k can be expressed as a convex combination between the limit point x and some other point $\hat{x}_k$ in P .

A second special case of interest (to be considered in the proof of Proposition 12 below) occurs when α_k converges to zero. In this case, we have a sequence $\{x_k\}_{k=1}^\infty$ converging to a point x in P with the property that $\alpha_k x_k + (1 - \alpha_k)x$ also belongs to P . Lemma 9 implies that x_k must itself belong to P for all k sufficiently large: In fact, for all k sufficiently large, each x_k can be expressed as a convex combination between the limit point x and some other point $\hat{x}_k$ in P .

Proof of Lemma 9. Note first that whenever $x_k = x$ we can let $\hat{x}_k = x$ and $\theta_k = 1/k$. Thus, we may assume that $x_k \neq x$ for all $k \in \mathbb{N}$.

Let $r > 0$ be such that $x + r(\tilde{x} - x)/\|\tilde{x} - x\| \in P$ for all $\tilde{x} \in P$, $\tilde{x} \neq x$. For convenience, let $\bar{x}_k = \alpha_k x_k + (1 - \alpha_k)x$ for all k . Note that there exists $K \in \mathbb{N}$ such that $\|x_k - x\| < r$ for all $k \geq K$. For all k , define $\gamma_k = \|\bar{x}_k - x\|/r$ and $\hat{x}_k = (1/\gamma_k)\bar{x}_k + (1 - 1/\gamma_k)x = x + r(\bar{x}_k - x)/\|\bar{x}_k - x\|$. Clearly, $\hat{x}_k \in P$ for all k . Then, $\bar{x}_k = \gamma_k \hat{x}_k + (1 - \gamma_k)x$ and so $x_k = (\gamma_k/\alpha_k)\hat{x}_k + (1 - \gamma_k/\alpha_k)x$ for all k .

Thus, define $\theta_k = \gamma_k/\alpha_k$ for all k , which immediately implies that $x_k = \theta_k \hat{x}_k + (1 - \theta_k)x$ for all k . Furthermore, $\theta_k = \|\bar{x}_k - x\|/r\alpha_k = \|x_k - x\|/r \in (0, 1)$ for all $k \geq K$ and so $\lim_k \theta_k = 0$.

Finally, consider the case where $\{\alpha_k\}_{k=1}^\infty$ is bounded away from zero. In this case, let $K \in \mathbb{N}$ be such that $\|\bar{x}_k - x\|^{1/2} < \min\{r, 1\}$ for all $k \geq K$. For all k , define $\gamma_k = \|\bar{x}_k - x\|^{1/2}$ and, as before, $\hat{x}_k = (1/\gamma_k)\bar{x}_k + (1 - 1/\gamma_k)x$ and $\theta_k = \gamma_k/\alpha_k$. Clearly, $\theta_k > 0$ for all k and $\lim_k \theta_k = 0$, implying that $\theta_k \in (0, 1)$ for sufficiently large k .

Note that $\|\hat{x}_k - x\| = \|\bar{x}_k - x\|/\gamma_k = \|\bar{x}_k - x\|^{1/2}$ and so $\lim_k \hat{x}_k = x$. It remains to show that $\hat{x}_k \in P$ for all $k \geq K$. Since $\|\bar{x}_k - x\|^{1/2} \leq r$, then $1/\|\bar{x}_k - x\|^{1/2} \leq r/\|\bar{x}_k - x\|$, which implies that $\hat{x}_k$ can be expressed as a convex combination of x and $x + r(\bar{x}_k - x)/\|\bar{x}_k - x\|$. Since the latter point belongs to P , then $\hat{x}_k$ also belongs to P . ■

Lemma 9, together with Ziad's original argument, allows us to prove Theorem 7.

Proof of Theorem 7. Let $i \in N$ and let $\beta_i : X_{-i} \rightrightarrows X_i$ be defined by $\beta_i(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) > V_i(x_{-i}) - \varepsilon\}$. Note that $BR_i^\varepsilon(x_{-i}) = \text{cl}(\beta_i(x_{-i}))$ for all $x_{-i} \in X_{-i}$. Furthermore, it follows by Aliprantis and Border (1999, Lemma 16.22, p. 535), that BR_i^ε is lower hemi-continuous if and only if β_i is lower hemi-continuous.

