Understanding Some Recent Existence Results for Discontinuous Games*

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Abstract

We introduce a new condition, weak better-reply security, and show that every compact, locally convex, metric, quasiconcave and weakly better-reply secure game has a Nash equilibrium. This result is established using simple generalizations of classical ideas. Furthermore, we show that, when players' action spaces are metric and locally convex, it implies the existence results of Reny (1999) and Carmona (2009) and that it is equivalent to a recent result of Barelli and Soza (2009). Our general existence result also implies a new existence result for weakly upper reciprocally semicontinuous and weakly payoff secure games that satisfy a strong quasiconcavity property.

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1 Introduction

The classical approach to the existence of Nash equilibrium in a normal form game consists in applying a fixed point theorem to its best-reply correspondence (e.g., Nash (1950)). When discontinuities on players' payoff functions prevent this approach from being used, the classical solution consists in finding conditions that guarantee that the original game can be suitably approximated by a sequence of sufficiently well-behaved games in the following sense: a fixed point theorem can be applied to the best-reply correspondence of each of the approximating games, and the limit points the resulting Nash equilibria are themselves a Nash equilibrium of the original game (e.g., Dasgupta and Maskin (1986), Simon (1987) and Simon and Zame (1990)). Clearly, an equally suitable approach is to use a fixed point theorem to guarantee the existence of ε -equilibrium in the approximating games, and to show that limit points of the resulting ε -equilibria, with ε converging to zero, are themselves a Nash equilibrium of the original game (e.g., Fudenberg and Levine (1986)).

In this note, we combine and generalize these ideas in order to obtain an existence theorem that, when players' action spaces are metric and locally convex, implies the main results of Reny (1999) and Carmona (2009), and is equivalent to the generalization of Reny's theorem obtained by Barelli and Soza (2009). In this way, these results are established in a relatively simple way using the classical ideas mentioned above.

The use of those classical ideas requires first a generalization of the notion of an approximate equilibrium. This is done by noting that a Nash equilibrium is a strategy yielding, to each player, a payoff at least as high as the value of his value function evaluated at the equilibrium strategies of the other players. Then, our notion of approximate equilibrium is obtained by replacing, in this definition of Nash equilibrium, each player's value function by a function strictly below it and defined also on the action space of the other players.

The next step in our approach is to define a condition, weak better-reply security, which guarantees that the original game is suitably approximated by a well-behaved

game in the following sense: First, players' value functions in the approximating game are lower semicontinuous and can therefore be "closely" approximated from below by continuous functions; second, a fixed point theorem can be applied to a sub-correspondence of the approximate best-reply correspondence (in the approximating game) defined using those continuous functions; and, third, every limit point of the resulting approximate equilibria of the approximating game is a Nash equilibrium of the original game.

Clearly, this approach corresponds closely to the classical ideas discussed above, the main difference being that a single game, instead of a sequence of games, is used to approximate the original game (there are, however, important differences on how the main steps of the proof are established). As its name suggest, our condition generalizes the notion of better-reply security of Reny (1999) and is, in fact, equivalent to the notion of generalized better-reply security of Barelli and Soza (2009). Furthermore, weak better-reply security also implies the (combined) notion of weak payoff security and weak upper semicontinuity considered in Carmona (2009). Thus, in the case where players' action spaces are metric and locally convex, Theorem 3.1 in Reny (1999), Corollary 6.8 in Barelli and Soza (2009) and Corollary 2 in Carmona (2009) can be established using our general existence result, and more importantly, be seen as a consequence of the classical approach on which our result is based.

The above results of Reny (1999), Barelli and Soza (2009) and Carmona (2009) are obtained by showing that any game that satisfies the conditions of any of these results can be suitably approximated by a particular well-behaved game, which differs from the original only in the players' payoff functions. In contrast, weak better-reply security allows for any appropriately close well-behaved game to serve as an approximation of the original game. Despite this apparent extra generality, we show that there is a best possible approximation leading to weak better-reply security and that it coincides with the approximation implicit in the notion of generalized better-reply security of Barelli and Soza (2009). This characterization of weak better-reply-security establishes the equivalence between our existence result and that of Barelli and Soza (2009). We also show that the approximation implicit in the notion

of better-reply security of Reny (1999) is the best possible approximation leading to weak better-reply security under the additional requirement that the approximating game be payoff secure. Finally, we characterize the class of weak better-reply secure games that dispense with the need for approximation and show that they include those that satisfy the conditions in Carmona (2009).

Although our general existence result does not cover the results of Radzik (1991), Ziad (1997) and Carmona (2010), the techniques of these papers can be combined with the ones of this paper to obtain a new existence result. This new result establishes the existence of a Nash equilibrium in games where players' action spaces are polytopes and players' payoff functions are locally joint quasiconcave, weakly reciprocal upper semicontinuous at all strategies that are not a Nash equilibrium and have continuous value functions.

The interest of this new existence result can be understood as follows. Note first that compact, quasiconcave, weakly reciprocal upper semicontinuous and weakly payoff secure (i.e., with lower semicontinuous value functions) games may fail to have a Nash equilibrium (see Carmona (2009)). Note also that if each player's value function is locally polyhedral quasi-concave, then it is also upper semicontinuous. Hence, the above existence result can be understood as providing strengthened quasiconcavity conditions that are sufficient to guarantee the existence of Nash equilibria in weakly reciprocal upper semicontinuous and weakly payoff secure games.

We note that our approach is a synthesis of several recent developments in the literature.¹ The idea of approximating the original game by a better behaved game is based on Reny (1999) and the approximation of players' value functions is achieved using Reny's approximation lemma. Furthermore, our sufficient condition for the existence of equilibrium is inspired by Reny's better-reply security. The idea of approximating players' value functions in the approximating game and of applying a fixed point to a subcorrespondence of the resulting approximate best-reply correspondence is based on Prokopovych (2010). The application of a fixed point theorem

¹See also Yannelis and Prabhakar (1983) and Lebrun (1996) for early developments of similar ideas.

to a subcorrespondence of the approximate best-reply correspondence requires, first, allowing players to use a well-behaved correspondence defined locally to secure payoffs and, second, using a partition of the unity to obtain, from the well-behaved correspondences defined locally, a well-behaved correspondence defined globally; both of these ideas are based on Barelli and Soza (2009). Finally, the idea that limit points of approximate equilibria of the approximating game are themselves Nash equilibria of the original game is based on Bagh (2009), Carmona (2009) and Prokopovych (2010). The strength of our approach is that these ideas are organized in such a way that it allows for a general existence result based on (simple generalizations of) classic ideas and which is strong enough to imply several known existence theorems. This, in turn, allows for a clear understanding of both the new and the previous existence results.

