Persistence of relative equilibria in Hamiltonian systems with non-compact symmetry

Claudia Wulff
Mathematics Institute, University of Warwick
Coventry CV47AL
United Kingdom*

Abstract

We prove new results on the persistence of Hamiltonian relative equilibria with generic velocity-momentum pairs in the case of non-compact non-free group actions and taking into account time reversibility. Our starting point is a relative equilibrium which is non-degenerate modulo isotropy which, in the case of a generic momentum implies persistence of the given relative equilibrium to all nearby momentum values with the same isotropy. We show that the analysis of the persistence problem involves the study of a singular algebraic variety which is determined solely by the symmetry group of the problem. We present persistence results for relative equilibria with velocity-momentum pairs which are regular points of this variety and give sufficient conditions for a velocity-momentum pair to be regular. We apply our results to relative equilibria of Euclidean equivariant systems, including models of rigid bodies in fluids.

1 Introduction

Relative equilibria are ubiquitous in Hamiltonian systems with symmetry modelling for example rigid bodies [1, 25, 17, 18], molecules [13, 28], systems of vortices [35, 23, 33], problems in elasticity theory [22, 26] and liquid drops [15, 20, 21]. In most cases the system under consideration is not only Hamiltonian, but also time-reversible. In contrast to the situation for general systems [8, 14] the local bifurcation theory for relative equilibria of Hamiltonian systems has been developed to a much lesser extent. There are only partial stability, persistence and bifurcation results so far, as detailed below. This is due to the fact that the conservation of momenta and symplectic structure changes the generic behaviour dramatically and has to be taken into account. Most persistence results for Hamiltonian relative equilibria which can be found in the literature only apply to compact symmetry groups, and so do not apply when there are translational symmetries present; but this is frequently the case in applications, eg in the case of translating bodies in fluids [18], vortices [33], and elasticity theory [26]. Moreover, so far time reversing symmetries have not been taken into account although they are almost always present in Hamiltonian systems.

In this paper we prove some persistence results for Hamiltonian relative equilibria in the case of noncompact group actions and taking into account reversibility. The results are new even for free non-compact group actions, but we also deal with relative equilibria with fixed isotropy in the case of a non-free group action. We assume that the relative equilibrium under consideration is *non-degenerate modulo isotropy*. This means that it satisfies a non-degeneracy condition on the fixed point space of its

*Permanent address: Department of Mathematics and Statistics, University of Surrey, Guildford GU2 7XH, United Kingdom, email: c.wulff@eim.surrey.ac.uk

isotropy subgroup, see Section 2 below. This assumption is satisfied by most relative equilibria of generic Hamiltonian systems and quite common in the literature, see [19, 27, 30, 32, 37]. Moreover we assume, as in [19, 30, 32, 37], that the relative equilibrium has a *velocity-momentum pair* which is *generic modulo isotropy*, see Section 4 below.

If the non-degeneracy condition is dispensed with then the bifurcation results which are available mainly deal with Hamiltonian equilibria rather than relative equilibria, see for example [10], [7]. An exception is the paper of Patrick & Roberts [34] where the non-degeneracy condition is replaced by a weaker transversality condition, see below.

Under the non-degeneracy assumption the following persistence results have been obtained. Patrick [32] proved a persistence result for relative equilibria of free compact group actions with generic velocity-momentum pairs. Roberts et al. [37] show persistence to all nearby momenta of non-degenerate relative equilibria with generic momenta for non-compact group actions. For related results see [6]. Montaldi [27] shows that persistence follows from formal stability of relative equilibria in the case of compact group actions. An extension of this result to special cases of noncompact groups can be found in [29]. In [27] Montaldi also proves persistence results for free compact group actions and non-generic momenta using the fact that relative equilibria are critical points of the Hamiltonian constrained to (compact) coadjoint orbits and employing topological results on the number of critical points of functions on compact manifolds. In the case of non-free group actions the persistence results mentioned above can be applied in the fixed point space of the isotropy subgroup of the relative equilibrium thus giving persistence of relative equilibria with prescribed isotropy, see Ortega and Ratiu [30]. For similar results see Roberts and de Sousa Dias [36] and Lerman and Singer [19]. Montaldi and Roberts [28] study persistence to relative equilibria with smaller isotropy under the assumption that isotropies are finite.