In order to establish that β_i is lower hemi-continuous, let $F \subseteq X_i$ be closed, $x_{-i} \in X_{-i}$, $\{x_{-i}^k\}_{k=1}^\infty \subseteq \{x'_{-i} \in X_{-i} : \beta_i(x'_{-i}) \subseteq F\}$ be such that $\lim_k x_{-i}^k = x_{-i}$ and $x_i \in \beta_i(x_{-i})$. We will show that $x_i \in F$, which implies that $\beta_i(x_{-i}) \subseteq F$.

Let $0 < \eta < \varepsilon$ be such that $u_i(x_i, x_{-i}) > V_i(x_{-i}) - \varepsilon + \eta$. Since u_i is upper semi-continuous, so is V_i . Thus, let $\delta > 0$ be such that $V_i(x'_{-i}) < V_i(x_{-i}) + \eta/2$ for all $x'_{-i} \in X_{-i}$ satisfying $\|x_{-i} - x'_{-i}\| < \delta$. By Lemma 9 (with $\alpha_k = 1$ for all k), let $\{\hat{x}_{-i}^k\}_{k=1}^\infty$ and $\{\theta_k\}_{k=1}^\infty$ be such that $\lim_k \hat{x}_{-i}^k = x$, $\lim_k \alpha_k = 0$, $\hat{x}_{-i}^k \in X_i$, $\theta_k \in (0, 1)$ and $x_{-i}^k = \theta_k \hat{x}_{-i}^k + (1 - \theta_k)x_{-i}$ for all k sufficiently large. Taking a subsequence if necessary, we may assume that $\{V_i(\hat{x}_{-i}^k)\}_{k=1}^\infty$ converges to $\limsup_k V_i(\hat{x}_{-i}^k)$ and that $\hat{x}_{-i}^k \in X_i$, $\theta_k \in (0, 1)$ and $x_{-i}^k = \theta_k \hat{x}_{-i}^k + (1 - \theta_k)x_{-i}$ for all $k \in \mathbb{N}$.

Let $\hat{x}_i^k \in X_i$ be such that $u_i(\hat{x}_i^k, \hat{x}_{-i}^k) > V_i(\hat{x}_{-i}^k) - \varepsilon + \eta/2$ and define $x_i^k = \theta_k \hat{x}_i^k + (1 - \theta_k)x_i$ for all k . Since $\lim_k \theta_k = 0$, it follows that $\lim_k x_i^k = x_i$.

We claim that $x_i^k \in \beta_i(x_{-i}^k)$ for all k sufficiently large. Suppose, in order to reach a contradiction, that there exists an infinite sequence $\{k_j\}_{j \in \mathbb{N}}$ such that $u_i(x_i^{k_j}, x_{-i}^{k_j}) \leq V_i(x_{-i}^{k_j}) - \varepsilon$. Taking a further subsequence if needed, we may assume that there exists $l \in \{1, \dots, L_i\}$ such that $\hat{x}_{-i}^{k_j} \in X_{-i}^l$ for all j . Since X_{-i}^l is compact, then $x_{-i} \in X_{-i}^l$. Thus, the quasi-concavity of u_i in $X_i \times X_{-i}^l$ implies that

$$\begin{aligned} u_i(x_i^{k_j}, x_{-i}^{k_j}) &\geq \min\{u_i(x_i, x_{-i}), u_i(\hat{x}_i^{k_j}, \hat{x}_{-i}^{k_j})\} \\ &> \min\{V_i(x_{-i}) - \varepsilon + \eta, V_i(\hat{x}_{-i}^{k_j}) - \varepsilon + \eta/2\} = V_i(\hat{x}_{-i}^{k_j}) - \varepsilon + \eta/2, \end{aligned} \tag{2}$$

for all j such that $|\hat{x}_{-i}^{k_j} - x_{-i}| < \delta$ (since in this case, $V_i(\hat{x}_{-i}^{k_j}) < V_i(x_{-i}) + \eta/2$). Hence, $V_i(\hat{x}_{-i}^{k_j}) + \eta/2 < V_i(x_{-i}^{k_j})$ for all j sufficiently large, and so

$$\limsup_k V_i(\hat{x}_{-i}^k) + \frac{\eta}{2} = \lim_j V_i(\hat{x}_{-i}^{k_j}) + \frac{\eta}{2} \leq \limsup_j V_i(x_{-i}^{k_j}) \leq \limsup_k V_i(x_{-i}^k), \quad (3)$$

contradicting the i -upper semi-continuity of u_i . This contradiction establishes that $x_i^k \in \beta_i(x_{-i}^k)$ for all k sufficiently large. Since $\beta_i(x_{-i}^k) \subseteq F$ for all k and F is closed, then $x_i = \lim_k x_i^k \in F$, as desired. Thus, β_i , and so BR_i^ε , is lower hemi-continuous.

