Atom in a coherently controlled squeezed vacuum

Itay Rabinak,* Eran Ginossar, and Shimon Levit

Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

(Received 19 October 2006; published 20 July 2007)

A broadband squeezed vacuum photon field is characterized by a complex squeezing function. We show that by controlling the wavelength dependence of its phase it is possible to change the dynamics of the atomic polarization interacting with the squeezed vacuum. Such a phase modulation effectively produces a finite range temporal interaction kernel between the two quadratures of the atomic polarization yielding the change in the decay rates as well as the appearance of additional oscillation frequencies. We show that decay rates slower than the spontaneous decay rate can be achieved even for a squeezed bath in the classical regime. For “piecewise linear” and quadratic phase modulations the power spectrum of the scattered light exhibits narrowing of the central peak due to the modified decay rates. For strong phase modulations, side lobes appear symmetrically around the central peak reflecting additional oscillation frequencies.

DOI: [10.1103/PhysRevA.76.013821](https://doi.org/10.1103/PhysRevA.76.013821)

PACS number(s): 42.50.Ct, 42.50.Ar, 42.50.Dv, 42.50.Lc

The effect of the interaction of an atom with a squeezed light has been studied extensively. Gardiner [1] has considered the behavior of a two-level atom damped by an infinite bandwidth squeezed vacuum. He showed that the two quadratures of the atomic polarization decay at different rates. Decay rates smaller than the decay rates of spontaneous emission can be achieved for nonclassical squeezing. The presence of the two decay rates modifies the fluorescence spectrum [2]. In Refs. [3–5], the interaction of a finite bandwidth squeezed vacuum with a two-level atom was investigated. Review of these and related studies is found in Ref. [6]. Recently, these works were extended in Ref. [7] to include interactions with semiconductor microstructures.

In the field of quantum coherent control pulse shapers [8] were used to attain prescribed phase modulation of a down converted light in order to control two photon absorption. It was shown [9] that pulses can be shaped in a way that will stretch them temporally affecting the transition probability [10]. Experiments [11,12] have also been performed on two photon absorption with coherent, narrow band down-converted light, demonstrating nonclassical features which appear at very low powers [13–15] and result from time and energy correlations (entanglement) between the down-converted photon pairs.

The interaction of amplitude or phase modulated classical light with a two-level system has been studied extensively both theoretically and experimentally (see, for example, [16–20]). In this work, we wish to investigate the effect of controlling and modulating the relative phase of the modes of a squeezed reservoir interacting with an atom. This can be easily done by a pulse shaper arrangement as shown in Refs. [9,21]. We will demonstrate that by controlling in this manner the phases of the correlations in the squeezed reservoir it is possible to control the dynamics of atomic polarization and in particular to further reduce its decay rates.

We will use the standard model to describe the interaction of a two-level atom with a broadband radiation field. The atom is assumed to be coupled to a one-dimensional set of

radiation modes. The Hamiltonian is given in the electric-dipole and rotating-wave approximations by $H=H_0+H_I$ as (we take $\hbar=1$)

$$H_0 = \omega_a \sigma_z + \sum_q \omega_q b_q^\dagger b_q,$$

$$H_I = \Gamma \sigma_+ + \sigma_- \Gamma^\dagger, \quad \Gamma = \sum_q g_q b_q. \quad (1)$$

The pseudospins σ_+ , σ_- , and σ_z that describe the atom are defined as

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

The atomic resonance frequency is ω_a , b_q , and $b_q^\dagger$ describe radiation modes with wave vector q , and frequencies ω_q and g_q are the mode-atom couplings.