Finally, we note that our approach leaves out the existence results based on the existence of a maximal element of binary relations, such as the one of Baye, Tian, and Zhou (1993).² A unifying approach to both these and the existence results we have focused on can be found in Barelli and Soza (2009). Moreover, our approach also leaves out the existence results of Simon and Zame (1990) and Jackson, Simon, Swinkels, and Zame (2002) for games with endogenous sharing rules. Although it would be interesting to have a sufficiently general existence theorem to include all such results, we stress that our goal is simply to show that several of them can be obtained by an appropriate generalization of classical ideas.

2 Notation and Definitions

A *normal form game* $G = (X_i, u_i)_{i \in N}$ consists of a finite set of players $N = \{1, \dots, n\}$ and, for all players $i \in N$, a pure strategy space X_i and a payoff function $u_i : X \rightarrow \mathbb{R}$, where $X = \prod_{i \in N} X_i$. We assume that, for all $i \in N$, X_i is a subset of a locally convex

²The existence theorem in Baye, Tian, and Zhou (1993) can be easily proven by applying Theorem 2 in Tian (1993) on the existence of maximal elements to the binary relation on the joint action space defined by $x \succ y$ if and only if $U(x, y) > U(y, y)$, where $U(x, y) = \sum_{i \in N} u_i(x_i, y_{-i})$ in a game $G = (X_i, u_i)_{i \in N}$.

metric vector space.

We say that a game G is *compact* if, for all $i \in N$, X_i is compact and u_i is bounded. Given a player $i \in N$, the symbol $-i$ denotes “all players but i ”. Also, $X_{-i} = \prod_{j \neq i} X_j$. Furthermore, we say that G is *quasiconcave* if, for all $i \in N$, X_i is convex and $u_i(\cdot, x_{-i})$ is quasiconcave for all $x_{-i} \in X_{-i}$.

Let $G = (X_i, u_i)_{i \in N}$ be a game and let, for all $i \in N$, $v_i : X_{-i} \rightarrow \mathbb{R}$ be defined by $v_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$; the function v_i is player i ’s value function. A strategy $x^* \in X$ is *Nash equilibrium* if $u_i(x^*) \geq v_i(x_{-i}^*)$ for all $i \in N$.

We define the notion of an approximate equilibrium by replacing, in the inequality above, v_i with a function strictly below it. Let $F(G)$ denote the set of all functions $f = (f_1, \dots, f_n)$ such that f_i is a real-valued function on X_{-i} and $f_i(x_{-i}) < v_i(x_{-i})$ for all $x_{-i} \in X_{-i}$ and $i \in N$. For all $f \in F(G)$, we say a strategy $x^* \in X$ is an *f-equilibrium* if $u_i(x^*) \geq f_i(x_{-i}^*)$ for all $i \in N$. Note that this notion generalizes the concept of ε -equilibrium since, for all $\varepsilon > 0$, $x^* \in X$ is an ε -equilibrium if it is an *f-equilibrium* for $f = (v_1 - \varepsilon, \dots, v_n - \varepsilon)$.

3 Existence of Nash equilibrium

Below we establish a general existence result (Subsection 3.1). Minimal assumptions for this result are obtained in Subsection 3.2. The general existence result of Subsection 3.1 and its version with minimal assumptions is used, in the case where players’ action spaces are metric and locally convex, to derive and understand the main results of Reny (1999) and Carmona (2009), and the generalization of Reny’s Theorem obtained by Barelli and Soza (2009) (Subsection 3.3). In particular, it is shown that our existence result is equivalent to the one of Barelli and Soza (2009). Furthermore, our existence result is used to establish the existence of Nash equilibrium in weakly reciprocal upper semicontinuous and weakly payoff secure games that satisfy strengthened quasiconcavity assumptions.

3.1 A general existence result

Our general existence result (Theorem 1 below) states that all compact and quasiconcave games satisfying a weak form of better-reply security have a Nash equilibrium.

Our notion of weak better-reply security guarantees that any game satisfying it can be approximated by a well-behaved game (in the sense that a fixed point argument can be used to establish the existence of approximate equilibria) and that the approximation can be done in such a way that limit points of approximate equilibria of the approximating game, with the level of approximation suitably converging to zero, are themselves Nash equilibria of the original game.

Weak better-reply security combines two properties. The first is entitled better-reply closedness. This property is the one responsible for the limit result and is formally defined as follows. Let $\underline{u}$ be a $\mathbb{R}^n$ -valued function on X , $\underline{v}_i(x_{-i}) = \sup_{x_i \in X_i} \underline{u}_i(x_i, x_{-i})$ for all $i \in N$ and $x_{-i} \in X_{-i}$, $G = (X_i, u_i)_{i \in N}$ be a game and Γ be the closure of the graph of $u = (u_1, \dots, u_n)$. We say that G is *better-reply closed relative to $\underline{u}$* if $\underline{u}_i \leq u_i$ for all $i \in N$ and if $(x^*, u^*) \in \Gamma$ and $u_i^* \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ imply that x^* is a Nash equilibrium of G . In this case, we say that $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is an approximating game of G .

The second property that weak better-reply security incorporates is a generalization of the notion of generalized payoff security introduced by Barelli and Soza (2009). This property is responsible for the existence of approximate equilibria for any approximating game with a level of approximation suitably converging to zero. A game $G = (X_i, u_i)_{i \in N}$ is *generalized payoff secure* if for all $i \in N$, $\varepsilon > 0$ and $x \in X$ there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} and an upper hemicontinuous correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ with nonempty, closed and convex values such that $u_i(x') \geq u_i(x) - \varepsilon$ for all $x' \in \text{graph}(\varphi_i)$. Given a $\mathbb{R}^n$ -valued function $\underline{u}$ on X , a game $G = (X_i, u_i)_{i \in N}$ is *generalized payoff secure relative to $\underline{u}$* if, for all $i \in N$, $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $x_{-i} \in X_{-i}$ and $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is generalized payoff secure.³

³In previous versions of this paper, we allowed players to secure payoffs in $\underline{G}$ with a lower hemicontinuous correspondence, also with nonempty, closed and convex values. However, this extra generality is only apparent. Because such correspondence takes values in a compact, hence complete

The notion of weak better-reply security combines generalized payoff security and better-reply closedness. Formally, a game G is *weakly better-reply secure* if there exists a bounded $\mathbb{R}^n$ -valued function $\underline{u}$ on X such that G is both generalized payoff secure and better-reply closed relative to $\underline{u}$. In this case, we also say that G is *weakly better-reply secure relative to $\underline{u}$* .

Our existence result states that compact, quasiconcave games that are weakly better-reply secure have a Nash equilibrium.

