Computation of the dominant Lyapunov exponent via spatial integration using matrix norms

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In a previous paper (Aston, P. J. & Dellnitz, M. 1999 *Comput. Meth. Appl. Mech. Engng* **170**, 223–237) we introduced a new method for computing the dominant Lyapunov exponent of a chaotic map by using spatial integration involving a matrix norm. We conjectured that this sequence of integrals decayed proportional to $1/n$. We now prove this conjecture and derive a bound on the next term in the asymptotic expansion of the terms in the sequence. The Hénon map and a system of coupled Duffing oscillators are explored in detail in the light of these theoretical results.

Keywords: Lyapunov exponents; invariant measure;
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1. Introduction

When considering chaotic systems, various averaged quantities are used to describe particular properties of the system. One such quantity is the dominant Lyapunov exponent which characterizes the average rate of growth of differences between two nearby orbits. A more recent important application, which involves the dominant Lyapunov exponent, is the phenomenon of *blowout bifurcations* in coupled chaotic oscillators (Ashwin *et al.* 1996; Ott & Sommerer 1994; Yamada & Fujisaka 1984). For such a bifurcation to occur, a chaotic synchronized state of the system must lose stability to non-synchronized motion. The stability of the chaotic synchronized state is determined by the sign of the dominant *normal Lyapunov exponent* which is associated with perturbations normal to the synchronous subspace. If all the normal Lyapunov exponents are negative, then there is a positive measure set of initial conditions in a neighbourhood of the synchronous subspace which is in the basin of attraction of the chaotic attractor contained within this subspace (Alexander *et al.* 1992).

For the chaotic iterated map

$$x_{n+1} = g(x_n), \quad g : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell,$$

the dominant Lyapunov exponent can be calculated in two ways, using either a vector norm or a matrix norm. In this paper, we are interested in computations using the

matrix norm. In a later paper, we will consider the vector norm case (Aston & Dellnitz 2003). Using the matrix norm, the dominant Lyapunov exponent is given by

$$\lambda_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Dg^n(x)\| \quad (1.1)$$

for μ -almost all x , where $Dg^n(x)$ is the Jacobian matrix of the n times iterated map $g^n(x)$ and μ is an invariant measure associated with the chaotic attractor of the system.

This time-averaging approach to the computation of λ_1 is used very widely since it is an easy formula to use. However, in practice, time averaging for chaotic systems has a number of pitfalls.

- (i) Convergence is very slow for even modest accuracy.
- (ii) You can never be sure whether the time average has converged or not. There are examples of systems where the average appears to settle down to the wrong value but with longer computation times, the correct value is obtained. This may occur in systems which have so-called *almost invariant sets* (see Dellnitz & Junge 1999). An example illustrating this situation can be found in Deuffhard *et al.* (1998). Similar results would be obtained for the computation of Lyapunov exponents.
- (iii) The time required to obtain an accurate value varies considerably for different values of x and, of course, there is no easy way to know what is a good value of x to choose.
- (iv) The formula (1.1) holds only for μ -almost all x . It may be that the value of x chosen corresponds to an unstable periodic orbit and so a different value of λ_1 is computed. If the periodic orbit cannot be computed exactly due to rounding errors, or if the initial condition is chosen very close to an unstable periodic point, then convergence may be very slow.
- (v) It is impossible to compute long orbits of a chaotic map accurately. Shadowing results can be applied in some cases but there are systems where shadowing breaks down.

Since there are so many drawbacks with time averaging, the obvious alternative is to use an integration approach. The Birkhoff ergodic theorem (Pollicott 1993) provides a simple formula for converting a time average to an integral (or spatial average), and states that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(x_k) = \int \phi(x) d\mu, \quad (1.2)$$

for any integrable function ϕ and for μ -almost all x . For a scalar map, the Lyapunov exponent is given by

$$\lambda_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \log |(g^n)'(x_0)|$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |g'(x_k)| \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |g'(x_k)| \\
&= \int \log |g'(x)| d\mu,
\end{aligned}$$

where prime denotes differentiation, and using (1.2). Thus, there is a simple integral formula for the Lyapunov exponent of a scalar map. However, for a higher-dimensional map,

$$Dg^n(x_0) = \prod_{k=0}^{n-1} Dg(x_k),$$

and in this case it is not possible to write the time-average formula (1.1) as a sum to which the Birkhoff ergodic theorem can be applied. Thus, there is no simple integral formula in this case.

There have been few previous attempts at computing the dominant Lyapunov exponent using the invariant measure. The first attempt was by Kim & Hsu (1986) based on a cell-mapping approach. They computed the invariant measure in a similar way to us, but using a uniform grid and without the sophistication of our subdivision algorithm. They then computed approximate Lyapunov vectors which were used to compute the dominant Lyapunov exponent. Froyland *et al.* (1993, 1995) also computed the invariant measure and the Lyapunov vectors and used these to approximate the Lyapunov exponents for the Hénon map. However, in this case, they tried to generate several vectors for each cell in order to get better results. Neither of these approaches used integration with respect to the invariant measure.

Our approach is to consider the sequence of integrals defined by

$$a_n = \frac{1}{n} \int \log \|Dg^n(x)\| d\mu.$$

It is easily verified that this sequence converges to λ_1 (see theorem 3.1) and we study the convergence of this sequence in detail. We note that this sequence bears a close resemblance to the time-average formula (1.1). Indeed, the terms in the sequence are essentially the same as the terms in the time-average formula but integrated with respect to the invariant measure. Thus, instead of taking one initial condition and a large number of terms in the sequence, our approach is to average over all initial conditions on the attractor and take only a small number of terms in the sequence. In this way, we can obtain more accurate results than time averaging at less computational cost when moderate to high accuracy is required. Moreover, this method avoids *all* the pitfalls of time averaging which we described earlier. Our method is also much more efficient than time averaging when working out the so-called *normal Lyapunov exponents* associated with a blowout bifurcation since the invariant measure of the chaotic attractor only has to be computed once and then the integrals are computed for each different value of the coupling parameter. We give an example of two coupled Duffing oscillators to illustrate this approach in § 5.