We assume that the radiation acts as a reservoir with correlations of a two mode broadband squeezed vacuum [1,3]

$$\langle b_q^\dagger b_{q'} \rangle = N(\omega_q) \delta_{q,q'},$$

$$\langle b_q^\dagger b_{q'}^\dagger \rangle = M(\omega_q) \delta_{2q_0-q,q'},$$

$$\langle b_q \rangle = \langle b_q^\dagger \rangle = 0, \quad (3)$$

where $N(\omega_q)$ describes the average occupations of photonic modes while the magnitude $|M(\omega_q)|$ gives the squeezing strength of mode pairs centered around the frequency $\omega_0 \equiv \omega(q_0)$. We define $Q \equiv 2q_0 - q$ and get $\omega_Q = 2\omega_0 - \omega_q$. The phase of $M(\omega_q)$ describes the “direction” of squeezing in the phase space of squeezed mode pairs. In the common method of generating squeezed vacuum radiation by nonlinear down conversion the phase of $M(\omega_q)$ is constant over the frequency range. Letting the radiation pass through a pulse shaper makes it possible to change this phase into a prescribed function of ω_q ; cf. Refs. [9,21].

We shall for simplicity assume that N , $|M|$, g_q are constants within a bandwidth $\omega_0 \pm B/2$ and that B is $\ll \omega_0$ but is much larger than any other frequency in the system.

*itay.rabinak@weizmann.ac.il

Using the equations of motion

$$\begin{aligned}\dot{\sigma}_+ &= -i[\sigma_+, H] = i\omega_a\sigma_+ - 2i\Gamma^\dagger\sigma_z, \\ \dot{\sigma}_z &= -i[\sigma_z, H] = -i\Gamma\sigma_+ + i\Gamma^\dagger\sigma_-, \\ \dot{b}_q &= -i[b_q, H] = -i\omega_q b_q - ig_q^*\sigma_-, \end{aligned} \quad (4)$$

and transforming them to the rotating frame of the laser frequency, we integrate over time the equations for σ_z and b_q and substitute the result back into the equation for σ_+ . We then average the resulting equation over the initial state and assume that the bath and the atom are (approximately) decorrelated, i.e., that the photon atom correlators factorize at all times; e.g., $\langle b_q(t)\sigma_+(t') \rangle \approx \langle b_q(t) \rangle \langle \sigma_+(t') \rangle$, etc. The decorrelation assumption is valid in the regime of weak coupling between the system and the photon bath. Using it we obtain a closed equation for the atomic polarizations $\langle \sigma_\pm \rangle$

$$\begin{aligned} \frac{d}{dt} \langle \sigma_+ \rangle &= \left(i(\omega_a - \omega_0) - \frac{\gamma}{2} \right) \langle \sigma_+ \rangle \\ &- 2 \int_0^t dt' \left\{ \sum_q |g_q|^2 N(\omega_q) e^{i(\omega_q - \omega_0)(t-t')} \langle \sigma_+(t') \rangle \right. \\ &\left. - g_q^* g_Q^* M(\omega_q) e^{i(\omega_q - \omega_0)(t-t')} \langle \sigma_-(t') \rangle \right\}, \end{aligned} \quad (5)$$

where $\gamma = \rho(\omega)|g_q|^2$ is the vacuum atomic decay rate and $\rho(\omega)$ is the density of the radiation modes. We assume that $\rho(\omega)$ is flat over the bandwidth B . We now transform the sums over q into integrals. We define the following parameters:

$$\begin{aligned} \gamma\mathcal{N} &= \rho(\omega)|g_q|^2 N(\omega_q), \\ \gamma\mathcal{M} &= \rho(\omega)g_q^* g_Q^* |M(\omega_q)|, \end{aligned} \quad (6)$$

and the following function

$$k(t-t') = \frac{1}{\pi} \int_{-B/2}^{B/2} d\omega e^{if(\omega)} e^{i\omega(t-t')}, \quad (7)$$

where $f(\omega)$ is the phase of $M(\omega) = |M|e^{if(\omega-\omega_0)}$. Note that $f(\omega) = f(-\omega)$. Without loss of generality, we can take $f(0) = 0$ using the freedom to absorb its nonzero value in the phase of $\mathcal{M}$.