It is clear that $BR_i^\varepsilon(x_{-i})$ is nonempty, closed and convex. Therefore, $BR_\varepsilon(x) = BR_1^\varepsilon(x_{-1}) \times \cdots \times BR_n^\varepsilon(x_{-n})$ is also nonempty, closed and convex. Furthermore, $BR_\varepsilon : X \rightrightarrows X$ is lower hemi-continuous. Thus, by Michael's selection theorem, there exists a continuous selection f of BR_ε . Hence, by Brouwer's fixed point theorem, f has a fixed point x and x is an ε -equilibrium of G . ■

We note that the only difference between the statement of Theorem 7 and the original statement in Ziad (1997) is that players' action spaces are now required to be polytopes (and not just a convex and compact subset of a euclidean space). As the following example shows, requiring the action spaces to be convex and compact is not sufficient to guarantee the lower hemi-continuity of the ε -best reply correspondence.

Example 10 Let $G = (X_1, X_2, u_1, u_2)$ be the following 2-player game. Let

$$X_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } (1-x)^2 \leq y \leq 1-x\} \quad (4)$$

and $X_2 = [0, 1]$. We denote $x_1 = (x, y)$ and $x_2 = z$.

Let player 1's payoff function u_1 be defined by $u_1(x, y, z) = 0$ for all $(x, y, z) \in X_1 \times X_2$. We define player 2's payoff function as follows. Define $A = X_1 \times \{0\} \subseteq X_1 \times X_2$, $B = \{(1, 0, 1)\} \subseteq X_1 \times X_2$ and $C = \text{co}(A \cup B)$. The payoff function for player 2 is

$$u_2(x, y, z) = \begin{cases} 2 & \text{if } (x, y, z) \in B, \\ 1 & \text{if } (x, y, z) \in C \setminus B, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Note that $X_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } f(x) \leq y \leq 1-x\}$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = (1-x)^2$. Since f is strictly convex and continuous, X_1 is convex

and compact. Clearly, X_2 is also compact and convex. Note that u_2 is quasi-concave on $X_1 \times X_2$. Furthermore, $A \cup B$, hence $\text{co}(A \cup B)$, is compact, and so u_2 is upper semi-continuous. Clearly, u_1 is both upper semi-continuous and quasi-concave.

We claim that u_i is also i -upper semi-continuous for all i . It is clear that $V_1(z) = 0$ for all $z \in X_2$ and $V_2(x, y) = 1$ for all $(x, y) \in X_1$, except when $(x, y) = (1, 0)$, in which case V_2 equals 2. Condition (1) is trivially satisfied for player 1 and one can easily show that V_2 also satisfies it. To see this, let $(x, y) \in X_1$, $\{(x_k, y_k)\}_{k=1}^\infty$ converging to (x, y) and $\{\alpha_k\}_{k=1}^\infty \subseteq (0, 1]$ converging to zero. If $(x, y) \neq (1, 0)$, we readily obtain that $\limsup_k V_2(\alpha_k(x_k, y_k) + (1 - \alpha_k)(x, y)) = \limsup_k V_2(x_k, y_k) = 1$. If $(x, y) = (1, 0)$, the fact that $\limsup_k V_2(x_k, y_k) \geq 1$ implies that it is enough to consider the case when $\limsup_k V_2(\alpha_k(x_k, y_k) + (1 - \alpha_k)(x, y)) = 2$. However, this condition holds if and only if there exists a subsequence $\{(x_{k_j}, y_{k_j})\}_{j=1}^\infty$ such that $(x_{k_j}, y_{k_j}) = (1, 0)$. But, clearly, in this case we also have $\limsup_k V_2(x_k, y_k) = 2$. Thus, it follows that V_2 satisfies condition (1) and so u_2 is i -upper semi-continuous.