Of course, our approach also has its drawbacks. The main restriction is that the method is suitable only for systems with a low-dimensional attractor, as the number of boxes required to compute the invariant measure for a high-dimensional attractor make the method infeasible in this case. There may also be cases where the numerical computation of the invariant measure is problematic, such as where natural invariant measures are not Sinai–Ruelle–Bowen (SRB) or do not even exist. Of course, these latter scenarios are also likely to cause difficulties for the time-averaging approach as well.

In a previous paper (Aston & Dellnitz 1999) the first few terms of the sequence of integrals for the Hénon map were computed. The sequence was found to converge very slowly and many terms in the sequence were required in order to obtain even modest accuracy, at which point the numerical process started to break down. Thus, the method appeared not to be a useful one in practice. However, it was observed that the sequence appeared to converge to first order proportional to $1/n$ and, based on that assumption, other sequences were derived which converged to λ_1 much faster. The purpose of this paper is to *prove* results about the convergence of the sequence, thus providing a sound theoretical basis for the methods described in Aston & Dellnitz (1999).

Other approaches to computing Lyapunov exponents have been tried. One approach based on the computation of periodic orbits (Artuso *et al.* 1990; Cvitanovic 1995) works very well for nice hyperbolic systems but does not converge very quickly for other systems. An alternative approach, again using periodic orbits, has been applied to expanding maps of the interval which have an absolutely continuous invariant measure (Pollicott & Jenkinson 2000). In such cases, convergence of the approximation is super-exponentially fast. Our approach is more general than these and will in principle work for any low-dimensional dissipative map. It is also particularly useful in the context of blowout bifurcations.

In §2 the numerical method for computing attractors and invariant measures is briefly reviewed. Section 3 contains a review of results related to the dominant Lyapunov exponent and associated Lyapunov vectors. In §4, new results on the rate of convergence of a sequence of integrals to the dominant Lyapunov exponent are obtained. Upper bounds are also obtained for the rate of convergence of faster converging sequences. Numerical results are obtained in §5 for the Hénon map to illustrate the theory and the method is applied to a blowout bifurcation problem in a system of two coupled Duffing oscillators in §6.

2. Computation of the invariant measure

Our aim is to use integration with respect to the invariant measure for chaotic systems to evaluate the dominant Lyapunov exponent. The first step of this process is necessarily to compute an approximation to the invariant measure. The methods for doing this have been described in detail in other papers and so here we give only a brief summary of the basic approach with references where more details can be found. There are essentially two steps to this process.

Computing a box covering of the attractor. Before finding the invariant measure for an attractor it is necessary to find the attractor itself. The methods that are used here to find a box covering of the attractor are the subdivision algorithm

developed by Dellnitz & Hohmann (1997) and its adaptive version by Dellnitz & Junge (1998). Roughly speaking, the algorithms start with a coarse box covering of the attractor and determine which boxes cannot contain part of the global attractor. These boxes are then discarded and the remaining boxes are subdivided into smaller boxes. This process is repeated until a sufficiently fine covering of the attractor is obtained.

Computing the invariant measure. Given a box covering of the attractor, the Perron–Frobenius operator is discretized on this box covering with the variables being the total measure per box. The discretization gives a matrix eigenvalue problem from which an approximation of the invariant measure is found from the eigenvector associated with the eigenvalue +1. Details on the computation of invariant measures can be found in Dellnitz & Junge (1999).

It is important to emphasize that *none of the computations that we do involves long orbits of a chaotic map* but requires only the computation of the first image of each box, which can be calculated accurately.

3. The dominant Lyapunov exponent and Lyapunov vectors

Consider the iterated map

$$x_{n+1} = g(x_n), \quad g : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell.$$

We assume that for a positive (Lebesgue) measure set of initial conditions, this map has an ω -limit set consisting of a chaotic attractor with a natural ergodic invariant measure μ , an *SRB-measure*, say. The invariance of the measure is characterized by the property

$$\mu(g^{-1}(B)) = \mu(B) \text{ for all (measurable) subsets } B \subset \mathbb{R}^\ell.$$

Moreover, the invariant measure is ergodic if $\mu(B)$ is either zero or one for all g -invariant sets $B \subset \mathbb{R}^\ell$.

As we explained in §1, there is no simple integration formula for computing the dominant Lyapunov exponent λ_1 for this map. However, there is a sequence of integrals which converges to λ_1 and it is this sequence which we analyse in detail in the next section. The following result confirms convergence of the sequence. We will then consider the rate of convergence of the sequence.

Theorem 3.1 (Pollicott 1993). *Let g be a diffeomorphism on a compact manifold M and let μ be an ergodic measure. Then the dominant Lyapunov exponent λ_1 is given by*

$$\lambda_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|Dg^n(x)\| \, d\mu = \inf \left\{ \frac{1}{n} \int \log \|Dg^n(x)\| \, d\mu \mid n \geq 1 \right\}.$$

In a previous paper (Aston & Dellnitz 1999), we computed the first few terms of the sequence of integrals for the Hénon map. We noted that the sequence converged very slowly and that to obtain any sort of reasonable accuracy would require many terms in the sequence, at which point the numerical process starts to break down. However, the observation was made that the sequence appeared to converge to first order

proportional to $1/n$ and, based on that assumption, we obtained other sequences which converged much faster to λ_1 . The purpose of this paper is to prove results about the convergence of the sequence, thus providing a sound theoretical basis for the methods described in Aston & Dellnitz (1999).

The method that we use for computing the dominant Lyapunov exponent by evaluating a sequence of integrals does not require any knowledge of the Lyapunov vectors of the system. However, in the proof of the convergence properties of the sequence, extensive use is made of these vectors and so we now summarize the results which we will require later.

The existence of the Lyapunov vectors is guaranteed by the multiplicative ergodic theorem, which was originally due to Oseledec (1968) but was strengthened by Ruelle (1979) for diffeomorphisms.

Theorem 3.2 (multiplicative ergodic theorem). *Let g be a diffeomorphism on a compact manifold M of dimension ℓ and let μ be an ergodic measure. Then there exist real numbers $\lambda_1 > \lambda_2 > \dots > \lambda_k$ ($k \leq \ell$), positive integers $\ell_1, \dots, \ell_k$ which satisfy $\sum_{i=1}^k \ell_i = \ell$ and a measurable decomposition $T_x M = W^{(1)}(x) \oplus \dots \oplus W^{(k)}(x)$ with $\dim(W^{(i)}(x)) = \ell_i$ and $Dg(W^{(i)}(x)) = W^{(i)}(g(x))$ such that for μ -almost all x*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Dg^n(x)v\| = \lambda_j,$$

provided that $v \in (W^{(j)}(x) \oplus \dots \oplus W^{(k)}(x))$ and $v \notin (W^{(j+1)}(x) \oplus \dots \oplus W^{(k)}(x))$.