With this notation the equation for the atomic polarizations becomes

$$\begin{aligned} \frac{d}{dt} \langle \sigma_+ \rangle &= \left[i\delta + i\Delta - \gamma \left(\mathcal{N} + \frac{1}{2} \right) \right] \langle \sigma_+ \rangle + \gamma\mathcal{M} \int_0^t dt' k(t-t') \\ &\times \langle \sigma_-(t') \rangle, \end{aligned} \quad (8)$$

where we defined the detuning $\delta = \omega_a - \omega_0$ and Δ is the atomic frequency shift caused by the principal part of the integral over the intensity of the radiation field. The contribution of the detuning δ and the atomic frequency shift Δ is similar. Therefore, it is convenient to define their sum as the combined detuning $\tilde{\delta} = \delta + \Delta$. Equation (8) demonstrates the non-Markovian nature of the phase modulated reservoir. In

contrast to the unmodulated squeezed vacuum reservoir we can see from Eq. (8) that, due to the retarded kernel $k(t-t')$, there are contributions to the polarization dynamics at time t from previous states of the conjugate polarization at earlier times t' .

Before formally solving Eq. (8) let us qualitatively discuss the expected behavior of the solutions. For this it is useful to recall the interpretation of the first two of the Eqs. (4) as Bloch equations for a spin in a fluctuating magnetic field Γ

$$\frac{d\sigma}{dt} = 2\Gamma \times \sigma. \quad (9)$$

The phase $e^{if(\omega)}$ of the squeezing parameter $M(\omega)$ controls the correlation functions of the components of Γ . Moving to a rotating frame with a frequency ω_0 and passing to Hermitian components,

$$\Gamma_x = \frac{1}{2}(\Gamma + \Gamma^\dagger), \quad \Gamma_y = \frac{i}{2}(\Gamma - \Gamma^\dagger), \quad (10)$$

we find that in the classical limit the time correlators are

$$\langle \Gamma_x(t) \Gamma_x(t') \rangle = \gamma\mathcal{N}\delta(t-t') + \gamma\frac{\mathcal{M}}{2}\text{Re}\{k(t-t')\},$$

$$\langle \Gamma_y(t) \Gamma_y(t') \rangle = \gamma\mathcal{N}\delta(t-t') - \gamma\frac{\mathcal{M}}{2}\text{Re}\{k(t-t')\},$$

$$\langle \Gamma_x(t) \Gamma_y(t') \rangle = -\gamma\frac{\mathcal{M}}{2}\text{Im}\{k(t-t')\}, \quad (11)$$

and zero averages $\langle \Gamma_x \rangle = \langle \Gamma_y \rangle = 0$.

When the phase of $M(\omega_q)$ is not modulated, $f(\omega) = 0$, the kernel function $k(t)$ becomes $k(t) = 2\delta(t)$ so that

$$\langle \Gamma_x(t) \Gamma_x(t') \rangle = \gamma(\mathcal{N} + \mathcal{M})\delta(t-t'),$$

$$\langle \Gamma_y(t) \Gamma_y(t') \rangle = \gamma(\mathcal{N} - \mathcal{M})\delta(t-t'),$$

$$\langle \Gamma_x(t) \Gamma_y(t') \rangle = 0. \quad (12)$$

The polarization component that is interacting with Γ_x sees stronger fluctuations than in the nonsqueezed ($M=0$) case and thus decays faster, while the component which interacts with Γ_y sees weaker fluctuations and decays slower. This is the essence of the effect described by Gardiner in Ref. [1].

Properly chosen nonzero phase $f(\omega)$ will amplify this effect. Consider, for instance, the simple piecewise linear phase modulation

$$f(\omega) = T|\omega|, \quad (13)$$

where T is a real positive parameter. To simulate the finite bandwidth effect in a simple way we will add a small positive imaginary part to T , i.e., assume

$$\text{Re}(T) \gg \text{Im}(T) = \lambda > 0. \quad (14)$$

The resulting kernel is

FIG. 1. Real and imaginary parts of the piecewise linear phase kernel as a function of normalized time. Finite bandwidth $B \cdot T \approx T/\lambda = 20$.