Patrick and Roberts [34] describe the stratification of the set of relative equilibria by their velocity-momentum pair orbit type for compact free group actions under a transversality assumption which is satisfied for all equilibria of generic Hamiltonians. They show that under this assumption the structure of the set of relative equilibria is determined by a singular algebraic variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$, the *velocity-momentum pair space*, where G is the symmetry group of the system and $\mathfrak{g}$ its Lie algebra, see Section 3 below. This variety is determined only by the structure of the Lie algebra $\mathfrak{g}$ and so is independent of the given Hamiltonian vectorfield. Generically the isotropy subgroup of the velocity-momentum pair is a maximal torus (if G is connected), and exactly in this case the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is locally a manifold. Moreover the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ and the set of relative equilibria determined by $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ are stratified by the orbit types of the action of the compact group G on the velocity-momentum pairs of the relative equilibria.

The main purpose of this article is to generalize the persistence results of Patrick [32] for relative equilibria with generic velocity-momentum pairs to noncompact group actions. The main difficulty in doing this is that the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ has quite different properties for noncompact groups than it does for compact groups. Here are the main differences:

- (i) For compact symmetry groups the isotropy subgroups of all generic velocity-momentum pairs are conjugate maximal tori. This is not true for noncompact groups: Example 4.10 c) below shows that in the case of the special Euclidean group of the plane $SE(2)$ the groups $\mathbb{R}$ and $SO(2)$ are both isotropy subgroups of generic velocity-momentum pairs;
- (ii) For compact symmetry groups the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ of velocity-momentum pairs is irreducible. Example 4.10 c) shows that this is not true for noncompact symmetry groups;
- (iii) In the case of a reducible variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ the different top-dimensional strata may even have different dimensions, see Example 4.10 e), resulting in several disjoint manifolds of relative equilibria with generic velocity-momentum pairs of different dimensions.

We show, however, that the results of Patrick [32] on the persistence of relative equilibria with generic velocity-momentum pairs are still true. To do this we employ the normal form equations near Hamiltonian relative equilibria derived in [37] combined Lie group theory, symplectic geometry and some basic algebraic geometry applied to the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$. We apply our results to the persistence of relative equilibria of Euclidean equivariant systems modelling, for example, rigid bodies in fluids. We will

see that in such systems relative equilibria with different generic velocity-momentum pairs can perform qualitatively different motions.

The rest of the paper is organized as follows. In Section 2 we introduce the setting that we work in and recall the normal form equations of Hamiltonian systems near group orbits of [37]. In Section 3 we introduce the notion of relative equilibria, non-degeneracy, velocity-momentum pairs and the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$. In Section 4 we investigate the local structure of the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ near regular velocity-momentum pairs for general Lie groups. In Section 5 of this paper we use the theory presented in the previous sections to extend the persistence results of Patrick [32] and Ortega and Ratiu [30] on relative equilibria with velocity-momentum pairs which are generic modulo isotropies to non-compact group actions. In Section 6 we prove first results on the persistence of reversible relative equilibria, extending the techniques presented in Section 5 to the time reversible case.

2 Slice equations near group orbits

In this section we describe the setting that we work in and summarize the results of [37] on the equations of Hamiltonian systems in slice coordinates. We will use these results heavily in the following sections.

We consider a Hamiltonian system

$$\dot{x} = f_H(x) \quad (2.1)$$

on a finite-dimensional symplectic manifold $\mathcal{M}$ with symplectic form $\omega(\cdot, \cdot)$:

$$\omega(f_H(x), v) = DH(x)v, \quad x \in \mathcal{M}, v \in \mathcal{T}_x\mathcal{M}.$$

We assume that a finite-dimensional Lie group G acts on $\mathcal{M}$ properly and semi-symplectically. This means that there is a *reversible sign* $s : G \rightarrow \{\pm 1\}$ such that

$$\omega_{gx}(gu, gv) = s(g)\omega_x(u, v), \quad x \in \mathcal{M}, u, v \in \mathcal{T}_x\mathcal{M}, g \in G.$$

We assume that the Hamiltonian H is G -invariant. This implies that (2.1) is G -semiequivariant, i.e.