The numbers $\lambda_1 > \lambda_2 > \dots > \lambda_k$ are the Lyapunov exponents belonging to the ergodic measure μ . From now on we assume for the sake of convenience that there are ℓ distinct Lyapunov exponents $\lambda_1 > \lambda_2 > \dots > \lambda_\ell$, although our results all hold provided that only the dominant Lyapunov exponent λ_1 is distinct. In this case $\dim(W^{(i)}(x)) = 1$ for $1 \leq i \leq \ell$ and we denote by $w_i(x)$ a normalized vector spanning $W^{(i)}(x)$. These are the Lyapunov vectors that we require and which have the property that

$$Dg(x)w_i(x) = a^{(i)}(x)w_i(g(x)), \quad i = 1, \dots, \ell, \quad (3.1)$$

where $a^{(i)}(x)$ is a scalar function of x . Note that, since g is a diffeomorphism, $a^{(i)}(x) \neq 0$. It can also be shown that (Froyland *et al.* 1993)

$$\lambda_i = \int \log |a^{(i)}(x)| d\mu, \quad i = 1, \dots, \ell. \quad (3.2)$$

As a simple example, consider μ to be a δ Dirac measure associated with a fixed point x^* of g . Then the vectors $w_i(x^*)$ are the normalized eigenvectors of $Dg(x^*)$ while $a^{(i)}(x^*)$ are the corresponding eigenvalues.

As a consequence of the decomposition of $T_x M$ given in the multiplicative ergodic theorem, any fixed vector $v \in \mathbb{R}^\ell$ can be represented in terms of the Lyapunov vector basis as

$$v = \sum_{i=1}^{\ell} \alpha_i(x) w_i(x), \quad (3.3)$$

for some scalars $\alpha_i(x)$, $i = 1, \dots, \ell$, which are measurable functions of x .

4. Rate of convergence

We consider the sequence of integrals given by

$$a_n = \frac{1}{n} \int \log \|Dg^n(x)\| \, d\mu, \quad n \geq 1. \quad (4.1)$$

We know from theorem 3.1 that this sequence converges to its infimum, although the sequence is not necessarily monotonic. We use two approaches to investigate the rate of convergence of this sequence.

The first method involves the use of Lyapunov vectors and can be used to prove that the dominant term in the error $a_n - \lambda_1$ is of the form c_1/n . A bound on the next term in the asymptotic expansion of a_n is also derived.

The second method involves a linear recurrence relation which expresses a_n in terms of a_{n-1} . From this relation, an expression for a_n as a function of n can be derived.

(a) Method 1: working with Lyapunov vectors

The first approach involves working with the columns of the matrix $Dg^n(x)$ and expressing them in terms of the Lyapunov basis vectors. We first define

$$c_1 = \lim_{n \rightarrow \infty} n(a_n - \lambda_1). \quad (4.2)$$

If this limit exists so that c_1 is finite, then the sequence $\{a_n\}_{n=1}^\infty$ has the asymptotic form

$$a_n = \lambda_1 + \frac{c_1}{n} + o\left(\frac{1}{n}\right). \quad (4.3)$$

If $c_1 = \infty$, then convergence is slower than $1/n$. Note that, since the sequence converges to its infimum, c_1 cannot be negative. Thus, our aim is to show that $c_1 \in [0, \infty)$. This will enable us to eliminate the slowly converging c_1/n term in (4.3) to give a much faster converging sequence.

We first define

$$A_n^{(j)}(x) = \prod_{k=0}^{n-1} a^{(j)}(g^k(x)), \quad G_n(x) = \frac{1}{A_n^{(1)}(x)} Dg^n(x), \quad (4.4)$$

where the scalar functions $a^{(i)}(x)$ are defined in (3.1). Thus, $A_n^{(j)}(x)$ is the total expansion or contraction associated with the direction of the Lyapunov vector $w_j(x)$ over n iterations. Note that, since g is a diffeomorphism, $a^{(j)}(x) \neq 0$ for all j and for all $x \in M$. Therefore, $A_n^{(j)}(x) \neq 0$ for all j and for all x .

We now note that

$$\begin{aligned} na_n &= \int \log \|Dg^n(x)\| \, d\mu \\ &= \int \log \left| \prod_{k=0}^{n-1} a^{(1)}(g^k(x)) \right| + \log \|G_n(x)\| \, d\mu \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \int \log |a^{(1)}(g^k(x))| \, d\mu + \int \log \|G_n(x)\| \, d\mu \\
&= n\lambda_1 + \int \log \|G_n(x)\| \, d\mu
\end{aligned}$$

using (3.2) and the invariance of the measure in the final step. Thus,

$$n(a_n - \lambda_1) = \int \log \|G_n(x)\| \, d\mu. \quad (4.5)$$

Hence we must consider the matrices $G_n(x)$ in more detail.

The i th column of $Dg^n(x)$ is $Dg^n(x)e_i$, where e_i is the i th column of the identity matrix. We express the fixed vector e_i in terms of the unit Lyapunov vectors $w_j(x)$ by

$$e_i = \sum_{j=1}^{\ell} \alpha_{ij}(x) w_j(x).$$

Using the relation (3.1) we then find that

$$\begin{aligned}
Dg^n(x)e_i &= \sum_{j=1}^{\ell} \alpha_{ij}(x) \left(\prod_{k=0}^{n-1} a^{(j)}(g^k(x)) \right) w_j(g^n(x)) \\
&= \sum_{j=1}^{\ell} \alpha_{ij}(x) A_n^{(j)}(x) w_j(g^n(x)).
\end{aligned}$$

Dividing through by $A_n^{(1)}(x) \neq 0$ gives

$$G_n(x)e_i = \alpha_{i1}(x) w_1(g^n(x)) + \sum_{j=2}^{\ell} \alpha_{ij}(x) \left(\frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right) w_j(g^n(x)).$$