$$\pi k(t) = \frac{\lambda - i(t-T)}{\lambda^2 + (t-T)^2} + \frac{\lambda + i(t+T)}{\lambda^2 + (t+T)^2}, \quad (15)$$

where the bandwidth is $B \approx 1/\lambda$. In Fig. 1 we plot the real and imaginary parts of this kernel. As one can see, the kernel is not concentrated around $t=0$ but peaks at a retardation time T . The time derivatives of the polarizations are thus determined by the earlier values of the polarizations themselves. Since the polarizations decay with time the interaction at $t'=t-T$ increases the effect as the polarization at t' has larger value. The accumulated effect of earlier higher values of the correlators in Eq. (11) is expected to lead to the amplification of the difference between the effect of Γ_x and Γ_y on the corresponding spin components. This will indeed be confirmed by the calculations below (cf. Fig. 6).

As another example, we consider the quadratic phase modulation

$$f(\omega) = T^2 \omega^2, \quad (16)$$

where T is a real positive parameter. For this phase modulation the kernel is

$$k(t) = \frac{e^{-it^2/4T^2}}{\sqrt{\pi} \sqrt{-iT^2}}. \quad (17)$$

We simulate the finite bandwidth effect B by adding a positive imaginary part to T as in Eq. (14)

$$\text{Re}(T) \gg \text{Im}(T) \approx 1/B > 0. \quad (18)$$

In Fig. 2 we plot the real and imaginary parts of the resulting kernel. One can see that once again in contrast to the Markov case the kernel is not concentrated around $t=0$ but is rather long ranged, with characteristic retardation time T .

Let us discuss the physical effect of the linear phase modulation, Eq. (13), in some detail. In the rotating wave picture the modulated field $\Gamma(t)$ can be written as a sum $\Gamma(t) = \Gamma_A(t) + \Gamma_R(t)$ of two components which are respectively retarded and advanced by the time T

FIG. 2. Real and imaginary parts of the quadratic phase kernel as a function of normalized time. Finite bandwidth $B \cdot T \approx 100$.

$$\Gamma_A(t) = \int_{-\infty}^0 d\omega e^{-i\omega(t+T/2)} \Gamma_\omega,$$

$$\Gamma_R(t) = \int_0^\infty d\omega e^{-i\omega(t-T/2)} \Gamma_\omega, \quad (19)$$

where $\Gamma_\omega = \rho(\omega) g_q b_q |_{\omega(q)=\omega}$. For the squeezed light, these components are correlated in time due to the squeezing correlations in the mode space [cf. Eq. (3)]. Therefore, the time correlation kernel $k(t-t') \sim \langle \Gamma(t) \Gamma(t') \rangle$ is a sum of the cross correlations $\langle \Gamma_A(t) \Gamma_R(t') \rangle$ and $\langle \Gamma_R(t) \Gamma_A(t') \rangle$, which connect the time t and the shifted $t \pm T$. For infinite bandwidth the kernel, Eq. (15) assumes the form

$$2\pi k(\tau) = \pi \delta(\tau+T) + i \text{PV} \left(\frac{1}{\tau+T} \right) + \pi \delta(\tau-T) - i \text{PV} \left(\frac{1}{\tau-T} \right), \quad (20)$$

where the imaginary parts represent the Cauchy principal value. In the limit of $T \rightarrow 0$, we recover the Markovian case, where the two principal value terms cancel. The appearance of these terms in the squeezing correlator is a direct result of the application of a *symmetric* phase modulation which affects differently the positive and negative frequency parts. They greatly complicate the discussion. However, useful physical intuition can be gained by taking into account only the real part of the kernel, which already contains the memory effect. In such a case the equations for $\sigma_{x,y}$ (putting $\tilde{\delta}=0$ for simplicity and considering $t > T$) become

$$\frac{d}{dt} \langle \sigma_x(t) \rangle = -\gamma \left(\mathcal{N} + \frac{1}{2} \right) \langle \sigma_x(t) \rangle + \gamma \mathcal{M} \langle \sigma_x(t-T) \rangle, \quad (21)$$

$$\frac{d}{dt} \langle \sigma_y(t) \rangle = -\gamma \left(\mathcal{N} + \frac{1}{2} \right) \langle \sigma_y(t) \rangle - \gamma \mathcal{M} \langle \sigma_y(t-T) \rangle. \quad (22)$$

They are decoupled and represent one of the simplest non-Markovian types of dynamical system, called *time-delay system*. These types of equations have been studied extensively; cf. Ref. [22]. In general, the solutions are found by using the Laplace transform method and exhibit modified decay times as well as the appearance of oscillations (cf. Figs. 8 and 9). As we shall see, similar qualitative features also appear in our system. For illustrative purposes, we show in the Appendix solutions of the above equations for a range of system parameters.