Theorem 1 *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave and weakly better-reply secure, then G has a Nash equilibrium.*

Theorem 1 is established with the help of three lemmas. The first of these lemmas states that if G is compact, quasiconcave and generalized payoff secure relative to a bounded $\underline{u}$, then the approximating game $\underline{G}$ has an f -equilibrium provided that f is continuous. Note that, under those assumptions, $\underline{G}$ is a compact, quasiconcave and generalized payoff secure game. Thus, Lemma 1 can be understood as stating that every compact, quasiconcave and generalized payoff secure game has an f -equilibrium for all continuous f . When stated in this way it becomes clear that Lemma 1 is a generalization of Reny (1996, Theorem 3) and Prokopovych (2010, Theorem 2) since the assumption of payoff security used in these results is replaced by generalized payoff security.

Lemma 1 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. If G is generalized payoff secure relative to a bounded $\underline{u}$, then $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ has an f -equilibrium for all continuous $f \in F(\underline{G})$.*

Proof. Since $\underline{u}$ is bounded, so is $\underline{v}$. Hence, $F(\underline{G})$ is nonempty and, in fact, there exist continuous functions $f \in F(\underline{G})$. Let f be such a function and define $B : X \rightrightarrows X$ subset of a metric space, it follows by Michael's selection theorem (see Theorem 17.66 in Aliprantis and Border (2006) and its proof) that it has a continuous selection. This implies that players can secure payoffs with the upper hemicontinuous correspondence with nonempty, closed and convex values defined by such continuous selection.

by $B(x) = \{y \in X : \underline{u}_i(y_i, x_{-i}) > f_i(x_{-i}) \text{ for all } i \in N\}$. Note that B is nonempty-valued since $f_i < \underline{v}_i$ for all $i \in N$ and is convex-valued since $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $i \in N$ and $x_{-i} \in X_{-i}$.

Next, we show that for all $x \in X$, there exist an open neighborhood V_x of x and an upper hemicontinuous correspondence $\varphi_x : V_x \rightrightarrows X$ with nonempty, closed and convex values such that $\varphi_x(x') \subseteq B(x')$ for all $x' \in V_x$.

In order to establish the above claim, let $x \in X$ and consider $y \in B(x)$. Then, for all $i \in N$, $\underline{u}_i(y_i, x_{-i}) > f_i(x_{-i}) + 2\eta$ for some $\eta > 0$ sufficiently small. Since $\underline{G}$ is generalized payoff secure and f is continuous, it follows that for all $i \in N$, there exist an open neighborhood $V_{x_{-i}}$ of x_{-i} and an upper hemicontinuous correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ with nonempty, closed and convex values such that $\underline{u}_i(x'_i) \geq \underline{u}_i(y_i, x_{-i}) - \eta$ for all $x'_i \in \text{graph}(\varphi_i)$ and $f_i(x_{-i}) > f_i(x'_{-i}) - \eta$ for all $x'_{-i} \in V_{x_{-i}}$. Define, for all $i \in N$, $V_i = X_i \times V_{x_{-i}}$, $V = \bigcap_{i \in N} V_i$ and $\varphi : V \rightrightarrows X$ by $\varphi(x') = \prod_{i \in N} \varphi_i(x'_{-i})$ for all $x' \in V$. Let $x' \in V$ and $y' \in \varphi(x')$. Then, for all $i \in N$, it follows that $x'_{-i} \in V_{x_{-i}}$ and $y'_i \in \varphi_i(x'_{-i})$. Then $\underline{u}_i(y'_i, x'_{-i}) \geq \underline{u}_i(y_i, x_{-i}) - \eta > f_i(x_{-i}) + \eta > f_i(x'_{-i})$ for all $i \in N$. Hence, $y' \in B(x')$ and so $\varphi(x') \subseteq B(x')$.

In the light of the above claim, we obtain a family $\{V_x\}_{x \in X}$ where V_x is an open neighborhood of x , and a family $\{\varphi_x\}_{x \in X}$ where $\varphi_x : V_x \rightrightarrows X$ is an upper hemicontinuous correspondence with nonempty, closed and convex values satisfying $\varphi_x(x') \subseteq B(x')$ for all $x' \in V_x$. Since X is compact, there exists a finite open cover $\{V_{x_j}\}_{j=1}^m$ and a partition of the unity $\{\beta_j\}_{j=1}^m$ subordinate to $\{V_{x_j}\}_{j=1}^m$. Define $\phi : X \rightrightarrows X$ by $\phi(x) = \sum_{j=1}^m \beta_j(x) \varphi_{x_j}(x)$. Then, it is easy to see that ϕ is an upper hemicontinuous correspondence with nonempty, closed and convex values. Thus, ϕ has a fixed point x^* by the Kakutani-Fan-Glicksberg fixed point theorem (see Aliprantis and Border (2006, Corollary 17.55)).

Let $1 \leq j \leq m$ be given and note that if $x^* \in V_{x_j}$, then $\varphi_{x_j}(x^*) \subseteq B(x^*)$ while if $x^* \notin V_{x_j}$, then $\beta_j(x^*) = 0$. Since B is convex-valued, then $x^* = \phi(x^*) = \sum_{j: \beta_j(x^*) > 0} \beta_j(x^*) \varphi_{x_j}(x^*) \in B(x^*)$. Hence, for all $i \in N$, $\underline{u}_i(x^*) > f_i(x^*_{-i})$ and so x^* is an f -equilibrium of $\underline{G}$. ■

Lemma 2 also concerns generalized payoff secure games. It states that if a game G

is generalized payoff secure relative to a bounded $\underline{u}$, then each player's value function in the approximating game $\underline{G}$ is lower semicontinuous.

Lemma 2 *If $G = (X_i, u_i)_{i \in N}$ is generalized payoff secure relative to a bounded $\underline{u}$, then $\underline{v}_i$ is lower semicontinuous for all $i \in N$.*

Proof. Since $\underline{u}$ is bounded, it follows that $\underline{v}_i$ is real-valued for all $i \in N$. Let $i \in N$, $x_{-i} \in X_{-i}$ and $\varepsilon > 0$ be given. Let $0 < \eta < \varepsilon$ and let $x_i \in X_i$ be such that $\underline{u}_i(x_i, x_{-i}) - \eta > \underline{v}_i(x_{-i}) - \varepsilon$. Since G is generalized payoff secure relative to $\underline{u}$, there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} and a nonempty-valued correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ such that $\underline{u}_i(x') \geq \underline{u}_i(x) - \eta$ for all $x' \in \text{graph}(\varphi_i)$. Then, for all $x'_{-i} \in V_{x_{-i}}$, letting $x'_i \in \varphi_i(x'_{-i})$, we obtain that $\underline{v}_i(x'_{-i}) \geq \underline{u}_i(x'_i) \geq \underline{u}_i(x) - \eta > \underline{v}_i(x_{-i}) - \varepsilon$. Hence, $\underline{v}_i$ is lower semicontinuous. ■

Since $\underline{v}_i$ is lower semicontinuous for all $i \in N$, it follows by Reny (1999, Lemma 3.5) that there exists a sequence $\{v_i^k\}_{k=1}^\infty$ of continuous real-valued functions on X_{-i} such that $v_i^k(x_{-i}) \leq \underline{v}_i(x_{-i})$ and $\liminf_k v_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i})$ for all $k \in \mathbb{N}$, $i \in N$, $x_{-i} \in X_{-i}$ and all sequences $\{x_{-i}^k\}_{k=1}^\infty$ converging to x_{-i} . When G is a compact game, this form of approximation together with better-reply closedness relative to a bounded $\underline{u}$ is enough for every limit point of every sequence of approximate equilibria of $\underline{G}$ to be a Nash equilibrium of G .