$$f_H(gx) = s(g)gf_H(x), \quad x \in \mathcal{M}, g \in G.$$

So whenever $x(t)$ is a solution of (2.1) then so is $gx(s(g)t)$. We call $\Gamma = \ker(s)$ the *symmetries* of (2.1) and each $\rho \in G \setminus \Gamma$ a *reversing symmetry* of (2.1). Let $\mathfrak{g}$ denote the Lie algebra of G . By Noether's theorem locally there is a conserved quantity $\mathbf{J}^\xi$ for each $\xi \in \mathfrak{g}$ such that $\mathbf{J}^\xi$ is the Hamiltonian for the symplectic flow $x \rightarrow \exp(\xi t)x$ [1, 25]. Moreover $\mathbf{J}^\xi$ is linear in ξ , so that $\mathbf{J}$ maps to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G . Let Ad_g , $g \in G$, denote the adjoint action of G on $\mathfrak{g}$: $\text{Ad}_g\xi = g\xi g^{-1}$, $\xi \in \mathfrak{g}$, $g \in G$, and consider the s -coadjoint action of G on $\mathfrak{g}^*$ given by

$$g\mu = s(g)(\text{Ad}_g^*)^{-1}\mu, \quad g \in G. \quad (2.2)$$

We assume throughout the paper that $\mathbf{J}$ is defined on the whole of $\mathcal{M}$ and G -equivariant with respect to the usual G -action on $\mathcal{M}$ and the s -coadjoint action on $\mathfrak{g}^*$.

Let $p \in \mathcal{M}$ be fixed. By the Palais slice theorem [31] we can write a G -invariant neighbourhood $\mathcal{U}$ of the group orbit Gp as $\mathcal{U} = G \times_{G_p} \mathcal{N}$. Here $\mathcal{N}$ is a G_p -invariant slice transversal to Gp at p , and we use the coordinate transformation $u = \tau(g, v) = g\psi(v)$, where $\psi : \mathcal{T}_p\mathcal{M} \rightarrow \mathcal{M}$ is a G_p -equivariant chart near p , $v \in \mathcal{N}$, and $g \in G$. Then $p \simeq (\text{id}, 0)$. The symbol $\times_{G_p}$ stands for the identifications $u = (g, v) = (gg_p^{-1}, g_pv)$, $v \in G_p$. If the system is Hamiltonian and we let $\mu = \mathbf{J}(p)$ denote the momentum value at p then the tangent space $\mathcal{T}_p\mathcal{M}$ can be decomposed into

$$\mathcal{T}_p\mathcal{M} = \mathcal{T} \oplus \mathcal{N}, \quad \mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1, \quad \mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$$

where $\mathcal{T}_i$ and $\mathcal{N}_i$, $i = 0, 1$, are G_p -invariant,

$$\mathcal{T}_0 = \mathfrak{g}_\mu p \simeq \mathfrak{g}_\mu / \mathfrak{g}_p, \quad \mathcal{N}_0 \simeq (\mathfrak{g}_\mu / \mathfrak{g}_p)^*, \quad \mathcal{T}_1 \simeq \mathfrak{g} / \mathfrak{g}_\mu,$$

$\mathcal{T}_0 \oplus \mathcal{N}_0$ is a symplectic space with standard symplectic form, $\mathcal{T}_1$ is a symplectic space, endowed with the Kostant-Kirillov-Souriau symplectic form, and $\mathcal{N}_1$ is the symplectic normal space at p , see [2, 37]. Here $\mathfrak{g}_\mu^*$ is the dual of the Lie algebra $\mathfrak{g}_\mu$ of the momentum isotropy G_μ of $\mu = \mathbf{J}(p)$ with respect to the s -coadjoint group action (2.2). This decomposition of $\mathcal{T}_p\mathcal{M}$ is called the *Witt-decomposition*. We have $\omega|_{\mathcal{T}_p\mathcal{M}} = \omega|_{\mathcal{T}_0 \oplus \mathcal{N}_0} + \omega|_{\mathcal{N}_1} + \omega|_{\mathcal{T}_1}$, and the model space $G \times_{G_p} \mathcal{N}$ carries a natural symplectic structure [24, 12, 37]. In the important special case where G_p is finite the space $\mathcal{N}_0 \equiv \mathfrak{g}_\mu^*$ can be interpreted as a section transverse to the group orbit $G\mu$ at μ , as we detail in Lemma 2.1 below. Moreover in this case the momentum level set $\mathbf{J}^{-1}(\mu) \subseteq \mathcal{M}$ is a manifold near p so that $\mathcal{N}_1$ can be interpreted as a section transverse to the group orbit $G_\mu p$ inside the momentum level set $\mathbf{J}^{-1}(\mu)$.