One can get an intuitive understanding of these features by taking a close look at Eq. (21). The time derivative of the polarization $\langle \sigma_x(t) \rangle$ has two terms: The first term induces decay of the polarization and the second one is a delayed feedback, which can be argued to amplify the polarization. This can be seen in the following way: Picking a random moment when the polarization is decreasing, it is clear that unless the delayed feedback is following up at the same rate, it will increase the derivative until it becomes positive. The opposite can happen when the polarization is increasing: At

some point, the first term will win over and the derivative will become negative. While such arguments cannot explain all the dynamics of the equation, it is clear that a competition between the terms leads to the appearance of oscillations.

Turning now to the analysis of the original problem we note first that the non-Markovian dynamics described by Eq.

(8) is still linear due to the atom-bath decoupling assumption. One can envisage situations in which the non-Markovian dynamics may lead to the breakdown of this assumption but we will not consider such cases here.

To conduct a quantitative analysis we apply the Laplace transform to Eq. (8) and obtain

$$\begin{pmatrix} \langle \tilde{\sigma}_-(s) \rangle \\ \langle \tilde{\sigma}_+(s) \rangle \end{pmatrix} = \frac{\begin{pmatrix} s + i\tilde{\delta} + \gamma(\mathcal{N} + 1/2) & \gamma\mathcal{M}\tilde{k}(s) \\ \gamma\mathcal{M}^*\tilde{k}^*(s^*) & s - i\tilde{\delta} + \gamma(\mathcal{N} + 1/2) \end{pmatrix} \begin{pmatrix} \langle \sigma_-(0) \rangle \\ \langle \sigma_+(0) \rangle \end{pmatrix}}{[(s + \gamma\mathcal{N} + 1/2)]^2 + \tilde{\delta}^2 - |\mathcal{M}|^2\tilde{k}(s)\tilde{k}^*(s^*)}, \quad (23)$$

where we assumed nonzero initial conditions at $t=0$ for $\langle \sigma_{\pm}(t) \rangle$ and denoted $\langle \tilde{\sigma}_{\pm}(s) \rangle$, $\tilde{k}(s)$ the Laplace transform of the polarizations and the kernel function, respectively. The nonzero $\langle \sigma_{\pm}(0) \rangle$ is the easiest situation to analyze although it can only be realized with a specially designed initial pulse applied before letting the squeezed reservoir interact with the atom. We will discuss later the implications for the fluorescence spectrum.

In the following, we will assume for simplicity the exact resonance case $\tilde{\delta}=0$. Later we will show how the results are affected for different values of the combined detuning $\tilde{\delta}$.

The poles of the Laplace transform are the solutions of

$$s/\gamma = -\mathcal{N} - 1/2 \pm |\mathcal{M}| \sqrt{\tilde{k}(s)\tilde{k}^*(s^*)}. \quad (24)$$

For the unmodulated phase $f(\omega)=0$, we obtain the usual Markov result $k(t)=2\delta(t)$, $\tilde{k}(s)=1$, in which case $s_{\pm}=\gamma(\mathcal{N} + 1/2 \pm |\mathcal{M}|)$, representing the splitting of the decay rate into the fast and the slow components under the influence of the squeezing [1]. For the modulated phase it is convenient to consider the graphic solution; cf. Fig. 3. The symmetry of $f(\omega)$ implies that $\tilde{k}(0)=1$. Therefore, for analytic $\tilde{k}(s)$ to the

first order in s , $\tilde{k}(s) \approx 1 - s \int_0^{\infty} t k(t) dt \equiv 1 - s k_1$. The right-hand side of Eq. (24) reduces to $-\mathcal{N} - 1/2 \pm |\mathcal{M}|(1 - s \text{Re } k_1)$ and if $\text{Re } k_1 > 0$ then the slow component of the decay rate

$$s_- = -\gamma(1 - \gamma|\mathcal{M}|\text{Re } k_1)(\mathcal{N} + 1/2 - |\mathcal{M}|) + \dots \quad (25)$$

can become even slower than in the Markov case.