Combining all the vectors into a matrix, we then obtain

$$G_n(x) = W_{1n}(x) + \sum_{j=2}^{\ell} \left(\frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right) W_{jn}(x), \quad (4.6)$$

where

$$\left. \begin{aligned} W_{jn}(x) &= w_j(g^n(x)) \alpha_j(x)^T, \\ \alpha_j(x)^T &= [\alpha_{1j}(x), \alpha_{2j}(x), \dots, \alpha_{\ell j}(x)], \end{aligned} \right\} \quad j = 1, 2, \dots, \ell.$$

We now consider $\lim_{n \rightarrow \infty} \|G_n(x)\|$. To determine this, we note that

$$\lim_{n \rightarrow \infty} \frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} = 0, \quad j = 2, 3, \dots, \ell, \quad (4.7)$$

for μ -almost all x since λ_1 is supposed to be simple. Now $\lim_{n \rightarrow \infty} W_{1n}(x)$ does not exist in general since it involves $w_1(g^n(x))$. However, $\|w_1(g^n(x))\| = 1$ for all n and so we now show that $\lim_{n \rightarrow \infty} \|W_{1n}(x)\|$ exists.

Proposition 4.1. For μ -almost all x , $\omega_1(x) \equiv \|W_{1n}(x)\|$ exists and is independent of n . In addition,

$$\lim_{n \rightarrow \infty} \|G_n(x)\| = \omega_1(x). \quad (4.8)$$

For particular norms we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|G_n(x)\|_1 &= \|\alpha_1(x)\|_\infty, \\ \lim_{n \rightarrow \infty} \|G_n(x)\|_2 &= \|\alpha_1(x)\|_2, \\ \lim_{n \rightarrow \infty} \|G_n(x)\|_\infty &= \|\alpha_1(x)\|_1. \end{aligned}$$

Proof. From the definition of $W_{jn}(x)$, we have that, for $j = 1, 2, \dots, \ell$,

$$\begin{aligned} \|W_{jn}(x)\| &= \|w_j(g^n(x))\alpha_j(x)^T\| \\ &= \max_{\|v\|=1} \|w_j(g^n(x))\alpha_j(x)^T v\| \\ &= \max_{\|v\|=1} |\alpha_j(x)^T v| \|w_j(g^n(x))\| \\ &= \max_{\|v\|=1} |\alpha_j(x)^T v| \end{aligned}$$

since $\|w_j(g^n(x))\| = 1$ for all n . Thus, $\|W_{jn}(x)\|$ is independent of n as claimed.

We now prove that $\lim_{n \rightarrow \infty} \|G_n(x)\| = \|W_{1n}(x)\| (= \omega_1(x))$. To do this, we note that from (4.6) for μ -almost all x

$$\begin{aligned} \lim_{n \rightarrow \infty} \|G_n(x)\| &= \lim_{n \rightarrow \infty} \left\| W_{1n}(x) + \sum_{j=2}^{\ell} \left(\frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right) W_{jn}(x) \right\| \\ &\leq \|W_{1n}(x)\| + \lim_{n \rightarrow \infty} \sum_{j=2}^{\ell} \left| \frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right| \|W_{jn}(x)\| \\ &= \|W_{1n}(x)\|, \end{aligned}$$

using (4.7) and the above result that $\|W_{jn}(x)\|$ is independent of n . Similarly, for μ -almost all x

$$\begin{aligned} \lim_{n \rightarrow \infty} \|G_n(x)\| &= \lim_{n \rightarrow \infty} \left\| W_{1n}(x) - \sum_{j=2}^{\ell} \left((-1) \frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right) W_{jn}(x) \right\| \\ &\geq \|W_{1n}(x)\| - \lim_{n \rightarrow \infty} \sum_{j=2}^{\ell} \left| \frac{A_n^{(j)}(x)}{A_n^{(1)}(x)} \right| \|W_{jn}(x)\| \\ &= \|W_{1n}(x)\|. \end{aligned}$$

Combining these two inequalities gives the desired result.

The results for particular norms are easily verified either using the formula

$$\|W_{1n}(x)\| = \max_{\|v\|=1} |\alpha_1(x)^T v|,$$

which was derived at the beginning of the proof, or by using the definitions of the different matrix norms. ■

We now consider the rate of convergence of the sequence $\{a_n\}_{n=1}^\infty$.

Theorem 4.2. *If the map g is a diffeomorphism with a uniformly hyperbolic attractor and corresponding SRB-measure μ , then the sequence $\{a_n\}_{n=1}^\infty$ has the asymptotic form*

$$a_n = \lambda_1 + \frac{c_1}{n} + o\left(\frac{1}{n}\right).$$

Moreover, the value of c_1 is

$$c_1 = \int \log \omega_1(x) \, d\mu < \infty. \quad (4.9)$$

Proof. From the definition of c_1 in (4.2) and equation (4.5) we have that

$$c_1 = \lim_{n \rightarrow \infty} n(a_n - \lambda_1) = \lim_{n \rightarrow \infty} \int \log \|G_n(x)\| \, d\mu.$$

From proposition 4.1 we also have that

$$\lim_{n \rightarrow \infty} \|G_n(x)\| = \omega_1(x) \text{ for } \mu\text{-almost all } x.$$

The final step is to bring the limit in the equation for c_1 inside the integral. This can be done using the dominated convergence theorem (Ledermann & Vajda 1982), which can be applied since $\|G_n(x)\|$ converges to $\omega_1(x)$ for μ -almost all x (see proposition 4.1) and since $\log \|G_n(x)\|$ is uniformly bounded above and below. To verify this last point, we note that the vector $\alpha_1(x)$ cannot be close to the zero vector and so $\omega_1(x)$ is bounded below. On the other hand, the attractor is uniformly hyperbolic and therefore the entries in the vector $\alpha_1(x)$ must be bounded above. Thus,

$$\begin{aligned} c_1 &= \lim_{n \rightarrow \infty} \int \log \|G_n(x)\| \, d\mu \\ &= \int \lim_{n \rightarrow \infty} \log \|G_n(x)\| \, d\mu \\ &= \int \log \omega_1(x) \, d\mu, \end{aligned}$$

using (4.8). Finally, as $\log \|G_n(x)\|$ is uniformly bounded, then c_1 must be finite. ■

Remark 4.3. If the attractor is non-uniformly hyperbolic then all of the above theory holds except for taking the limit inside the integral. In this case, there are three distinct possibilities:

- (i) the limit can come inside the integral in the same way as for the uniformly hyperbolic case;
- (ii) the limit cannot come inside the integral but c_1 is still finite;
- (iii) the limit cannot come inside the integral and c_1 is infinite.