We will need the differential equations near a group orbit Gp expressed in bundle coordinates $u = (g, v)$, $v = (\nu, w)$, $g \in G$, $\nu \in \mathcal{N}_0$, $w \in \mathcal{N}_1$. To describe these we first introduce some more notation: Let $\mathbf{n}_\mu$ denote a G_p -invariant complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. Then we can identify

$$\mathfrak{g}_\mu^* \simeq \text{ann}(\mathbf{n}_\mu), \quad (\mathfrak{g}/\mathfrak{g}_\mu)^* \simeq \text{ann}(\mathfrak{g}_\mu)$$

where $\text{ann}(\mathbf{n}_\mu)$ and $\text{ann}(\mathfrak{g}_\mu)$ denote the annihilators of $\mathbf{n}_\mu$ and $\mathfrak{g}_\mu$ in $\mathfrak{g}^*$. We denote the identity component of a subgroup $\tilde{G}$ of G by $\tilde{G}^0$.

We say that μ is *split* [37, 11], if there is a G_μ^0 -invariant complement $\mathbf{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$. This is always the case if G is compact, but it is in general not satisfied if G is non-compact, for example if $G = \text{SE}(3)$ is the Euclidean group of motions in three-space, see [37].

Let $P_{\text{ann}(\mathfrak{g}_\mu)}$ be the projection from $\mathfrak{g}^*$ to $\text{ann}(\mathfrak{g}_\mu)$ with kernel $\mathfrak{g}_\mu^* = \text{ann}(\mathbf{n}_\mu)$. Then for any $\xi \in \mathfrak{g}_\mu$ and any small $\nu \in \text{ann}(\mathbf{n}_\mu)$ there is a unique $\eta \in \mathbf{n}_\mu$ such that

$$P_{\text{ann}(\mathfrak{g}_\mu)}(\text{ad}_{\xi+\eta}^*(\mu + \nu)) = 0$$

as is shown in [37, Prop. 2.5]. Moreover $\eta = \eta_\mu(\nu)\xi$ is linear in ξ . If μ is split then $\mathbf{n}_\mu$ is G_μ^0 -invariant, and therefore $\mathfrak{g}_\mu^* = \text{ann}(\mathbf{n}_\mu)$ is G_μ^0 -invariant, which implies that $\eta \equiv 0$. Define $j_\mu(\nu) \in \text{Mat}(\mathfrak{g}_\mu, \mathfrak{g})$ by $j_\mu(\nu) = \text{id} + \eta_\mu(\nu)$.

We define $\tilde{Z}_{\mu,\nu}$, where $\nu \in \mathfrak{g}_\mu^*$ is small, by the condition

$$g \in \tilde{Z}_{\mu,\nu} \iff P_{\text{ann}(\mathfrak{g}_\mu)}(\text{Ad}_{g^{-1}}^*(\mu + \nu) - \mu) = 0 \quad (2.3)$$

and $Z_{\mu,\nu}$ to be the path connected component of $\tilde{Z}_{\mu,\nu}$ containing the identity. Clearly $G_{\mu+\nu}^0 \subseteq Z_{\mu,\nu}$ for every $\nu \in \text{ann}(\mathbf{n}_\mu)$.

Lemma 2.1

- a) Let $\mu \in \mathfrak{g}^*$. Then $\mu + \mathfrak{g}_\mu^*$ with $\mathfrak{g}_\mu^* = \text{ann}(\mathbf{n}_\mu)$ is a section transverse to $G\mu$ at μ . More precisely, there are neighbourhoods U of μ in $\mathfrak{g}^*$, $U_G(\text{id})$ of id in G and $U_{\mathfrak{g}_\mu^*}(0)$ of 0 in $\mathfrak{g}_\mu^*$ such that

$$\text{for all } \hat{\mu} \in U \text{ there are } \gamma \in U_G(\text{id}), \nu \in U_{\mathfrak{g}_\mu^*}(0) \text{ with } \hat{\mu} = \gamma(\mu + \nu), \quad (2.4)$$

and (γ, ν) is locally unique modulo the action of $U_G(\text{id}) \cap Z_{\mu,\nu}$ given by

$$\gamma_\mu(\gamma, \nu) := (\gamma\gamma_\mu^{-1}, \gamma_\mu \circ \nu), \quad \gamma_\mu \circ \nu := \gamma_\mu(\mu + \nu) - \mu, \quad \gamma_\mu \in U_G(\text{id}) \cap Z_{\mu,\nu}. \quad (2.5)$$

Moreover if we choose $\gamma = \exp(\chi)$ where $\chi \in \mathbf{n}_\mu$, $\chi \approx 0$, then χ and ν are locally unique.