We wish to remark that in addition to the real poles, the complex poles of Eq. (23) in the $\text{Re } s < 0$ half-plane also play a role in determining the dynamics of the atomic polarizations. Their imaginary parts represent additional oscillation frequencies. If their real part is smaller than the real poles they may dominate the long time decay. Finally, possible cuts may also make a contribution. We will discuss such contributions in the examples below.

The simplest case to calculate is the quadratic phase (16). The corresponding Laplace transformed memory kernel in the $B \rightarrow \infty$ limit is

$$\tilde{k}(s) = e^{-iT^2 s^2} \text{erfc}(\sqrt{-i}Ts). \quad (26)$$

The solution of Eq. (24) with this $\tilde{k}(s)$ is shown graphically by plotting the two sides of this equation in Fig. 3. One sees the increase of the fast and the decrease of the slow decay rates caused by the phase modulation. In this example, $\gamma \text{Re } k_1$ in Eq. (25) is $\sqrt{2}/\pi T\gamma$.

For $\tilde{k}(s)$ given by Eq. (26) one finds numerous complex poles of Eq. (23) in addition to the real ones. The positions of the complex poles depend on the parameters $\mathcal{N}$, $\mathcal{M}$, T , and γ . Their influence can be seen in Fig. 4 where we plot the real part of

$$\langle \tilde{\sigma}_-(s=i\omega) \rangle = \frac{[i\omega + \gamma(\mathcal{N} + 1/2)]\langle \sigma_-(0) \rangle}{[i\omega + \gamma(\mathcal{N} + 1/2)]^2 - |\mathcal{M}|^2\tilde{k}(i\omega)\tilde{k}^*(-i\omega)} \quad (27)$$

normalized to unity at the central frequency. In addition to the significant narrowing of the central part of the peak relative to the Markov case $T=0$, one observes the appearance of the side lobes which reflect a complicated pole structure in the $\text{Re } s < 0$ half-plane.

FIG. 3. Graphical solution of Eq. (24) with $\mathcal{N}=\mathcal{M}=1$ for the quadratic phase modulation, Eq. (26). The wavy lines show the right-hand side of the equation for $T\gamma=5$ (solid line) and $T\gamma=1$ (dashed line), while the horizontal dotted lines are for $T=0$ —the standard Markov case.

FIG. 4. The fluorescence spectrum for the quadratic phase modulation, Eq. (16), and $\mathcal{N}=\mathcal{M}=1$. Solid, dashed, and dotted lines are for $T\gamma=5$, $T\gamma=1$, and $T=0$ (the Markov case), respectively.

The quantity plotted in Fig. 4 can be related to the fluorescence spectrum of light emitted by the atom into empty modes of the radiation field. This spectrum is given by

$$S(\omega) = \frac{\gamma}{2\pi} \text{Re} \left\{ \int_0^\infty d\tau e^{-i\omega\tau} \langle \sigma_+(t_0 + \tau) \sigma_-(t_0) \rangle_{ss} \right\}, \quad (28)$$

