New upper bounds on separating codes

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Abstract — Separating codes, initially introduced to test automaton, have revived lately in the study of fingerprinting codes, which are used for copyright protection. Separating codes play their role in making the fingerprinting scheme secure against coalitions of pirates. We provide here better bounds on such codes.

Index Terms— separating code, copyright protection.

I. INTRODUCTION

Separating codes were introduced in 1969 and have been the topic of several papers with various motivations. Many initial results are due to Sagalovich; see [5] for a survey, and also [4, 8]. New applications of separating codes have appeared during the last decade, namely *traitor tracing* and *fingerprinting*.

Fingerprinting is a proposed technique for copyright protection. The vendor has some copyrighted work of which he wants to sell copies to customers. If he is not able to prevent the customer from duplicating his copy, he may individually mark every copy sold with a unique fingerprint. If an illegal copy (for which the vendor has not been paid) subsequently appears, it may be traced back to one legal copy and one pirate via the fingerprint. A pirate is here any customer guilty of illegal copying of the copyrighted work.

Traitor tracing is the same idea applied to broadcast encryption keys. E.g. the vendor broadcasts encrypted pay-TV, and each customer buys or leases a decoder box to be able to decrypt the programmes. If the vendor is not able to make the decoder completely tamperproof, he may fingerprint the decryption keys which are stored in the box.

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The set of fingerprints in use, is called the fingerprinting code. Separating codes are used in the study of *collusion secure* fingerprinting codes. If several pirates collude, they possess several copies with different fingerprints. By comparing their copies, they will find differences which must be part of the fingerprint. These identified ‘marks’ may be changed to produce a false fingerprint. A collusion secure code should aim to identify at least one of the pirates from this false fingerprint.

We shall introduce two useful concepts regarding collusion secure code. If the code is *t*-frameproof, it is impossible for any collusion of at most *t* pirates to produce a false fingerprint which is also a valid fingerprint of an innocent user. In other words, no user may be framed by a coalition of *t* pirates or less. A *t*-frameproof code is the same as a $(t, 1)$ -separating code, which will be defined formally in the next section.

If the code is *t*-identifying, the vendor is always able to identify at least one pirate from any coalition of size at most *t*, given a false fingerprint created by the coalition. Good *t*-identifying codes are rare, so we are also interested in probabilistic (t, ϵ) -identifying codes, where the vendor is able to identify a pirate with probability $1 - \epsilon$ for some small ϵ . Barg et al. [1] apply (t, t) -separating codes to construct (t, ϵ) -identifying codes.

II. DEFINITIONS

For any positive real number x we denote by $\lceil x \rceil$ the smallest integer at least equal to x . Let $GF(q)$ be a finite field of q elements, and $GF(q)^n$ the n -dimensional vector space thereover. A subset $C \subseteq GF(q)^n$ is called an (n, M) -code if $|C| = M$. For any vector $\mathbf{x} \in GF(q)^n$, we write x_i for the i -th component, so that $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

Consider a subset $\mathcal{C} \subseteq C$. For any position i , we define the *projection* $P_i(\mathcal{C}) = \bigcup_{\mathbf{a} \in \mathcal{C}} \{a_i\}$. The *feasible*

set of C is

$$F(C) = \{x \in GF(q)^n : \forall i, x_i \in P_i(C)\}.$$

If C is the fingerprints held by some pirate coalition, then $F(C)$ is the set of fingerprints they may produce. If two non-intersecting coalitions can produce the same descendant, i.e., if their feasible sets intersect, it will be impossible to trace with certainty even one pirate. This motivates the following definition.

Definition 1 A code C is (t, t') -separating if, for any pair (T, T') of disjoint subsets of C where $|T| = t$ and $|T'| = t'$, the feasible sets are disjoint, i.e. $F(T) \cap F(T') = \emptyset$.

Since the separation property is preserved by translation, we shall always assume that $\mathbf{0} \in C$. The separation property can be rephrased as follows when $q = 2$: For any ordered $t + t'$ -tuple of codewords, there is a coordinate where the $t + t'$ -tuple $(1..10..0)$ of weight t or its complement occurs.