Lemma 3 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and better-reply closed game relative to a bounded $\underline{u}$. If $x^* \in X$, $\{x_k\}_{k=1}^\infty \subseteq X$ and $\{f_k\}_{k=1}^\infty \subseteq F(\underline{G})$ are such that $x^* = \lim_k x_k$, $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and x_k is a f_k -equilibrium of $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ for all $k \in \mathbb{N}$, then x^* is a Nash equilibrium of G .*

Proof. Since u is bounded, taking a subsequence if necessary, we may assume that $\{u(x_k)\}_{k=1}^\infty$ converges. Let $u^* = \lim_k u(x_k)$ and note that $(x^*, u^*) \in \Gamma$.

Let $i \in N$. Since x_k is a f_k -equilibrium of $\underline{G}$ for all $k \in \mathbb{N}$, then $u_i(x_k) \geq \underline{u}_i(x_k) \geq f_i^k(x_{-i}^k)$ for all $k \in \mathbb{N}$, and so $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ implies that $u_i^* \geq \underline{v}_i(x_{-i}^*)$. Since $u_i^* \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and G is better-reply closed relative to $\underline{u}$, it follows that x^* is a Nash equilibrium of G . ■

We finally turn to the proof of Theorem 1, which is obtained easily from Lemmas 1—3.

Proof of Theorem 1. Let $\underline{u}$ be a bounded $\mathbb{R}^n$ -valued function on X such that G is both generalized payoff secure and better-reply closed relative to $\underline{u}$.

For all $k \in \mathbb{N}$ and $i \in N$, let $f_i^k : X_{-i} \rightarrow \mathbb{R}$ be defined by $f_i^k(x_{-i}) = v_i^k(x_{-i}) - 1/k$ for all $x_{-i} \in X_{-i}$, where $\{v_i^k\}_{k=1}^\infty$ is a sequence of continuous real-valued functions on X_{-i} such that $v_i^k(x_{-i}) \leq \underline{v}_i(x_{-i})$ and $\liminf_k v_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i})$ for all $k \in \mathbb{N}$, $i \in N$, $x_{-i} \in X_{-i}$ and all sequences $\{x_{-i}^k\}_{k=1}^\infty$ converging to x_{-i} (as remarked above, the existence of this sequence follows from Lemma 2 and Reny (1999, Lemma 3.5)). Since f_k is continuous and $f_i^k < \underline{v}_i$ for all $i \in N$, Lemma 1 implies that $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ has a f_k -equilibrium, x_k , for all $k \in \mathbb{N}$.

Since X is compact, we may assume that $\{x_k\}_{k=1}^\infty$ converges. Letting $x^* = \lim_k x_k$, we have that $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and Lemma 3 implies that x^* is a Nash equilibrium of G . ■

3.2 Minimal Weak Better-Reply Security

Weak better-reply security requires players' payoff functions to be suitably approximated from below. In this subsection, we show in Theorem 2 below that there exists a best possible approximation. This means that only one (bounded, $\mathbb{R}^n$ -valued) function needs to be considered when checking for weak better-reply security because it yields the weakest form of weak better-reply security.

The function that characterizes weak better-reply security is defined as follows: For all $i \in N$ and $x_{-i} \in X_{-i}$, let $N(x_{-i})$ denote the set of all open neighborhoods of x_{-i} . Furthermore, for all $i \in N$, $x \in X$ and $U \in N(x_{-i})$, let $W_U(x)$ be the set of all upper hemicontinuous correspondences $\varphi_i : U \rightrightarrows X_i$ with nonempty, convex and closed values that satisfy $x \in \text{graph}(\varphi_i)$. For all $i \in N$ and $x \in X$, define

$$\underline{u}_i^u(x) = \sup_{U \in N(x_{-i})} \sup_{\varphi_i \in W_U(x)} \inf_{z \in \text{graph}(\varphi_i)} u_i(z).$$

Theorem 2 shows that weak better-reply security relative to $\underline{u}^u$ is the minimal form of weak better-reply security. Furthermore, it shows that weak better-reply

security relative to $\underline{u}^u$ amounts to better-reply closedness relative to $\underline{u}^u$.

Theorem 2 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. Then, the following conditions are equivalent:*

1. G is weakly better-reply secure.
2. G is weakly better-reply secure relative to $\underline{u}^u$.
3. G is better-reply closed relative to $\underline{u}^u$.

Theorem 2 is obtained from Lemma 4 below. This lemma shows that $\underline{u}^u$ approximates u from below and yields a generalized payoff secure game $\underline{G}^u = (X_i, \underline{u}_i^u)_{i \in N}$. Furthermore, Lemma 4 also shows that the value function $\underline{v}_i^u$ of $\underline{u}_i^u$, $i \in N$, is the pointwise supremum of the set of value functions $\underline{v}_i$ defined from the functions $\underline{u}$ that approximate u from below and yield generalized payoff secure games. In other words, if $\underline{u}$, like $\underline{u}^u$, approximates u from below and yields a generalized payoff secure game, then, for all $i \in N$, the value function $\underline{v}_i$ of $\underline{u}_i$ is (everywhere) below the value function $\underline{v}_i^u$ of $\underline{u}_i^u$. The following notation is used in its statement. Let $G = (X_i, u_i)_{i \in N}$ be a game and define $L(G)$ as the set of all bounded functions $\underline{u} : X \rightarrow \mathbb{R}^n$ such that $\underline{u}_i \leq u_i$ for all $i \in N$ and $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is generalized payoff secure.

Lemma 4 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. Then, $\underline{u}^u \in L(G)$ and $\underline{v}_i^u(x_{-i}) \geq \sup_{x_i \in X_i} \underline{u}_i(x_i, x_{-i})$ for all $\underline{u} \in L(G)$, $i \in N$ and $x_{-i} \in X_{-i}$.*

Proof. We first show that $\underline{u}^u \in L(G)$. Since u is bounded, so is $\underline{u}^u$. Note that $u_i(x) \geq \underline{u}_i^u(x)$ for all $i \in N$ and $x \in X$. In fact, for all $U \in N(x_{-i})$ and $\varphi_i \in W_U(x_i, x_{-i})$, we have that $x \in \text{graph}(\varphi_i)$. Thus, $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) \leq u_i(x)$ and so $\underline{u}_i^u(x) \leq u_i(x)$.