where the subscript ss denotes steady state. This expression involves the average of the two time product of polarization operators $\langle \sigma_+(t_0 + \tau) \sigma_-(t_0) \rangle$. It is not difficult to show that for $t_0=0$ and $t_0 + \tau = t$ this average satisfies the same dynamical Eq. (8) as a single time average $\langle \sigma_+(t) \rangle$ provided that one uses the atom-bath decorrelation assumption. Irrespective of the initial condition, the system relaxes to a steady state regime. One can therefore replace the initial time, i.e., the low limit $t=0$, in the integral in Eq. (8) by t_0 provided that it is chosen after the steady state is reached. The initial condition is then given by $\langle \sigma_+(t_0) \sigma_-(t_0) \rangle_{ss} = 1/2 + \langle \sigma_z(t_0) \rangle_{ss}$. The Laplace transformed solution of this set is given by Eq. (23) with $\langle \sigma_\pm \sigma_\mp \rangle(s)$ replacing $\tilde{\sigma}_\pm(s)$ in the left-hand side and $\langle \sigma_\pm(0) \rangle$ replaced by

$$\langle \sigma_-(0) \rangle \rightarrow \langle \sigma_+(t_0) \sigma_-(t_0) \rangle_{ss} = \frac{1}{2} - \frac{1/2}{2N+1},$$

FIG. 5. Sensitivity of the fluorescence spectrum to the degree of squeezing. $T\gamma=2$, $\mathcal{M}=0.5$ (solid line), $T=0$, $\mathcal{M}=0.5$ (dashed), and $\mathcal{M}=0$ (dotted), white noise reservoir. For all graphs, $\mathcal{N}=1$.

FIG. 6. The fluorescence spectrum for the linear phase modulation (13) with $\mathcal{N}=1$, $\mathcal{M}=1$ and $T\gamma=5$ (solid line), $T\gamma=1$ (dashed), and $T=0$ (dotted, the unmodulated Markov case).

$$\langle \sigma_+(0) \rangle \rightarrow \langle \sigma_+(t_0) \sigma_+(t_0) \rangle_{ss} = 0. \quad (29)$$

The fluorescence spectrum is given by $S(\omega) = (\gamma/2\pi) \text{Re} \{ \langle \sigma_+ \sigma_- \rangle(i\omega) \}$. The fluorescence spectrum normalized to one at peak intensity is the quantity plotted in Fig. 4.

We note that the discussion above essentially means that under the set of the adopted assumptions the quantum regression theorem can be applied despite the finite memory effects induced by the phase modulation.

In Fig. 5 we show how the phase modulation increases the sensitivity to squeezing. The graphs represent three possible reservoirs: Squeezed phase modulated, squeezed unmodulated, and white noise reservoir. One observes that the narrowing of the central peak is present even for the reduced values of M provided a strong phase modulation is applied.

We will now briefly discuss the fluorescence spectrum for the piecewise linear modulation (13). In this case, $\tilde{k}(s)$ is multivalued with $s=0$ as a branch point. The pole structure is now more complicated and should be discussed together with the branch cuts in the complex $\text{Re}s < 0$ half-plane. In particular, the discussion following Eq. (24) should be modified. We will not discuss it here but rather present in Fig. 6

FIG. 7. The fluorescence spectrum for the quadratic phase modulation and finite detuning, Eq. (16), $\mathcal{N}=\mathcal{M}=1$ and $T\gamma=5$. Solid, dashed, and dotted lines are for $\tilde{\delta}=\gamma$, $\tilde{\delta}=\gamma/2$, and $\tilde{\delta}=0.01\gamma$.

FIG. 8. (Color online) The dynamics of polarization $\langle \sigma_x \rangle$ in the time delay dynamics, Eq. (21) for $\gamma = \mathcal{N} = \mathcal{M} = 1$ and the following delays: $T=0.1$ (dotted line), $T=2$ (dashed line), and $T=5$ (solid line).

the quantity $\langle \tilde{\sigma}_-(s=i\omega) \rangle / \langle \tilde{\sigma}_-(s=i\omega_0) \rangle$ for this phase modulation. As discussed above, this quantity represents the fluorescence spectrum. We observe features similar to those found in the quadratic modulation case and in particular the narrowing of the spectrum around the central frequency and the development of the side lobes. From an experimental point of view, the advantages of the “piecewise linear” modulation is the relative ease of achieving it in practice.