III. UPPER BOUNDS ON $(t, 1)$ SEPARATING CODES

The case $t' = 1$ corresponds to “frameproof” codes introduced in [2]. Körner (personal communication) has a simplified proof of $R \leq 1/2$ for $(1, 2)$ -separation in the binary case. We generalize it to any t and q as follows.

Partition $\{1, 2, \dots, n\}$ into t almost equal parts $P_1, \dots, P_t$ of size approximately n/t . Say a codeword c is unmatched on P_i if no other codeword projects onto the same n/t -tuple on P_i . Denote by U_i the subset of codewords unmatched on P_i .

Lemma 1 If C is $(1, t)$ -separating, then every codeword c of C is unmatched on at least one P_i .

Proof: Otherwise, c could not be separated from the at most t codewords matching it on the P_i 's.

Combining $C = \bigcup U_i$ with $|U_i| \leq q^{\lceil n/t \rceil}$, we get

Theorem 1 If C is $(1, t)$ -separating, then $|C| \leq tq^{\lceil n/t \rceil}$.

For constant t , this asymptotically gives $R \leq 1/t$. This is essentially tight, as we now show.

To get infinite constructions of separating codes, we need the following constructive result from Tsfasmann [10].

Lemma 2 For any $\alpha > 0$ there is an infinite family of q -ary codes with parameters $(N, q^{NR}, N\delta)$ for $N \geq N_0(\alpha)$ and

$$R + \delta \geq 1 - (\sqrt{q} - 1)^{-1} - \alpha.$$

Together with the following easily checked sufficient condition in terms of minimum distance for a code to be t -separating:

$$td > (t - 1)n,$$

this yields

Theorem 2 For fixed t and large enough q , the largest possible rate of a q -ary family of t -separating codes satisfy

$$R = (1/t)(1 + o(1)).$$

IV. THE GENERAL CASE

A Separating Weights

Given a (t, t') -configuration (T, T') we define the separating set $\Theta(T, T')$ to be the set of coordinate positions where (T, T') is separated. Let $\theta(T, T') := \#\Theta(T, T')$ be the separating weight. Clearly $\theta(T, T') \geq 1$ is equivalent with (T, T') being separated. The minimum (t, t') -separating weight $\theta_{t,t'}(C)$ is the least separating weight of any (t, t') -configuration of C . The minimum separating weights have previously been studied by Sagalovich [5]. Clearly $\theta_{1,1}(C) = d_1(C)$.

B A stronger property in the binary case

Definition 2 (Completely Separating Code)

A binary code is said to be (t, t') -completely separating $((t, t')$ -CSS) if for any set ordered set of $t + t'$ codewords, there is at least one column with 1 in the t upper positions, and 0 elsewhere, and one column with 0 in the t upper positions and 0 in the t' lower ones.

We define $R_{SS}(t, t')$ as the largest asymptotical possible rate of a family of (t, t') -SS, and similarly define $R_{CSS}(t, t')$ for (t, t') -CSS. We clearly obtain that

$$R_{SS}(t, t') \geq R_{CSS}(t, t') \geq \frac{1}{2}R_{SS}(t, t'). \quad (1)$$

C Improved upper bounds on (t, t) -separating codes

Theorem 3 Any (t, t) -separating (θ_0, M, θ_1) code with separating weights $(\theta_1, \dots, \theta_t)$ gives rise to a completely (i, i) -separating $(\theta_{t-i}, M - 2t + 2i, 2\theta_{t+1-i})$ code with complete-separating weight θ_i , for any $i < t$.

Proof: Consider a pair of $(t - i)$ -tuples of vectors which are separated on θ_{t-i} positions. Pick any vector $\mathbf{c}$ from the first $(t - i)$ -tuple and replace the code C by its translation $C - \mathbf{c}$. Thus all the columns which separates the two tuples have the form $(0 \dots 01 \dots 1)$.