Let $0 < \varepsilon < 1$. Then, $BR_1^\varepsilon(z) = X_1$ for all $z \in X_2$ and

$$BR_2^\varepsilon(x, y) = \begin{cases} \{1\} & \text{if } (x, y) = (1, 0), \\ \{z \in [0, 1] : (x, y, z) \in C\} & \text{otherwise,} \end{cases} \quad (6)$$

for all $(x, y) \in X_1$. Clearly, if $g : X_1 \rightarrow X_2$ is a selection from BR_2^ε , then $g(x, (1 - x)^2) = 0$ for all $x \in [0, 1)$, while $g(1, 0) = 1$. Hence, it follows that BR_2^ε has no continuous selection. Since $BR_2^\varepsilon(x, y)$ is nonempty, convex and compact for all $(x, y) \in X_1$, it follows by Michael's selection theorem that BR_2^ε is not lower hemi-continuous.

Before we conclude this section, we note that the above example can be easily modified to show that, even when action spaces are polytopes, the ε -best reply correspondence may fail to be lower hemi-continuous if players' payoff functions are not i -upper semi-continuous:

Example 11 Let $G = (X_1, X_2, u_1, u_2)$ be as in Example 10 except that $X_1 = \{(x, y) \in [0, 1] \times [0, 1] : y \leq 1 - x\}$ and both u_1 and u_2 are extended by defining $u_1(x, y, z) = u_2(x, y, z) = 0$ for all $(x, y, z) \in X_1 \times X_2$ such that $y < (1 - x)^2$.

Clearly, both payoff functions are quasi-concave and upper semi-continuous. The same argument used above shows that BR_2^ε is neither lower hemi-continuous nor

does it have a continuous selection for all $0 < \varepsilon < 1$. Finally, note that u_2 is not i -upper semi-continuous. Letting $(x, y) = (1, 0)$, $(x_k, y_k) = (1 - 1/k, 1/2k^2)$ and $\alpha_k = 1/2k$ for all $k \in \mathbb{N}$, we obtain that $y_k < f(x_k)$ (recall that $f(x) = (1 - x)^2$), $\alpha_k y_k + (1 - \alpha_k)0 = 1/4k^3 \geq f(1 - 1/2k^2) = f(\alpha_k x_k + (1 - \alpha_k)1)$ for all k and so $\limsup_k V_i(\alpha_k(x_k, y_k) + (1 - \alpha_k)(1, 0)) = 1 > 0 = \limsup_k V_i(x_k, y_k)$.

3 Relationship between Radzik's and Ziad's Theorems

As we have shown in the previous section, the polyhedral convexity of the action spaces is essential to the lower hemi-continuity of the ε -best reply correspondence and, therefore, to any approach to the existence of ε -equilibria based on that property. The importance of polyhedral convexity is strengthened here by showing that piecewise polyhedral concavity of the players' value functions implies i -upper semi-continuity. This result is then used to show that Radzik's Theorem is a corollary of the version of Ziad's theorem presented in Theorem 7.

Proposition 12 *If V_i is piecewise polyhedral quasi-concave, then u_i is i -upper semi-continuous.*

Proof. Let $x_{-i} \in X_{-i}$, $\{x_{-i}^k\}_{k=1}^\infty \subseteq X_{-i}$ be a sequence converging to x_{-i} and $\{\alpha_k\}_{k=1}^\infty \subseteq (0, 1]$ converging to zero. Let $\bar{x}_{-i}^k = \alpha_k x_{-i}^k + (1 - \alpha_k)x_{-i}$ for all $k \in \mathbb{N}$.