We conclude by showing how the fluorescence spectrum is affected by nonzero combined detuning $\tilde{\delta}$. In Fig. 7 we plot the fluorescence spectrum for a quadratic phase modulation and different combined detuning $\tilde{\delta}$. We can see that for small combined detuning the spectrum is slightly asymmetric and that the basic features, such as the linewidth narrowing and the side lobes, still appear in the spectrum. When the combined detuning $\tilde{\delta}$ is equal to γ the spectrum is deformed.

It is a pleasure to acknowledge valuable discussions with Y. B. Levinson.

FIG. 9. (Color online) The dynamics of polarization $\langle \sigma_y \rangle$ in the time delay dynamics, Eq. (22) for $\gamma = \mathcal{N} = \mathcal{M} = 1$ and the following delays: $T=0.1$ (dotted line), $T=2$ (dashed line), and $T=5$ (solid line).

APPENDIX: TIME-DELAY EQUATIONS

Time-delay systems are a special type of functional differential equations, cf [22], which arise in many applications. The general solution of such equations consists of solving using the method of Laplace transform. In practice, however, the numerical solution is the only feasible way to solve such equations. In Figs. 8 and 9 we plot the solution for $\langle \sigma_x \rangle$ as a function of time for a constant initial polarization for the Markovian case and for several representative finite delay times T with the parameters $N, M=1$, which are similar to those used in Fig. 6. Note that in time the integro-differential Eq. (8), combined with the δ -function kernels in Eq. (20), leads to a pure decay in the initial interval $0 < t < T$, before entering the oscillatory phase. One can see that as the time delay (T) increases, the oscillations become pronounced and the decay slows down.

-
- [1] C. W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).
 - [2] H. J. Carmichael, A. S. Lane, and D. F. Walls, Phys. Rev. Lett. **58**, 2539 (1987).
 - [3] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).
 - [4] C. W. Gardiner, A. S. Parkins, and M. J. Collett, J. Opt. Soc. Am. B **4**, 1683 (1987).
 - [5] A. S. Parkins and C. W. Gardiner, Phys. Rev. A **40**, 3796 (1989).
 - [6] B. J. Dalton, Z. Ficek, and S. Swain, J. Mod. Opt. **46**, 379 (1999).
 - [7] E. Ginosar and S. Levit, Phys. Rev. B **72**, 075333 (2005).
 - [8] M. M. Wefers and K. A. Nelson, Opt. Lett. **20**, 400 (1995).
 - [9] D. Meshulach and Y. Silberberg, Nature (London) **396**, 402 (1998); Phys. Rev. A **60**, 1287 (1999).
 - [10] N. Dudovich, B. Dayan, S. M. Gallagher Faeder, and Y. Silberberg, Phys. Rev. Lett. **86**, 47 (2001).
 - [11] N. P. Georgiades, E. S. Polzik, K. Edamatsu, H. J. Kimble, and A. S. Parkins, Phys. Rev. Lett. **75**, 3426 (1995).
 - [12] N. P. Georgiades, E. S. Polzik, and H. J. Kimble, Phys. Rev. A **55**, R1605 (1997).
 - [13] J. Gea-Banacloche, Phys. Rev. Lett. **62**, 1603 (1989).
 - [14] J. Javanainen and P. L. Gould, Phys. Rev. A **41**, 5088 (1990).
 - [15] H.-B. Fei, B. M. Jost, S. Popescu, B. E. A. Saleh, and M. C. Teich, Phys. Rev. Lett. **78**, 1679 (1997).
 - [16] T. W. Mossberg and M. Lewenstein, Phys. Rev. A **39**, 163 (1989).
 - [17] J. H. Eberly and V. D. Popov, Phys. Rev. A **37**, 2012 (1988).

- [18] M. Janowicz, Phys. Rev. A **44**, 3144 (1991).
- [19] M. Wilkens and K. Rzazewski, Phys. Rev. A **40**, 3164 (1989).
- [20] P. Thomann, J. Phys. B **13**, 1111 (1980).
- [21] B. Dayan, A. Pe'er, A. A. Friesem, and Y. Silberberg, Phys. Rev. Lett. **93**, 023005 (2004).
- [22] J. H. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, Berlin, 1993), Chap. 1.