- b) For $\nu \in \mathfrak{g}_\mu^* = \text{ann}(\mathbf{n}_\mu)$ small there is some neighbourhood $U_G(\text{id})$ of the identity in G such that the set $Z_{\mu,\nu} \cap U_G(\text{id})$ is a manifold of dimension $\dim G_\mu$. Moreover, $T_{\text{id}}Z_{\mu,\nu} = j_\mu(\nu)\mathfrak{g}_\mu$.
- c) If μ is split and $\mathbf{n}_\mu$ is G_μ^0 -invariant then $j_\mu \equiv \text{id}$, $Z_{\mu,\nu} = G_\mu^0$, $\gamma_\mu \circ \nu = \gamma_\mu \nu$ in (2.5), and $G_{\mu+\nu}^0 \subseteq G_\mu^0$ for any small $\nu \in \mathfrak{g}_\mu^* = \text{ann}(\mathbf{n}_\mu)$.

Proof. Part a) and b) follow from the implicit function theorem and the definition of $Z_{\mu,\nu}$. To prove c) let μ be split. Then clearly $j_\mu(\nu) \equiv \text{id}$ for ν small, and therefore $G_\mu^0 \subset Z_{\mu,\nu}$. The invariance of $Z_{\mu,\nu}$ under right multiplication of G_μ^0 and part b) imply the equality $G_\mu^0 = Z_{\mu,\nu}$. Finally $G_{\mu+\nu}^0 \subseteq Z_{\mu,\nu} = G_\mu^0$. $\blacksquare$

Let $\mathbf{m}_\mu$ be a G_p -invariant complement to $\mathfrak{g}_p$ in $\mathfrak{g}_\mu$. We decompose

$$\mathfrak{g}_\mu^* \simeq (\mathfrak{g}_\mu/\mathfrak{g}_p)^* \oplus \mathfrak{g}_p^*, \quad \mathfrak{g}_p^* \simeq \text{ann}_{\mathfrak{g}_\mu^*}(\mathfrak{g}_p), \quad (\mathfrak{g}_\mu/\mathfrak{g}_p)^* \simeq \text{ann}_{\mathfrak{g}_\mu^*}(\mathfrak{g}_p).$$

For $\xi \in \mathbf{m}_\mu \simeq \mathfrak{g}_\mu/\mathfrak{g}_p$ let $\pi_{\mathbf{m}_\mu}$ be the projection from $\mathfrak{g}_\mu$ to $\mathbf{m}_\mu$ with kernel $\mathfrak{g}_p$ and define for $\eta \in \mathbf{m}_\mu$

$$\overline{\text{ad}}_\xi^\mu \eta = \pi_{\mathbf{m}_\mu} \text{ad}_\xi^\mu \eta$$

where ad^μ denotes the adjoint action of $\mathfrak{g}_\mu$ on $\mathfrak{g}_\mu$. Let $\mathbf{J}_{\mathcal{N}_1} : \mathcal{N}_1 \rightarrow \mathfrak{g}_p^*$ be a G_p -equivariant momentum map for the action of G_p on $\mathcal{N}_1$, and let $\zeta(t) = \nu(t) + \mathbf{J}_{\mathcal{N}_1}(w(t)) \in \mathfrak{g}_\mu^*$. Let $h(\zeta, w) = h(\nu, w)$ where $\zeta = \nu + \zeta_p$, $\nu \in (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$, $\zeta_p \in \mathfrak{g}_p^*$, denote the Hamiltonian in bundle coordinates. We have the following theorem:

Theorem 2.2 [37, Theorem 3.1]

The equation (2.1) in bundle coordinates $(g, \nu, w) \in \Gamma \times (\mathcal{N}_0 \oplus \mathcal{N}_1)$ takes the form

$$\dot{g} = g j_\mu(\zeta) D_\zeta h(\zeta, w), \quad \dot{\zeta} = \text{ad}_{j_\mu(\zeta) D_\zeta h(\zeta, w)}^*(\zeta + \mu), \quad \dot{w} = J_{\mathcal{N}_1} D_w h(\zeta, w) \quad (2.6)$$

where $J_{\mathcal{N}_1}$ is a complex structure for the symplectic form ω on $\mathcal{N}_1$ and $\zeta(t) = \nu(t) + \mathbf{J}_{\mathcal{N}_1}(w(t)) \in \mathfrak{g}_\mu^*$. If μ is split then the $(\dot{g}, \dot{\zeta})$ -equations simplify to $\dot{g} = g D_\zeta h$, $\dot{\zeta} = \text{ad}_{D_\zeta h}^*(\zeta)$.