Now consider any two i -tuples of vectors. Coupling each i -tuple with a $(t - 1)$ -tuple, we get two t -tuples which must be separated on θ_t positions, i.e. the two i -tuples must have at least θ_t columns of the form $(0 \dots 01 \dots 1)$. Now, observe that we can swap the two $(t - i)$ -tuples, and the two resulting t -tuples are still separated. This guarantees at least θ_t columns of the form $(1 \dots 10 \dots 0)$.

Deleting all the columns where the two $(t - i)$ -tuples are not separated, and the words of these two tuples must this leave us with an (i, i) -CSS with separating weight θ_i and parameters $(\theta_{t-i}, M - 2t + 2i, 2\theta_{t+1-i})$, as required.

Theorem 4 Any completely (t, t) -separating $(\theta_0, M, 2\theta_1)$ code with separating weights $(\theta_1, \dots, \theta_t)$ gives rise to a completely (i, i) -separating $(\theta_{t-i}, M - 2t + 2i, 2\theta_{t+1-i})$ code with complete-separating weight θ_i , for any $i < t$.

This is proved in the same way as the previous theorem.

Theorem 5 For any (t, t) -CSS, the rate R_t satisfies

$$R_t \leq \bar{R}(2R_t/\bar{R}_{t-1}),$$

where $\bar{R}(\delta)$ is any upper bound on the rate of error-correcting codes in terms of the normalised minimum distance, and $\bar{R}_{t-1}$ is the upper bound on the rate of any $(t - 1, t - 1)$ -CSS.

Proof: Let C_{t-1} be the $(t - 1, t - 1)$ -CSS which exists by Theorem 4, and let R_{t-1} be its rate. We have that

$$\delta_t = 2 \frac{\theta_1}{\theta_0} = 2 \frac{\log M}{\theta_0} \frac{\theta_1}{\log M} = 2R_t/R_{t-1}.$$

Now, obviously $R_t \leq \bar{R}(\delta_t)$, which is decreasing in δ_t , and this gives the result.

With a completely analogous proof, we also get the following.

Theorem 6 For any (t, t) -SS, the rate R satisfies

$$R \leq \bar{R}(R/\bar{R}_{t-1}),$$

where $\bar{R}(\delta)$ is any upper bound on the rate of error-correcting codes in terms of the normalised minimum distance, and $\bar{R}_{t-1}$ is the upper bound on the rate of any $(t - 1, t - 1)$ -CSS.

Putting equality in the bounds and solving the equations, we get the upper rate bounds given as 'Bound 1' in Table 1. Comparing this to the CSS bounds by D'yachkov et al., we see that we get an improvement from $(3, 3)$ -CSS and onwards. However, D'yachkov et al.[9] have a good bound on $(2, 2)$ -CSS, and using that as a seed for the recursive bounds of our theorems, we get the better bounds given as 'Bound 2' in the table.

Example 1 Let C_1 be an asymptotic class of $(\theta_0, 2^k, \theta_1)$ $(3, 3)$ -SS. Then there is an asymptotic class C_2 of $(\theta_1, 2^k, \theta_2)$ $(2, 2)$ -CSS. We have that $R_2 = k/\theta_1 \leq 0.161$, and

$$R_1 = k/\theta_0 = R_2\delta_1 \leq 0.161\delta_1,$$

which is equivalent to

$$\delta_1 \geq R_1/0.161.$$

We can use any upper bound $\bar{R}(\delta)$ on R_1 , and get

$$R_1 \leq \bar{R}(\delta_1) \leq \bar{R}(R_1/0.161).$$

Using the linear programming bound, we get $R_1 \leq 0.0663$.

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(t, t)	Bound 1		D'yachkov et al. CSS rate	Bound 2	
	CSS rate	SS rate		CSS rate	SS rate
(1,1)	1	1	1	1	1
(2,2)	0.1712	0.2835	0.161	—	—
(3,3)	0.03742	0.06998	0.0445	0.0354	0.0663
(4,4)	0.008843	0.01721	0.0123	0.00837	0.0163
(5,5)	0.002156	0.004261	0.00333	0.00204	0.00404

Table 1: Rate bounds on CSS and SS.

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