We next show that $\underline{u}_i^u(\cdot, x_{-i})$ is quasiconcave for all $i \in N$ and $x_{-i} \in X_{-i}$. Let $\alpha \in \mathbb{R}$, $x_i, x'_i \in \{y_i \in X_i : \underline{u}_i^u(y_i, x_{-i}) > \alpha\}$ and $\lambda \in (0, 1)$. Then, there exist $U, U' \in N(x_{-i})$, $\varphi_i \in W_U(x_i, x_{-i})$ and $\varphi'_i \in W_{U'}(x'_i, x_{-i})$ such that $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \alpha$ and $\inf_{z \in \text{graph}(\varphi'_i)} u_i(z) > \alpha$.

Define $\bar{x}_i = \lambda x_i + (1 - \lambda)x'_i$, $\bar{U} = U \cap U'$ and $\bar{\varphi}_i = \lambda\varphi_i + (1 - \lambda)\varphi'_i$ in $\bar{U}$. It follows that $\bar{U}$ is an open neighborhood of x_{-i} and $\bar{x}_i \in \bar{\varphi}_i(x_{-i})$. Furthermore, it is easy to see that $\bar{\varphi}_i$ is upper hemicontinuous with nonempty, convex and closed values. Hence, $\bar{U} \in N(x_{-i})$ and $\bar{\varphi}_i \in W_{\bar{U}}(\bar{x}_i, x_{-i})$.

Let $\bar{z} \in \text{graph}(\bar{\varphi}_i)$. Then, there exist $z_i \in \varphi_i(\bar{z}_{-i})$ and $z'_i \in \varphi'_i(\bar{z}_{-i})$ such that $\bar{z}_i = \lambda z_i + (1 - \lambda)z'_i$. The quasiconcavity of $u_i(\cdot, \bar{z}_{-i})$ implies that

$$u_i(\bar{z}) \geq \min\{u_i(z_i, \bar{z}_{-i}), u_i(z'_i, \bar{z}_{-i})\} \geq \min\left\{\inf_{y \in \text{graph}(\varphi_i)} u_i(y), \inf_{y \in \text{graph}(\varphi'_i)} u_i(y)\right\} > \alpha.$$

Hence, $\underline{u}_i^u(\bar{x}_i, x_{-i}) \geq \inf_{z \in \text{graph}(\bar{\varphi}_i)} u_i(z) > \alpha$. Thus, $\bar{x}_i = \lambda x_i + (1 - \lambda)x'_i \in \{y_i \in X_i : \underline{u}_i^u(y_i, x_{-i}) > \alpha\}$, proving that $\underline{u}_i^u(\cdot, x_{-i})$ is quasiconcave.

Finally, we show that $\underline{G}^u = (X_i, \underline{u}_i^u)_{i \in N}$ is generalized payoff secure. Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Then, there exists $U \in N(x_{-i})$ and $\varphi_i \in W_U(x)$ such that $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \underline{u}_i^u(x) - \varepsilon$. Then, for all $x' \in \text{graph}(\varphi_i)$, we have that $x'_{-i} \in U$ and $x'_i \in \varphi_i(x'_{-i})$, that is, $U \in N(x'_{-i})$ and $\varphi_i \in W_U(x')$. Thus, $\underline{u}_i^u(x') \geq \inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \underline{u}_i^u(x) - \varepsilon$.

We next establish that $\underline{v}_i^u(x_{-i}) \geq \underline{v}_i(x_{-i})$ for all $\underline{u} \in L(G)$, $i \in N$ and $x_{-i} \in X_{-i}$, where $\underline{v}_i(x_{-i}) = \sup_{x_i \in X_i} \underline{u}_i(x_i, x_{-i})$. Let $\underline{u} \in L(G)$, $i \in N$ and $x_{-i} \in X$ be fixed. Also, let $\varepsilon > 0$ and $x_i \in X_i$ be given. Since $\underline{u} \in L(G)$, then there exists $U \in N(x_{-i})$ and a upper hemicontinuous correspondence $\varphi_i : U \rightrightarrows X_i$ with nonempty, convex and closed values such that $\underline{u}_i(z) \geq \underline{u}_i(x) - \varepsilon$ for all $z \in \text{graph}(\varphi_i)$. Let $x'_i \in \varphi_i(x_{-i})$ and note that $\varphi_i \in W_U(x'_i, x_{-i})$. Since $u_i \geq \underline{u}_i$, then

$$\underline{v}_i^u(x_{-i}) \geq \underline{u}_i^u(x'_i, x_{-i}) \geq \inf_{z \in \text{graph}(\varphi_i)} u_i(z) \geq \inf_{z \in \text{graph}(\varphi_i)} \underline{u}_i(z) \geq \underline{u}_i(x) - \varepsilon$$

and so $\underline{v}_i^u(x) \geq \underline{u}_i(x) - \varepsilon$. Since $\varepsilon > 0$ and x_i are arbitrary, then $\underline{v}_i^u(x_{-i}) \geq \underline{v}_i(x_{-i})$. ■

Theorem 2 follows easily from Lemma 4.

Proof of Theorem 2. It is clear that condition 2 implies both condition 1 and condition 3. Hence, it suffices to show that condition 2 is implied by each one of the two remaining conditions.

Suppose that G is weakly better-reply secure relative to $\underline{u}$. Then, G is generalized payoff secure relative to $\underline{u}$ and so $\underline{G} = (X_i, \underline{u})_{i \in N}$ is generalized payoff secure. Since

$\underline{u}^u \in L(G)$, it follows that $\underline{G}^u = (X_i, \underline{u}_i^u)_{i \in N}$ is generalized payoff secure and so G is generalized payoff secure relative to $\underline{u}^u$. Furthermore, $\underline{u}^u \in L(G)$ also implies that $\underline{u}_i^u \leq u_i$ for all $i \in N$ and, by Lemma 4, we have that $\underline{v}_i \leq \underline{v}_i^u$ for all $i \in N$. Hence, if $(x^*, u^*) \in \Gamma$ is such that $u_i^* \geq \underline{v}_i^u(x_{-i}^*)$ for all $i \in N$, then $u_i^* \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and so x^* is a Nash equilibrium of G since G is better-reply closed relative to $\underline{u}$. Thus, G is weakly better-reply secure relative to $\underline{u}^u$.