Projecting the ζ -equation onto $\mathcal{N}_0 = (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ we get the following more explicit equation for the evolution of $\nu(t)$:

$$\dot{\nu} = \overline{\text{ad}}_{D_{\nu h}}^{\mu,*} \nu + \text{ad}_{D_{\nu h}}^{\mu,*} \mathbf{J}_{\mathcal{N}_1}(w) + P_{(\mathfrak{g}_\mu/\mathfrak{g}_p)^*}(\text{ad}_{\hat{\eta}_\mu(\nu, w)}^*(\nu + \mathbf{J}_{\mathcal{N}_1}(w))) \quad (2.7)$$

where $\hat{\eta}(\nu, w) = \eta_\mu(\nu) D_\nu h(\nu, w)$. Here $\text{ad}_\xi^{\mu,*}$ is the dual operator to ad_ξ^μ and $P_{(\mathfrak{g}_\mu/\mathfrak{g}_p)^*}$ the projection from $\mathfrak{g}^*$ onto $\text{ann}(\mathfrak{g}_p + \mathbf{n}_\mu) \simeq (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ with kernel $\text{ann}(\mathbf{m}_\mu)$.

3 Relative equilibria

Relative equilibria are steady states for the Hamiltonian system (2.1) defined on the space of Γ -group orbits. Note that the G -semiequivariance of (2.1) does not imply that the flow descends to a flow on the space of orbits for the full action of G , so it does not make sense to replace Γ by G . We say that a point $p \in \mathcal{M}$ lies on a relative equilibrium Γp if there is some $\xi \in \mathfrak{g}$, called *drift velocity* of the relative equilibrium, such that

$$f_H(p) = \xi p.$$

In other words, p is a critical point of the Hamiltonian $H^\xi = H - \mathbf{J}^\xi$ in the frame moving with velocity $\xi \in \mathfrak{g}$. We call the relative equilibrium Γp reversible if it is invariant under the action of $G/\Gamma \simeq \mathbb{Z}_2$, and non-reversible otherwise. Denote the isotropy of p in G by G_p , and $\Gamma_p = G_p \cap \Gamma$. We either have $G_p/\Gamma_p \simeq \mathbb{Z}_2$ which implies that the relative equilibrium Γp is reversible, or $G_p = \Gamma_p$ in which case it is non-reversible [37]. If $\rho \in G_p \setminus \Gamma_p$ we say the p is ρ -reversible.

Relative equilibria near Γp correspond to relative equilibria with respect to the Γ_p -action of the (ν, w) -system of (2.6), (2.7) given by

$$\dot{\nu} = \overline{\text{ad}}_{D_{\nu h}}^{\mu,*} \nu + \text{ad}_{D_{\nu h}}^{\mu,*} \mathbf{J}_{\mathcal{N}_1}(w) + P_{(\mathfrak{g}_\mu/\mathfrak{g}_p)^*}(\text{ad}_{\hat{\eta}_\mu(\nu, w)}^*(\nu + \mathbf{J}_{\mathcal{N}_1}(w))), \quad \dot{w} = J_{\mathcal{N}_1} D_w h \quad (3.1)$$

on the slice $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$. In order to find these relative equilibria we need to solve the equation

$$\left(\overline{\text{ad}}_{D_{\nu h}}^{\mu,*} \nu + \text{ad}_{D_{\nu h}}^{\mu,*} \mathbf{J}_{\mathcal{N}_1}(w) + P_{(\mathfrak{g}_\mu/\mathfrak{g}_p)^*}(\text{ad}_{\hat{\eta}_\mu(\nu, w)}^*(\nu + \mathbf{J}_{\mathcal{N}_1}(w))) + \text{ad}_{\xi_p}^* \nu \right) = 0 \quad (3.2)$$

for some $\xi_p \in \mathfrak{g}_p$ which is the drift velocity of the relative equilibrium of (3.1). These equations have already been formulated by Chossat et al. [6]. Instead of solving the rather complicated equation (3.2)

we will restrict our attention to persistence of non-degenerate relative equilibria Γp with fixed isotropy G_p . For a subgroup $\tilde{G}$ of G acting on $\mathcal{M}$ denote by

$$\text{Fix}_{\mathcal{M}}(\tilde{G}) = \{u \in \mathcal{M} \mid \tilde{g}u = u \ \forall \tilde{g} \in \tilde{G}\}$$

the fixed point space of $\tilde{G}$, and by $N(\tilde{\Gamma})$ the normalizer of $\tilde{\Gamma} = \tilde{G} \cap \Gamma$. Then $\text{Fix}_{\mathcal{M}}(\tilde{\Gamma})$ is symplectic, invariant under the time-evolution of (2.1), and $N(\tilde{\Gamma})$ acts on $\text{Fix}_{\mathcal{M}}(\tilde{\Gamma})$. Now we define what we mean by a nondegenerate relative equilibrium.