In the first two cases, the dominant term in the error $a_n - \lambda_1$ is still of the form c_1/n but in the second case, a formula for c_1 is not known. In the third case, the convergence of the sequence is slower than c_1/n . We have been unable to determine which of these cases occurs for the non-uniformly hyperbolic case, but numerical results indicate that convergence of the sequence is still proportional to $1/n$.

We now want to determine the next term in the asymptotic expansion of the sequence.

Theorem 4.4. *Suppose that the conditions of theorem 4.2 hold and that*

$$\left| \frac{A_n^{(2)}(x)}{A_n^{(1)}(x)} \right| \leq C\gamma^n \quad (4.10)$$

uniformly for μ -almost every x with constants $0 < \gamma < 1$, $C > 0$ and $n > N_0$ for some N_0 . Then the sequence $\{a_n\}_{n=1}^\infty$ has the asymptotic form

$$a_n = \lambda_1 + \frac{c_1}{n} + \frac{f_n}{n} + o\left(\frac{\gamma^n}{n}\right), \quad (4.11)$$

where

$$|f_n| \leq c_2 \gamma^n, \quad (4.12)$$

for some constant $c_2 > 0$ and n sufficiently large.

Proof. We know that (4.11) holds where c_1 is given in (4.9) and where, necessarily, f_n is a sequence for which $\lim_{n \rightarrow \infty} f_n = 0$. Since we know c_1 , we then have that

$$\begin{aligned} |f_n| &= |na_n - n\lambda_1 - c_1| \\ &= \left| \int \log \|G_n(x)\| \, d\mu - \int \log \omega_1(x) \, d\mu \right| \\ &= \left| \int \log \frac{\|G_n(x)\|}{\omega_1(x)} \, d\mu \right| \\ &\leq \int \left| \log \frac{\|G_n(x)\|}{\omega_1(x)} \right| \, d\mu. \end{aligned}$$

Using the decomposition of $G_n(x)$ given by (4.6), we then obtain

$$\begin{aligned} \|G_n(x)\| &= \left\| W_{1n}(x) + \left(\frac{A_n^{(2)}(x)}{A_n^{(1)}(x)} \right) \hat{W}_{2n}(x) \right\| \\ &\leq \|W_{1n}(x)\| + \left| \frac{A_n^{(2)}(x)}{A_n^{(1)}(x)} \right| \|\hat{W}_{2n}(x)\| \\ &\leq \omega_1(x) + \left| \frac{A_n^{(2)}(x)}{A_n^{(1)}(x)} \right| \hat{\omega}_2(x), \end{aligned}$$

by proposition 4.1, where

$$\hat{W}_{2n}(x) = \sum_{j=2}^{\ell} \left(\frac{A_n^{(j)}(x)}{A_n^{(2)}(x)} \right) W_{jn}(x) \quad \text{and} \quad \hat{\omega}_2(x) = \sup_n \|\hat{W}_{2n}(x)\|.$$

Using an argument similar to the one at the beginning of the proof of proposition 4.1, it can be shown that there is a constant $K > 0$ such that

$$\hat{\omega}_2(x) \leq K$$

for μ -almost every x . Using this bound and (4.10) we then obtain for μ -almost every x

$$\begin{aligned}\frac{\|G_n(x)\|}{\omega_1(x)} &\leq \frac{\omega_1(x) + C\gamma^n K}{\omega_1(x)} \\ &= 1 + \frac{\tilde{K}\gamma^n}{\omega_1(x)},\end{aligned}$$

where $\tilde{K} = CK$. Finally, since $\log(1 + y) \leq y$ for all $y \geq 0$ we have that

$$\begin{aligned}|f_n| &\leq \int \left| \log \left(1 + \frac{\tilde{K}\gamma^n}{\omega_1(x)} \right) \right| d\mu \\ &\leq \tilde{K}\gamma^n \int \frac{1}{\omega_1(x)} d\mu \\ &= c_2\gamma^n,\end{aligned}$$

where

$$c_2 = \tilde{K} \int \frac{1}{\omega_1(x)} d\mu > 0$$

as required. ■

Remark 4.5. For μ -almost all x ,

$$\lim_{n \rightarrow \infty} \left| \frac{A_n^{(2)}(x)}{A_n^{(1)}(x)} \right|^{1/n} = e^{\lambda_2 - \lambda_1}, \quad (4.13)$$

and so the bound γ might be expected to be close to this value. However, this may not be the case as the limit in (4.13) does not occur uniformly in x but we require a uniform bound to prove the result. Thus, γ may be larger than the optimal value of $e^{\lambda_2 - \lambda_1}$.

(b) *Method 2: a recurrence relation*

For the second approach to understanding the behaviour of the sequence (4.1), we define for simplicity

$$Y_n(x) = Dg^n(x), \quad U_n(x) = \frac{Y_n(x)}{\|Y_n(x)\|}.$$

We also define an error term

$$\delta_n = \int \log \|Dg(g^{n-1}(x))U_{n-1}(x)\| d\mu - \lambda_1$$

and use this expression to derive a recurrence relation for the elements of the sequence $\{a_n\}$.

Proposition 4.6. *The sequence $\{a_n\}_{n=1}^\infty$ satisfies $a_1 = \lambda_1 + \delta_1$ and for $n \geq 2$,*

$$a_n = \frac{n-1}{n}a_{n-1} + \frac{1}{n}(\lambda_1 + \delta_n). \quad (4.14)$$

Proof. The definition of δ_1 reduces to $\delta_1 = a_1 - \lambda_1$ from which the formula for a_1 is derived. For $n \geq 2$, using the chain rule gives

$$\begin{aligned} Y_n(x) &= Dg^n(x) \\ &= Dg(g^{n-1}(x))Dg^{n-1}(x) \\ &= Dg(g^{n-1}(x))Y_{n-1}(x) \\ &= \|Y_{n-1}(x)\|Dg(g^{n-1}(x))U_{n-1}(x). \end{aligned}$$

Taking logs and integrating then gives

$$\begin{aligned} a_n &= \frac{1}{n} \int \log \|Y_n(x)\| \, d\mu \\ &= \frac{1}{n} \int \log \|Y_{n-1}(x)\| + \log \|Dg(g^{n-1}(x))U_{n-1}(x)\| \, d\mu \\ &= \frac{n-1}{n} a_{n-1} + \frac{1}{n} (\lambda_1 + \delta_n) \end{aligned}$$

as claimed. ■

Corollary 4.7. For $n \geq 1$

$$a_n = \lambda_1 + \frac{1}{n} \sum_{i=1}^n \delta_i. \quad (4.15)$$

Under the assumptions of theorem 4.2, we can combine (4.2) and (4.15) to give

$$c_1 = \sum_{i=1}^{\infty} \delta_i, \quad (4.16)$$

and so the errors δ_i clearly must go to zero for $n \rightarrow \infty$ as c_1 is finite. We can also obtain bounds on δ_n and c_1 .