Suppose that G is better-reply closed relative to $\underline{u}^u$. Since $\underline{u}^u \in L(G)$, it follows that G is generalized payoff secure relative to $\underline{u}^u$ and thus that G is weakly better-reply secure relative to $\underline{u}^u$. ■

3.3 Corollaries of the general existence result

When players' action spaces are metric and locally convex, Theorem 1 allows us to obtain the existence results of Reny (1999, Theorem 3.1), Carmona (2009, Corollary 2) and Barelli and Soza (2009, Corollary 6.8). In fact, we show that all of these results provide sufficient conditions for a game to be weakly better reply-secure relative to a particular choice of a bounded $\mathbb{R}^n$ -function $\underline{u}$ on X .

The result in Barelli and Soza (2009) is the most general of the above results. It considers generalized better-reply secure games, which are defined as follows. A game $G = (X_i, u_i)_{i \in N}$ is *generalized better-reply secure* if whenever $(x^*, u^*) \in \Gamma$ and x^* is not a Nash equilibrium, there exists a player $i \in N$, an open neighborhood U of x_{-i}^* , an upper hemicontinuous correspondence $\varphi_i : U \rightrightarrows X_i$ with nonempty, closed and convex values, and a number $\alpha_i > u_i^*$ such that $u_i(x') \geq \alpha_i$ for all $x' \in \text{graph}(\varphi_i)$. Theorem 3 shows that generalized better-reply security is equivalent to weak better-reply security relative to $\underline{u}^u$ and, by Theorem 2, equivalent to weak better-reply security.

Theorem 3 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. Then, G is weakly better-reply secure relative to $\underline{u}^u$ if and only if G is generalized better-reply secure.*

Proof. (Necessity) Suppose that G is weakly better-reply secure relative to $\underline{u}^u$ and let $(x^*, u^*) \in \Gamma$ be such that x^* is not a Nash equilibrium. Since G is better-

reply closed relative to $\underline{u}^u$, it follows that there is $i \in N$ such that $\underline{v}_i^u(x_{-i}^*) > u_i^*$. Hence, there exists $\alpha_i > u_i^*$ and $x_i \in X_i$ such that $\underline{u}_i^u(x_i, x_{-i}^*) > \alpha_i$. This in turn implies that there exist $U \in N(x_{-i}^*)$ and $\varphi_i \in W_U(x_i, x_{-i}^*)$ such that $u_i(z) > \alpha_i$ for all $z \in \text{graph}(\varphi_i)$. Thus, G is generalized better-reply secure.

(Sufficiency) Since $\underline{u}^u \in L(G)$, it follows that $u_i \geq \underline{u}_i^u$ for all $i \in N$ and that G is generalized payoff secure relative to $\underline{u}^u$. Consider $(x^*, u^*) \in \Gamma$ such that $\underline{v}_i^u(x_{-i}^*) \leq u_i^*$ for all $i \in N$, and, in order to reach a contradiction, suppose that x^* is not a Nash equilibrium of G . By generalized better-reply security, there exist $i \in N$, $U \in N(x_{-i}^*)$, an upper hemicontinuous correspondence $\varphi_i : U \rightrightarrows X_i$ with nonempty, convex and closed values and $\alpha_i > u_i^*$ such that $u_i(z) \geq \alpha_i$ for all $z \in \text{graph}(\varphi_i)$. Let $x_i \in \varphi_i(x_{-i}^*)$. Then, $\varphi_i \in W_U(x_i, x_{-i}^*)$ and so $\underline{v}_i^u(x_{-i}^*) \geq \underline{u}_i^u(x_i, x_{-i}^*) \geq \inf_{z \in \text{graph}(\varphi_i)} u_i(z) \geq \alpha_i > u_i^*$. This is a contradiction. Hence, x^* is a Nash equilibrium of G . Thus, G is better-reply closed relative to $\underline{u}^u$ and so G is weakly better-reply secure relative to $\underline{u}^u$. ■

Theorem 3 allow us to obtain Corollary 6.8 in Barelli and Soza (2009) immediately from Theorem 1. In fact, these two results are equivalent.

Corollary 1 (Barelli and Soza) *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave and generalized better-reply secure, then it has a Nash equilibrium.*

Reny's existence theorem establishes the existence of Nash equilibria in better-reply games, which are defined as follows. A game $G = (X_i, u_i)_{i \in N}$ is *better-reply secure* if whenever $(x^*, u^*) \in \Gamma$ and x^* is not a Nash equilibrium, there exists a player $i \in N$, a strategy $\bar{x}_i \in X_i$, an open neighborhood U of x_{-i}^* and a number $\alpha_i > u_i^*$ such that $u_i(\bar{x}_i, x'_{-i}) \geq \alpha_i$ for all $x'_{-i} \in U$.

Better-reply security can be characterized in terms of weak better-reply security relative to a particular function $\underline{u}$. Furthermore, as Theorem 4 below shows, Theorem 3.1 in Reny (1999) is the best existence result based on weak better-reply security under the requirement that the approximating games be payoff secure.

Given a compact and quasiconcave game $G = (X_i, u_i)_{i \in N}$, let $\underline{u}^p : X \rightarrow \mathbb{R}^n$ be defined by

$$\underline{u}_i^p(x) = \sup_{U \in N(x_{-i})} \inf_{z_{-i} \in U} u_i(x_i, z_{-i})$$

for all $i \in N$ and $x \in X$. Also, let $L^p(G)$ be the set of all bounded functions $\underline{u} : X \rightarrow \mathbb{R}^n$ such that $\underline{u}_i \leq u_i$ for all $i \in N$ and $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is payoff secure. Analogously to Lemma 4, we have that $\underline{u}^p \in L^p(G)$ and $\underline{v}_i^p(x_{-i}) \geq \sup_{x_i \in X_i} \underline{u}_i(x_i, x_{-i})$ for all $\underline{u} \in L^p(G)$, $i \in N$ and $x_{-i} \in X_{-i}$. In fact, letting, for all $i \in N$, $x \in X$ and $U \in N(x_{-i})$, $W_U^p(x)$ be the set of all singleton-valued constant correspondences $\varphi_i : U \rightrightarrows X_i$ satisfying $x \in \text{graph}(\varphi_i)$, we can write

$$\underline{u}_i^p(x) = \sup_{U \in N(x_{-i})} \sup_{\varphi_i \in W_U^p(x)} \inf_{z \in \text{graph}(\varphi_i)} u_i(z)$$

for all $i \in N$ and $x \in X$ (since $W_U^p(x) = \{\hat{\varphi}_i\}$, where $\hat{\varphi}_i \equiv \{x_i\}$). The conclusion then follows from (a simple adaptation of) the arguments used in the proof of Lemma 4.