Definition 3.1 *A relative equilibrium Γp of (2.1) with drift velocity ξ , momentum μ and symplectic normal space $\mathcal{N}_1$ is said to be non-degenerate modulo isotropy if $D^2 H^\xi(p)|_{\text{Fix}_{\mathcal{N}_1}(G_p)}$ is a non-degenerate quadratic form on $\text{Fix}_{\mathcal{N}_1}(G_p)$.*

Since (2.1) defines a vectorfield on $\text{Fix}_{\mathcal{M}}(\Gamma_p)$ and we are only interested in relative equilibria with fixed isotropy G_p , we define

$$M = \text{Fix}_{\mathcal{M}}(\Gamma_p), \quad L = N(\Gamma_p)/\Gamma_p, \quad \Lambda = (N(\Gamma_p) \cap \Gamma)/\Gamma_p, \quad N = \text{Fix}_{\mathcal{N}}(\Gamma_p). \quad (3.3)$$

Then Λ acts symplectically on the symplectic manifold M , and so there is a momentum map $\mathbf{J}_M : M \rightarrow \mathfrak{l}^*$ where $\mathfrak{l}$ is the Lie algebra of L . The relation between $\mathbf{J}_M$ and $\mathbf{J}$ is as follows: the Lie algebra $\mathfrak{l}$ can be embedded into $\mathfrak{g}$ by the identification $\mathfrak{l} \simeq \text{Fix}_{\mathfrak{g}}(\Gamma_p) \cap \mathfrak{g}_p^\perp$ with respect to some G_p -invariant inner product on $\mathfrak{g}$. Analogously $\mathfrak{l}^* \simeq \text{Fix}_{\mathfrak{g}^*}(\Gamma_p) \cap \text{ann}(\mathfrak{g}_p)$. With these identifications $\mathbf{J}_M$ is the restriction of $\mathbf{J}$ to M and $\mathfrak{l}$, ie, $\mathbf{J}_M^{\xi_L}(x) = \mathbf{J}^\xi(x)$, $\xi_L \simeq \xi \in \mathfrak{l}$. Note that Λ acts freely on M near p . Since $\xi p = f_H(p)$, since f_H is G -equivariant and p is G_p -invariant we can always choose the drift velocity ξ of the relative equilibrium which is determined modulo $\mathfrak{g}_p$ such that $\xi \simeq \xi_L \in \mathfrak{l} \subseteq \text{Fix}_{\mathfrak{g}}(G_p) \subseteq \mathfrak{g}$, cf. [16]. We further define

$$N = N_0 \oplus N_1, \quad N_0 = \text{Fix}_{\mathcal{N}_0}(\Gamma_p) \equiv \mathfrak{l}_\mu^*, \quad N_1 = \text{Fix}_{\mathcal{N}_1}(\Gamma_p) \quad (3.4)$$

where $\mathfrak{l}_\mu$ is the Lie algebra of L_μ (here we consider the momentum $\mu \simeq \mu|_{\mathfrak{l}}$ as an element of $\mathfrak{l}^*$). Then Λ acts freely on M near p and the relative equilibrium Λp in M is non-reversible if $L_p = \{\text{id}\}$ and reversible if $L_p = \mathbb{Z}_2 := \{\text{id}, \rho\}$. Here $\rho \in G_p \setminus \Gamma_p$ is some reversing symmetry of the relative equilibrium in $\mathcal{M}$ and is of order two when viewed as element of L . An L -invariant neighbourhood U of Lp in M , expressed in bundle coordinates, becomes $U \simeq L \times_{L_p} N$. Relative equilibria near Γp in $\mathcal{M}$ with the same isotropy G_p correspond to relative equilibria near Λp in M with isotropy L_p . To find these relative equilibria on M we need to solve the equation

$$\begin{pmatrix} \text{ad}_{j_{\mu_L}(\nu)}^* D_\nu h(\nu, w)(\nu + \mu_L) \\ D_w h(\nu, w) \end{pmatrix} = 0 \quad (3.5)$$