Lemma 4.8. Suppose that the assumptions of theorem 4.4 hold. Then

- (i) $|\delta_n| \leq c_2 \frac{\gamma+1}{\gamma} \gamma^n$,
- (ii) $c_1 \leq c_2 \frac{1+\gamma}{1-\gamma}$.

Proof.

- (i) Substituting (4.11) into (4.14) gives

$$\delta_n = f_n - f_{n-1}.$$

The bound (4.12) can then be used to obtain the result.

- (ii) The constant c_1 is expressed in terms of an infinite sum of the errors δ_n in equation (4.16). Replacing the errors with the bound in (i) gives the desired result. ■

(c) *Acceleration of convergence of the sequence*

Now that we have some results concerning the asymptotic form of the sequence $\{a_n\}_{n=1}^\infty$, we can eliminate the first term in the error, which is only slowly converging, resulting in sequences which converge rapidly to the correct answer using only a few terms of the sequence.

In our previous paper (Aston & Dellnitz 1999), we assumed that the sequence had a dominant term in the asymptotic expansion given by c_1/n (which we have now proved) and this term was eliminated in two different ways. We define

$$\tilde{a}_n = na_n - (n-1)a_{n-1}, \quad n = 2, 3, 4, \dots, \quad (4.17)$$

$$A_m = 2a_{2^m} - a_{2^{m-1}}, \quad m = 1, 2, 3, \dots \quad (4.18)$$

Both these new sequences eliminate the c_1/n term in the asymptotic expansion of a_n . The second was suggested due to the fact that the subsequence a_{2^m} , $m = 1, 2, 3, \dots$, is a monotonically decreasing sequence.

Substituting for a_n and a_{n-1} in (4.17) with (4.11) and using (4.12) gives

$$\begin{aligned} |\tilde{a}_n - \lambda_1| &= |f_n - f_{n-1}| \\ &\leq |f_n| + |f_{n-1}| \\ &\leq c_2 \gamma^n (1 + \gamma^{-1}). \end{aligned}$$

Similarly, we note that

$$\begin{aligned} |2a_n - a_{n/2} - \lambda_1| &= \frac{2}{n} |f_n - f_{n/2}| \\ &\leq \frac{2}{n} (|f_n| + |f_{n/2}|) \\ &\leq \frac{2c_2 \gamma^{n/2}}{n} (\gamma^{n/2} + 1). \end{aligned}$$

Thus,

$$|A_m - \lambda_1| \leq \frac{2c_2 \gamma^{2^{m-1}} (\gamma^{2^{m-1}} + 1)}{2^m}.$$

The best approximation from each sequence having computed 2^m terms occurs when $n = 2^m$. In this case, $|\tilde{a}_n - \lambda_1|$ is bounded by terms of $O(\gamma^n)$, whereas $|A_m - \lambda_1|$ is bounded by terms of $O(\gamma^{2^{m-1}}/2^m) = O(\gamma^{n/2}/n)$. Thus, asymptotically, the sequence $\tilde{a}_n$ has the fastest rate of convergence. However, for a finite number of terms, the magnitude of the unknown constant involved will often be the most important factor.

5. Numerical results

We now apply these results to the Hénon map

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n, \\ y_{n+1} &= bx_n, \end{aligned}$$

with the usual parameter values of $a = 1.4$ and $b = 0.3$. The dominant Lyapunov exponent was computed by time averaging using 3×10^8 iterations, giving a value of

Figure 1. The first 16 terms of the sequence a_n for the Hénon map.

Figure 2. The first 15 terms of the sequence $\tilde{a}_n$ for the Hénon map.

$\lambda_1 = 0.41919$. For the Hénon map, the two Lyapunov exponents are related by the condition $\lambda_1 + \lambda_2 = \log b$, which thus gives $\lambda_2 = -1.62316$.

In order to get accurate results, subdivision level 26 was used and the attractor was covered by 242 711 boxes of size 1.83×10^{-4} by 4.88×10^{-5} . The first 16 terms of the sequence $\{a_n\}_{n=1}^{\infty}$ were computed and are shown in figure 1. As expected, convergence of the sequence is very slow. In order to obtain a faster converging sequence, we then computed $\tilde{a}_n$, $n = 2, \dots, 16$, and A_m , $m = 1, \dots, 4$, which are shown in figures 2 and 3. Clearly, good accuracy is achieved with $\tilde{a}_6$ and A_3 , requiring fewer than the 16 terms of the original sequence which we originally computed.

To analyse this example in more detail, we would like to approximate the values of c_1 , c_2 and γ defined in (4.11) and (4.12). To find the constant c_1 , we note that

$$c_1 = \lim_{n \rightarrow \infty} n(a_n - \lambda_1).$$

By computing the terms on the right-hand side for different values of n , a value of $c_1 = 0.3959$ was obtained.

Figure 3. The first 4 terms of the sequence A_m for the Hénon map. Figure 4. $\log |f_n|$ for the Hénon map.

Since $|f_n| = |na_n - n\lambda_1 - c_1|$ is bounded by $c_2\gamma^n$ (see (4.12)), in figure 4 we plot $\log |f_n|$ against n , which we would expect to be approximately linear. Clearly, this is the case for the first five terms only. There is then insufficient accuracy in the computed values of a_n for $n > 5$ to obtain any higher accuracy. The slope of this line should be bounded by $\log \gamma$, where γ is expected to be close to $e^{\lambda_2 - \lambda_1} = 0.129\,72$ (cf. Remark 4.5). By fitting a straight line through the two values for $n = 4$ and $n = 5$, we can estimate c_2 and γ , giving values

$$c_2 = 5.492\,77 \quad \text{and} \quad \gamma = 0.129\,12.$$

Clearly, the computed value of γ gives very good agreement with the predicted value of $e^{\lambda_2 - \lambda_1}$, which shows that our assertion at the end of Remark 4.5 does indeed hold for this example at least.