Theorem 4 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. Then, G is weakly better-reply secure relative to $\underline{u}^p$ if and only if G is better-reply secure.*

Proof. (Necessity) Suppose that G is weakly better-reply secure relative to $\underline{u}^p$ and let $(x^*, u^*) \in \Gamma$ be such that x^* is not a Nash equilibrium. Since G is better-reply closed relative to $\underline{u}^p$, it follows that there is $i \in N$ such that $\underline{v}_i^p(x_{-i}^*) > u_i^*$. Hence, there exists $\alpha_i > u_i^*$ and $x_i \in X_i$ such that $\underline{u}_i^p(x_i, x_{-i}^*) > \alpha_i$. This in turn implies that there exist a $U \in N(x_{-i}^*)$ such that $u_i(x_i, z_{-i}) > \alpha_i$ for all $z_{-i} \in U$. Thus, G is better-reply secure.

(Sufficiency) Since $\underline{u}^p \in L^p(G)$, it follows that $u_i \geq \underline{u}_i^p$ for all $i \in N$ and that G is generalized payoff secure relative to $\underline{u}^p$ (in fact, $\underline{G}^p = (X_i, \underline{u}_i^p)_{i \in N}$ is generalized payoff secure since $W_U^p(x) \subseteq W_U(x)$ for all $i \in N$, $x \in X$ and $U \in N(x_{-i})$). Consider $(x^*, u^*) \in \Gamma$ such that $\underline{v}_i^p(x_{-i}^*) \leq u_i^*$ for all $i \in N$, and, in order to reach a contradiction, suppose that x^* is not a Nash equilibrium of G . By better-reply security, there exist $i \in N$, $U \in N(x_{-i}^*)$, $x_i \in X_i$ and $\alpha_i > u_i^*$ such that $u_i(x_i, z_{-i}) \geq \alpha_i$ for all $z_{-i} \in U$. Thus, $\underline{v}_i^p(x_{-i}^*) \geq \underline{u}_i^p(x_i, x_{-i}^*) \geq \inf_{z_{-i} \in U} u_i(x_i, z_{-i}) \geq \alpha_i > u_i^*$, a contradiction. Hence, x^* is a Nash equilibrium of G . Thus, G is better-reply closed relative to $\underline{u}^p$ and so G is weakly better-reply secure relative to $\underline{u}^p$. ■

Theorem 4 allows us to obtain Reny's Theorem from Theorem 1.

Corollary 2 (Reny) *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave and better-reply secure, then it has a Nash equilibrium.*

Some existence results, such as Corollaries 3 and 4 below, are particularly simple because there is no need to approximate the original game, i.e., we can have $\underline{u} = u$. Theorem 5 below characterizes those games $G = (X_i, u_i)_{i \in N}$ that are weakly better-reply secure relative to u . This result is then used to obtain the existence result of Carmona (2009) and a new existence result based on the ideas of Radzik (1991), Ziad (1997) and Carmona (2010). Part of the characterization relies on the notion of weak upper semicontinuity introduced by Bagh and Jofre (2006). A game $G = (X_i, u_i)_{i \in N}$ is *weakly reciprocal upper semicontinuous* if for all (x, α) in the frontier of the graph of u , there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > \alpha_i$. Moreover, for all $x \in X$, we say that G is *weakly reciprocal upper semicontinuous at x* if for all $\alpha \in \mathbb{R}^n$ such that (x, α) is in the frontier of the graph of u , there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > \alpha_i$. Note that, clearly, G is weakly reciprocal upper semicontinuous if and only if it is weakly reciprocal upper semicontinuous at x for all $x \in X$.

Theorem 5 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. Then, G is weakly better-reply secure relative to u if and only if G is generalized payoff secure and weakly reciprocal upper semicontinuous at x^* for all x^* that are not Nash equilibria of G .*

Proof. By definition, G is weakly better-reply secure relative to u if and only if G is generalized payoff secure and better-reply closed relative to u . Thus, it suffices to show that G is better-reply closed relative to u if and only if G is weakly reciprocal upper semicontinuous at x^* for all x^* that are not Nash equilibria of G .

Suppose that G is better-reply closed relative to u . Let (x^*, u^*) be in the frontier of $\text{graph}(u)$ and be such that x^* is not a Nash equilibrium of G . Then, in particular, $(x^*, u^*) \in \Gamma$ and better-reply closedness relative to u implies that there is $i \in N$ such that $v_i(x_{-i}^*) > u_i^*$. Hence, there is an $x_i \in X_i$ such that $u_i(x_i, x_{-i}^*) > u_i^*$. Thus, G is weak reciprocally upper semicontinuous at x^* .

Conversely, suppose that G is weakly reciprocal upper semicontinuous at x^* for all x^* that are not Nash equilibria of G . Let $(x^*, u^*) \in \Gamma$ be such that $u_i^* \geq v_i(x_{-i}^*)$ for all $i \in N$. If x^* is not an equilibrium, then there is a $j \in N$ such that $v_j(x_{-j}^*) > u_j(x^*)$ and so $u_j^* > u_j(x^*)$. This implies that (x^*, u^*) belongs to the frontier of $\text{graph}(u)$. Since G is weak reciprocally upper semicontinuous at x^* , then there is an $i \in N$ such that $u_i^* < v_i(x_{-i}^*)$, a contradiction. Hence, x^* is a Nash equilibrium and G better-reply closed relative to u . ■

Carmona (2009) established an existence result independent of that of Reny (1999) and which is valid for games that are weakly upper semicontinuous and weakly payoff secure. Formally, a game $G = (X_i, u_i)_{i \in N}$ is *weakly payoff secure* if v_i is lower semicontinuous for all $i \in N$. Furthermore, G is *weakly upper semicontinuous* if, defining $\bar{u} : X \times X \rightarrow \mathbb{R}^n$ by $\bar{u}(x, y) = (u_1(x_1, y_{-1}), \dots, u_n(x_n, y_{-n}))$, then for all (x, y, α) in the frontier of the graph of $\bar{u}$, there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, y_{-i}) > \alpha_i$.

The result in Carmona (2009) is obtained from Theorem 1 by using $\underline{u} = u$, that is, by combining Theorem 1 and Theorem 5. In fact, in compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure games, the best-reply correspondence is upper hemicontinuous with nonempty, convex and closed values and it can be used to secure players' payoffs.

Corollary 3 (Carmona) *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure, then it has a Nash equilibrium.*

Proof. We first show that G is weakly reciprocal upper semicontinuous. Let (x^*, u^*) be in the frontier of the graph of u . Then, for some $i \in N$, (x^*, u_i^*) belongs to the closure of the graph of u_i and it follows from Carmona (2009, Theorem 1) that there exists $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}^*) > u_i^*$.