Let $\gamma = \limsup_k V_i(\bar{x}_{-i}^k)$. Then, there exists a subsequence $\{\bar{x}_{-i}^{k_j}\}_j$ of $\{\bar{x}_{-i}^k\}_k$ such that $\lim_{j \rightarrow \infty} V_i(\bar{x}_{-i}^{k_j}) = \gamma$. We may assume that $x_{-i}^{k_j} \neq x_{-i}$ for infinitely many indexes j , since otherwise $x_{-i}^{k_j} = \bar{x}_{-i}^{k_j} = x_{-i}$ for all j sufficiently large and so $\limsup_k V_i(x_{-i}^k) \geq \limsup_j V_i(x_{-i}^{k_j}) = V_i(x_{-i}) = \limsup_k V_i(\bar{x}_{-i}^k)$. Hence, taking a subsequence if necessary, we may assume that $x_{-i}^{k_j} \neq x_{-i}$ for all $j \in \mathbb{N}$; clearly, this implies that $\bar{x}_{-i}^{k_j} \neq x_{-i}$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$ and let $J_1 \in \mathbb{N}$ be such that $V_i(\bar{x}_{-i}^{k_j}) > \gamma - \varepsilon$ for all $j \geq J_1$.

Let $\{X_{-i}^l\}_{l=1}^{L_i}$ be a compact, convex cover of X_{-i} such that V_i is polyhedral quasi-concave on X_{-i}^l for all $l = 1, \dots, L_i$. Since the cover is finite, we may assume that there exists $l \in \{1, \dots, L_i\}$ such that $\bar{x}_{-i}^{k_j} \in X_{-i}^l$ for all $j \in \mathbb{N}$. Letting $P = \{y \in$

$X_{-i}^l : V_i(y) \geq \gamma - \varepsilon\}$, then P is a polytope and $\bar{x}_{-i}^{k_j} \in P$ for all $j \geq J_1$. Furthermore, $x_{-i} \in P$ since P is compact.

By Lemma 9, there exist $J > J_1$ and sequences $\{\hat{x}_{-i}^{k_j}\}_{k=1}^\infty$ and $\{\theta_{k_j}\}_{k=1}^\infty$ such that $\hat{x}_{-i}^{k_j} \in P$, $\theta_{k_j} \in (0, 1)$ and $x_{-i}^{k_j} = \theta_{k_j} \hat{x}_{-i}^{k_j} + (1 - \theta_{k_j}) x_{-i}$ for all $j \geq J$. Hence, for all $j \geq J$, it follows that $x_{-i}^{k_j} \in P$, which implies that $V_i(x_{-i}^{k_j}) \geq \gamma - \varepsilon$ and so $\limsup_k V_i(x_{-i}^k) \geq \gamma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\limsup_k V_i(x_{-i}^k) \geq \gamma = \limsup_k V_i(\bar{x}_{-i}^k)$. Thus, u_i is i -upper semi-continuous. ■

We note that for two-player games on a square with piecewise quasi-concave payoff functions, each player has a piecewise polyhedral quasi-concave value function and so an i -upper semi-continuous payoff function. Indeed, since, for all $i = 1, 2$, both X_i and X_{-i} are contained in $\mathbb{R}$, every convex, compact subset of X_i or X_{-i} is a polytope. In particular, X_i and X_{-i} are polytopes, if $\{X_{-i}^l\}_{l=1}^{L_i}$ is a compact, convex cover of X_{-i} , then X_{-i}^l is a polytope and so are the upper sets $\{x_{-i} \in X_{-i}^l : V_i(x_{-i}) \geq a\}$, $a \in \mathbb{R}$, of V_i in X_{-i}^l . Thus, combining this observation with Ziad's theorem, we obtain the main result in Radzik (1991).

Corollary 13 (Radzik) *If G is a two-player game such that, for all $i = 1, 2$, (1) X_i is a compact interval on the real line and (2) u_i is upper semi-continuous and piecewise quasi-concave, then G has an ε -equilibrium for all $\varepsilon > 0$.*

We can also use Proposition 12 to obtain existence results for n -person games played in polytopes that parallel the statement of Radzik's Theorem and are, again, a corollary of Ziad's. One such result is Proposition 6, stated in the introduction and reproduced in Corollary 14 below for convenience. Its statement parallels that of Radzik in that action spaces are polytopes and payoff functions are upper semi-continuous and piecewise quasi-concave, but it allows for more than two players and for action spaces that are subsets of any euclidean space. Since in this setting players' value functions are not necessarily piecewise polyhedral quasi-concave, this requirement must be made explicitly. A second existence result is then obtained by noting that if players' payoff functions are piecewise polyhedral quasi-concave, so are their value functions (player i 's payoff function is *piecewise polyhedral quasi-concave* if there

is a finite cover of X_{-i} by compact convex subsets X_{-i}^l such that u_i is polyhedral quasi-concave on $X_i \times X_{-i}^l$ for all l).