Using these computed values for c_2 and γ , the values of the bound on the error of the accelerated sequence $\tilde{a}_n$ are shown in table 1, together with the computed errors.

Table 1. Values of the bound on and the true value of the error $|\tilde{a}_n - \lambda_1|$

n	bound on $ \tilde{a}_n - \lambda_1 $	computed value of $ \tilde{a}_n - \lambda_1 $
2	0.800 794	0.075 145
3	0.103 398	0.010 644
4	0.013 351	0.002 826
5	0.001 724	0.001 330
6	0.000 223	0.000 078

Table 2. Values of the bound on and the true value of the error $|A_m - \lambda_1|$

m	bound on $ A_m - \lambda_1 $	computed value of $ A_m - \lambda_1 $
1	0.800 794	0.075 145
2	0.046 550	0.006 735
3	0.000 382	0.000 212

For larger values of n the computed errors do not decrease any further. Similar values are given in table 2 for the sequence A_m .

We note that, in all cases, the computed error is less than the bound. We claimed in the previous section that the bounds were lower for the sequence $\tilde{a}_n$ than for the corresponding terms in the sequence A_m , given a fixed number of terms in the original sequence. Now $\tilde{a}_2 = A_1$ and so there is no comparison to be made here. If four terms of the original sequence are computed, then it is possible to evaluate $\tilde{a}_4$ and A_2 and it can be seen from the tables that both the bound and the computed values of $|\tilde{a}_4 - \lambda_1|$ are less than the corresponding values for $|A_2 - \lambda_1|$. If eight terms of the original sequence are computed, then we should compare the errors in $\tilde{a}_8$ and A_3 . However, we are not able to compute $\tilde{a}_8$ to its full accuracy and so this comparison is not possible.

We conclude from these results that it is sufficient to compute only the first five terms of the sequence $\tilde{a}_n$, which requires the terms a_n , $n = 1, \dots, 6$, to compute λ_1 with an accuracy of less than 10^{-4} , provided the terms in the sequence are computed with sufficient accuracy. Clearly, computing more terms without increasing the accuracy is of no benefit.

The reason that very few terms in the sequence are required for the Hénon map is that there is quite a big difference between λ_1 and λ_2 resulting in a small value of γ . If this difference were smaller, then more terms in the sequence would be required to obtain an accurate answer.

6. Application to blowout bifurcations of coupled chaotic oscillators

One important application of our method for computing the dominant Lyapunov exponent is to the determination of blowout bifurcations in systems of identical coupled chaotic oscillators. If the oscillators are behaving chaotically and are synchro-

nized, then this motion is occurring in an invariant subspace for the full problem. The stability of this synchronized motion to perturbations out of the invariant subspace associated with loss of synchronization is determined by the sign of the dominant *normal* Lyapunov exponent (Ashwin *et al.* 1994, 1996). If this exponent is negative, the synchronized motion is stable, whereas if it is positive, then the synchronized motion is unstable to asynchronous perturbations. A blowout bifurcation occurs when the dominant Lyapunov exponent changes sign and results in a change in the stability of the synchronized motion.

Now if the coupling parameter is varied, the synchronized motion will be unaffected although the normal Lyapunov exponents will change. Thus, if the dominant normal Lyapunov exponent is computed using time averaging, the same chaotic attractor in the invariant subspace is computed for each different value of the coupling parameter which is clearly a waste of computational effort. However, with our approach, the box covering of the chaotic attractor and the invariant measure on this attractor can be computed only *once* and then for each different value of the coupling parameter, the sequence of integrals can be computed from which the dominant Lyapunov exponent is derived. This represents a considerable reduction in the computational effort required to compute the dominant normal Lyapunov exponent as a function of the coupling parameter.

To illustrate these ideas, we consider two coupled Duffing oscillators given by

$$\begin{aligned}\dot{x}_1 &= y_1, & \dot{y}_1 &= -ky_1 - x_1^3 + x_1 + A \cos \omega t + c(x_2 - x_1), \\ \dot{x}_2 &= y_2, & \dot{y}_2 &= -ky_2 - x_2^3 + x_2 + A \cos \omega t + c(x_1 - x_2).\end{aligned}$$

In the new coordinates defined by

$$\begin{aligned}X_S &= \frac{1}{2}(x_1 + x_2), & Y_S &= \frac{1}{2}(y_1 + y_2), \\ X_N &= \frac{1}{2}(x_1 - x_2), & Y_N &= \frac{1}{2}(y_1 - y_2),\end{aligned}$$

the equations become

$$\begin{aligned}\dot{X}_S &= Y_S, & \dot{Y}_S &= -kY_S - X_S^3 - 3X_SX_N^2 + X_S + A \cos \omega t, \\ \dot{X}_N &= Y_N, & \dot{Y}_N &= -kY_N - 3X_S^2X_N - X_N^3 + X_N - 2cX_N.\end{aligned}$$

The synchronized state corresponds to $X_N = Y_N = 0$. The synchronized motion is found by solving the first two equations in X_S and Y_S with $X_N = Y_N = 0$. The normal Lyapunov exponents are then associated with the linearization of the second pair of equations about $X_N = Y_N = 0$. These equations can be coupled together to give the six equations:

$$\begin{aligned}\dot{X}_S &= Y_S, \\ \dot{Y}_S &= -kY_S - X_S^3 + X_S + A \cos \omega t, \\ \dot{J}_{11} &= J_{21}, \\ \dot{J}_{21} &= -kJ_{21} - 3X_S^2J_{11} + J_{11} - 2cJ_{11}, \\ \dot{J}_{12} &= J_{22}, \\ \dot{J}_{22} &= -kJ_{22} - 3X_S^2J_{12} + J_{12} - 2cJ_{12}.\end{aligned}$$