Finally, we show that G is generalized payoff secure. Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Let $BR_i : X_{-i} \rightrightarrows X_i$ be defined by $BR_i(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) = v_i(x_{-i})\}$ for all $x_{-i} \in X_{-i}$. Since v_i is lower semicontinuous, let $U \in N(x_{-i})$ be such that $v_i(x'_{-i}) > v_i(x_{-i}) - \varepsilon$ and define $\varphi_i(x'_{-i}) = BR_i(x'_{-i})$ for all $x'_{-i} \in U$. Then, φ_i is an upper hemicontinuous correspondence with nonempty, closed and convex values

(Carmona (2009, Theorem 1)) and, for all $x' \in \text{graph}(\varphi_i)$, we have that $u_i(x') = v_i(x'_{-i}) > v_i(x_{-i}) - \varepsilon \geq u_i(x) - \varepsilon$. ■

Theorem 1 also allows us to obtain an existence result using some ideas from Radzik (1991), Ziad (1997) and Carmona (2010). We say that a game $G = (X_i, u_i)_{i \in N}$ is *weakly continuous* if v_i is continuous for all $i \in N$. The following notions strengthen the usual quasiconcavity assumption. A game $G = (X_i, u_i)_{i \in N}$ is *locally joint quasiconcave* if for all $i \in N$ and $x \in X$, X_i is a polytope (and so a subset of an Euclidean space) and there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} such that u_i is quasiconcave in $X_i \times V_{x_{-i}}$. The upper semicontinuity of each player's value function can be obtained by strengthening the previous notion, as follows. A game $G = (X_i, u_i)_{i \in N}$ is *strongly quasiconcave* if G is locally joint quasiconcave and for all $i \in N$ and $x_{-i} \in X_{-i}$, there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} such that v_i is polyhedral quasiconcave in $V_{x_{-i}}$ (i.e., for all $\alpha \in \mathbb{R}$, there exists a polytope P such that $\{x'_{-i} \in V_{x_{-i}} : v_i(x'_{-i}) \geq \alpha\} = P \cap V_{x_{-i}}$).

Corollary 4 *If $G = (X_i, u_i)_{i \in N}$ is compact, locally joint quasiconcave, better-reply closed relative to u and weakly continuous, then it has a Nash equilibrium.*

In particular, if $G = (X_i, u_i)_{i \in N}$ is compact, strongly quasiconcave, weakly reciprocal upper semicontinuous and weakly payoff secure, then it has a Nash equilibrium.

Proof. Clearly, since G is locally joint quasiconcave, then G is quasiconcave. Thus, it suffices to show that G is generalized payoff secure.

Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Let $0 < \eta < \varepsilon/2$ and, for all $\alpha > 0$, let $B_\alpha(x_{-i})$ denote the open ball of radius α around x_{-i} with respect to the sup norm of the Euclidean space containing X_{-i} . In particular, note that the closure of $B_\alpha(x_{-i})$ is a polytope.

Since v_i is continuous and u_i is locally quasiconcave, there is $\delta > 0$ such that u_i is quasiconcave in $X_i \times (X_{-i} \cap B_\delta(x_{-i}))$ and $|v_i(x'_{-i}) - v_i(\hat{x}_{-i})| < \eta$ for all $x'_{-i}, \hat{x}_{-i} \in X_{-i} \cap B_\delta(x_{-i})$. Let $U = X_{-i} \cap B_{\delta/2}(x_{-i})$ and note that $P := X_{-i} \cap \overline{B_{\delta/2}(x_{-i})}$ is contained in $X_{-i} \cap B_\delta(x_{-i})$.

Since P is a polytope, let $x_{-i}^0, \dots, x_{-i}^m \in P$ be such that $P = \text{co}(\{x_{-i}^0, \dots, x_{-i}^m\})$.

For convenience, let $\bar{x}_{-i} = (x_{-i}^0, \dots, x_{-i}^m)$. Consider the correspondence $\Theta_i : U \rightrightarrows \Delta_m$ defined by $\Theta_i(z_{-i}) = \{\theta \in \Delta_m : z_{-i} = \theta \cdot \bar{x}_{-i}\}$ for all $z_{-i} \in U$, where $\Delta_m = \{\theta \in \mathbb{R}^{m+1} : \sum_{j=0}^m \theta_j = 1 \text{ and } \theta_j \geq 0 \text{ for all } j\}$.

For all $j \in \{0, \dots, m\}$, let $x_i^j \in X_i$ be such that $u_i(x_i^j, x_{-i}^j) > v_i(x_{-i}^j) - \varepsilon/2 + \eta$ and let $\bar{x}_i = (x_i^0, \dots, x_i^m)$. Define $\varphi_i : U \rightrightarrows X_i$ by $\varphi_i(z_{-i}) = \{\theta \cdot \bar{x}_i : \theta \in \Theta_i(z_{-i})\}$ for all $z_{-i} \in U$. It is easy to show that φ_i is upper hemicontinuous and has nonempty, closed and convex values.

We claim that $v_i(x'_{-i}) - \varepsilon/2 < u_i(x'_i, x'_{-i})$ for all $x'_i \in \varphi_i(x'_{-i})$ and $x'_{-i} \in U$. If not, then $v_i(x'_{-i}) - \varepsilon/2 \geq u_i(x'_i, x'_{-i})$ for some $x' \in \text{graph}(\varphi_i)$. Since $x'_i \in \varphi_i(x'_{-i})$, then $(x'_i, x'_{-i}) = \theta \cdot (\bar{x}_i, \bar{x}_{-i})$ for some $\theta \in \Delta_m$. The quasiconcavity of u_i in $X_i \times (X_{-i} \cap B_\delta(x_{-i}))$ implies that

$$u_i(x'_i, x'_{-i}) = u_i(\theta \cdot (\bar{x}_i, \bar{x}_{-i})) \geq \min_j u_i(x_i^j, x_{-i}^j) > \min_j v_i(x_{-i}^j) - \varepsilon/2 + \eta.$$

Hence, $v_i(x'_{-i}) > \min_j v_i(x_{-i}^j) + \eta$, a contradiction since $x'_{-i}, x_{-i}^j \in B_\delta(x_{-i})$ for all j .

Thus, $\varphi_i : U \rightrightarrows X_i$ is an upper hemicontinuous correspondence with nonempty, closed and convex values and, for all $x' \in \text{graph}(\varphi_i)$, we have that $u_i(x') > v_i(x'_{-i}) - \varepsilon/2 > v_i(x_{-i}) - \varepsilon \geq u_i(x) - \varepsilon$. Hence, G is generalized payoff secure.

Finally, note that if G is strongly quasiconcave, then v_i is upper semicontinuous for all $i \in N$. Furthermore, it follows from Theorem 5 that if G is weakly reciprocal better-reply secure, then G is better-reply closed relative to u . Hence, all compact, strongly quasiconcave, weakly reciprocal upper semicontinuous and weakly payoff secure games have a Nash equilibrium. ■

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