Corollary 14 *If G is such that, for all $i \in N$, (1) X_i is a polytope, (2) u_i is upper semi-continuous and piecewise quasi-concave and (3) V_i is polyhedral piecewise quasi-concave, then G has an ε -equilibrium for all $\varepsilon > 0$.*

In particular, if G is such that, for all $i \in N$, (1) X_i is a polytope and (2) u_i is upper semi-continuous and piecewise polyhedral quasi-concave, then G has an ε -equilibrium for all $\varepsilon > 0$.

4 Concluding Remarks

The approach for the existence results discussed in this note relies on the lower hemi-continuity of the ε -best reply correspondence. As the example in Section 3 shows, this condition may fail even if the game satisfies the assumptions in Ziad (1997). Nevertheless, we have shown that this problem can be solved by assuming that players' action spaces are polytopes. Furthermore, we show that if an upper semi-continuous game is such that the upper sets of players' payoff functions are polytopes (or at least it satisfies a generalization of this condition), then it has an ε -equilibrium for all $\varepsilon > 0$.

The drawback of this note is that it does not answer the question of whether or not Ziad's theorem holds as it was stated originally. For instance, it is clear that the game presented in Section 2 has Nash equilibria. Moreover, so does the game presented at the end of Section 2 and obtained by modifying the original example so that players' payoff functions are not i -upper semi-continuous. In fact, one can conjecture that every game with nonempty, compact, convex and finite-dimensional action spaces and with upper semi-continuous and piecewise quasi-concave payoff functions have ε -equilibria for all $\varepsilon > 0$.

Although no answer to the above conjecture will be offered, we note that the problem can be simplified by reducing it to games with finite-valued payoff functions. In fact, if G is a game satisfying the above properties, we can first normalize payoffs

so that $u_i(X) \subseteq (0, 1)$ for all $i \in N$ and then define, for all $k \in \mathbb{N}$, $u_i^k(x) = (j-1)/2^k$ if $(j-1)/2^k \leq u_i(x) < j/2^k$, for all $j = 0, \dots, 2^k$ (so, $u_i^k(x)$ is defined by rounding down $u_i(x)$ to the nearest multiple of 2^{-k}). Then, it is easy to see that for all $i \in N$ and $k \in \mathbb{N}$, u_i^k is finite-valued, upper semi-continuous and piecewise quasi-concave and $\{u_i^k\}_{k=1}^\infty$ converges uniformly to u_i . In fact, if $a \in (0, 1)$ and $j = \min\{j' : (j'-1)/2^k \geq a\}$, then $\{x \in X_i \times Y_{-i} : u_i^k(x) \geq a\} = \{x \in X_i \times Y_{-i} : u_i(x) \geq (j-1)/2^k\}$ is closed if $Y_{-i} = X_{-i}$ and convex if $Y_{-i} = X_{-i}^l$ (where $\{X_{-i}^l\}_l$ is a compact convex cover of X_{-i} given by the piecewise quasi-concavity of u_i). Thus, u_i^k is upper semi-continuous and piecewise quasi-concave. Furthermore, one has that $\sup_{x \in X} |u_i^k(x) - u_i(x)| \leq 2^{-k}$ implying that $\{u_i^k\}_{k=1}^\infty$ converges uniformly to u_i . We finally claim that if one establishes that every game with finite-valued, upper semi-continuous and piecewise quasi-concave payoff functions (whose action spaces also satisfy the same properties as those of G) have a Nash equilibrium, then the properties of $\{u_i^k\}_{k=1}^\infty, i \in N$, imply that G has an ε -equilibrium for all $\varepsilon > 0$. Indeed, if $k \in \mathbb{N}$ is such that $\sup_{x \in X} |u_i^k(x) - u_i(x)| < \varepsilon/2$ for all $i \in N$, and x^* is a Nash equilibrium of $G_k = (X_i, u_i^k)_{i \in N}$, then, for all $i \in N$ and $x_i \in X_i$, $u_i(x^*) > u_i^k(x^*) - \varepsilon/2 \geq u_i^k(x_i, x_{-i}^*) - \varepsilon/2 > u_i(x_i, x_{-i}^*) - \varepsilon$ and so x^* is an ε -equilibrium of G .

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