If the initial conditions are

$$\begin{aligned} X_S(2n\pi/\omega) &= \alpha_n, & Y_S(2n\pi/\omega) &= \beta_n, \\ J_{11}(2n\pi/\omega) &= J_{22}(2n\pi/\omega) = 1, & J_{12}(2n\pi/\omega) &= J_{21}(2n\pi/\omega) = 0, \end{aligned}$$

then the Poincaré map is defined by

$$\begin{aligned} \alpha_{n+1} &= f(\alpha_n, \beta_n), \\ \beta_{n+1} &= g(\alpha_n, \beta_n), \end{aligned}$$

where

$$f(\alpha_n, \beta_n) = X_S(2(n+1)\pi/\omega), \quad g(\alpha_n, \beta_n) = Y_S(2(n+1)\pi/\omega),$$

and the normal linearization is given by the matrix

$$\begin{bmatrix} J_{11}(2(n+1)\pi/\omega) & J_{12}(2(n+1)\pi/\omega) \\ J_{21}(2(n+1)\pi/\omega) & J_{22}(2(n+1)\pi/\omega) \end{bmatrix}.$$

These equations were solved using a fourth-order Runge–Kutta scheme with constant step length $h = \pi/10$. Values of the parameters were chosen to be $k = 0.2$, $\omega = 1.0$ and $A = 0.36$ and we varied the coupling parameter c . In our previous paper (Aston & Dellnitz 1999) we found that a blowout bifurcation occurred at approximately $c = 0.1351$.

We now consider in more detail the particular case of $c = 0.18$. It was found previously that the sequence of integrals does not converge monotonically in this case and this suggests that convergence of the accelerated sequence is quite slow. Using a time average with 1×10^9 iterations of the Poincaré map, we approximated both the normal Lyapunov exponents in this case and obtained the values $\lambda_1 = -0.4595$ and $\lambda_2 = -0.8033$ giving $e^{\lambda_2 - \lambda_1} = 0.7091$, which is much closer to one than the corresponding value for the Hénon map considered previously, thus resulting in slower convergence of both the time average and the sequence of integrals.

In order to approximate the attractor and find the invariant measure, subdivision level 22 was used resulting in 176 095 boxes covering the attractor. The box covering is shown in figure 5. The sequence of integrals was then computed and is shown in figure 6.

In order to speed up the convergence of the sequence, the new sequences $\tilde{a}_n$ and A_m defined in equations (4.17) and (4.18) were computed and are shown in figures 7 and 8. For the sequence $\tilde{a}_n$, there is steady convergence towards λ_1 for the first 16 or so terms but clearly, later terms cannot be calculated with sufficient accuracy.

For the sequence A_m , it is found that the terms converge very well to the correct value with $|A_5 - \lambda_1| = 3.71 \times 10^{-3}$. The calculation of A_5 involves a combination of a_{16} and a_{32} , both of which are likely to involve a reasonable degree of error due to the limitations of this method. However, a combination of the two gives excellent results. It is often found in practice that the sequence A_m gives better results than $\tilde{a}_n$, even though this is not what is predicted by the theory. One possible reason for this is that, for small values of n ,

$$\gamma^n > \frac{\gamma^{n/2}}{n}$$

Figure 5. The box covering of the Duffing attractor at level 22.

Figure 6. The first 32 terms of the sequence a_n for the Duffing attractor.

particularly for values of γ close to 1 as we have in this case, even though the opposite is true asymptotically and hence for larger values of n . This would suggest that the sequence A_m may give better results for the low values of m that we are using than the sequence $\tilde{a}_n$, as we observe in this case (see § c). Of course, the magnitude of the constants involved is also a factor.

In order to get further improved results, we consider again the sequence $\tilde{a}_n$. We have proved that

$$|\tilde{a}_n - \lambda_1| \leq c_2 \gamma^n (1 + \gamma^{-1}).$$

If this bound was an equality, then the Aitkens δ^2 method could be used to accelerate the convergence of this sequence. We make two observations from figure 7:

- (i) each set of four terms in the sequence have a similar pattern;

Figure 7. The first 31 terms of the sequence $\tilde{a}_n$ for the Duffing attractor. Figure 8. The first 5 terms of the sequence A_m for the Duffing attractor.

- (ii) there is an upper and a lower envelope to the sequence of iterates that appears to be decaying exponentially.

The reason for the pattern of four terms in this sequence is not clear, although it can also be seen in the sequence a_n (see figure 6). It is likely to be due to a dominant period 4 orbit.

We choose to work with the lower envelope which corresponds to the terms $\tilde{a}_{4k}$, $k = 1, 2, \dots$, and we assume that

$$\tilde{a}_{4k} = \lambda_1 + c_3 \gamma^{4k} = \lambda_1 + c_3 (\gamma^4)^k, \quad k = 1, 2, \dots$$

If this assumption was correct, then the Aitkens δ^2 method could be used on this subsequence to obtain better results. However, there is still some variation in the terms of the subsequence so that this relation is not precise and good results are not obtained. Thus, we eliminate λ_1 by taking the difference of two successive terms

Figure 9. Acceleration of convergence for the sequence $\tilde{a}_n$. +, $\tilde{a}_{4k}$, $k = 1, \dots, 7$; x, $\tilde{a}_{4k} - c_3 \gamma^{4k}$; —, the exponential fit given by $\lambda_1 + c_3 \gamma^n$.

giving the new sequence

$$\hat{a}_k = \tilde{a}_{4k+4} - \tilde{a}_{4k} = c_3 \gamma^{4k} (\gamma^4 - 1).$$

Regression can then be used to obtain an exponential fit from which values of γ and c_3 can be derived. A new approximation to the dominant Lyapunov exponent is then given by

$$\lambda_1 \simeq \tilde{a}_{4k} - c_3 \gamma^{4k}. \quad (6.1)$$

Applying this approach gives $\gamma = 0.8632$ and $c_3 = -0.5692$. These results are illustrated in figure 9. Note that in this case γ is somewhat larger than the optimal value of $e^{\lambda_2 - \lambda_1} = 0.7091$ (see the discussion in Remark 4.5).

The approximation to λ_1 using (6.1) with $k = 8$ ($n = 28$) is then $\lambda_1 = -0.460173$, which has an error of 6.73×10^{-4} . Moreover, the last three values of λ_1 from (6.1) with $k = 6, 7, 8$ are all consistently close to the true value. We consider this to be a good result obtained from this difficult problem.

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