on M . In this case $D_\nu h(0) = \xi_L \in \mathfrak{l}_\mu$ is the drift velocity of the relative equilibrium Λp in M , ad denotes the adjoint action on $\mathfrak{l}$ and $j_{\mu_L}(\nu) : \mathfrak{l}_\mu \rightarrow \mathfrak{l}$, $\nu \in \mathfrak{l}_\mu^*$. In bundle coordinates the momentum map $\mathbf{J}_M$ on M is given by $\mathbf{j}_M(g, \nu) = g(\mu_L + \nu)$, $g \in L$, $\nu \in N_0 = \text{Fix}_{\mathcal{N}_0}(\Gamma_p) = \mathfrak{l}_\mu^*$ [24], so that in bundle coordinates on M the Hamiltonian $H^\xi = H - \mathbf{J}^\xi$ in the frame moving with velocity ξ becomes $h^{\xi_L}(\nu, w) = h(\nu, w) - (g(\mu_L + \nu))(\xi_L)$. Computing $D^2 H^\xi(p)|_{\text{Fix}_{\mathcal{N}_1}(G_p)}$ in slice coordinates we therefore get

$$D^2 H^\xi(p)|_{\text{Fix}_{\mathcal{N}_1}(G_p)} = D_w^2 h(0)|_{\text{Fix}_{N_1}(L_p)}$$

(since $g = \text{id}$ on the slice). Therefore the condition that the relative equilibrium is non-degenerate on $\text{Fix}_{\mathcal{M}}(G_p)$ translates into the condition that $D_w^2 h(0)|_{\text{Fix}_{N_1}(L_p)}$ is non-degenerate, where $N_1 = \text{Fix}_{\mathcal{N}_1}(\Gamma_p)$. Due to the non-degeneracy condition we can solve the second equation of (3.5) by the implicit function theorem for $\nu \in \text{Fix}_{N_0}(L_p) = \text{Fix}_{\mathcal{N}_0}(G_p)$ to get a locally unique path $w(\nu) \in \text{Fix}_{N_1}(L_p) = \text{Fix}_{\mathcal{N}_1}(G_p)$ satisfying $D_w h(\nu, w(\nu)) = 0$. Inserting this into the first equation of (3.5) we are left with the Lie group theoretic equation

$$F(\nu) = f(\xi(\nu), \mu(\nu)) = 0 \quad \text{where} \quad F : \text{Fix}_{\mathfrak{l}_\mu^*}(L_p) \rightarrow \text{Fix}_{\mathfrak{l}_\mu^*}(\Lambda_p), \quad (3.6)$$

$$\xi(\nu) = j_{\mu_L}(\nu)D_\nu h(\nu, w(\nu)), \quad \mu(\nu) = \mu_L + \nu, \quad (3.7)$$

and

$$f : \mathbf{l} \oplus \mathbf{l}^* \rightarrow \mathbf{l}^*, \quad f(\xi, \mu) = \text{ad}_\xi^* \mu. \quad (3.8)$$

This procedure is quite common [27, 36, 6, 37]. So we see that the set of relative equilibria close to Γp with fixed isotropy G_p is related to the algebraic variety

$$f^{-1}(0) = (\mathbf{l} \oplus \mathbf{l}^*)^c = \{(\xi, \mu) \in \mathbf{l} \oplus \mathbf{l}^*, \quad f(\xi, \mu) = \text{ad}_\xi^* \mu = 0\} \quad (3.9)$$

of pairs $(\xi, \mu) \in \mathbf{l} \oplus \mathbf{l}^*$ which ‘commute’ in the sense that $\text{ad}_\xi^* \mu = 0$. The drift velocity $\xi_L \in \mathbf{l}$ of a relative equilibrium Λp in $M = \text{Fix}_{\mathcal{M}}(\Gamma_p)$ with momentum $\mu_L = \mathbf{J}_M(p) \in \mathbf{l}^*$ satisfies $\xi_L \in \mathbf{l}_\mu$ so that $(\xi_L, \mu_L) \in (\mathbf{l} \oplus \mathbf{l}^*)^c$. We therefore call the elements of $(\mathbf{l} \oplus \mathbf{l}^*)^c$ *velocity-momentum-pairs* and $(\mathbf{l} \oplus \mathbf{l}^*)^c$ itself *velocity-momentum pair space*. In [34] the elements of $(\mathbf{l} \oplus \mathbf{l}^*)^c$ are called generator-momentum pairs.

The following proposition summarizes the relationship between solutions of (3.6) and relative equilibria of (2.1).

Proposition 3.2 *Let Γp be a relative equilibrium of (2.1) with momentum $\mu = \mathbf{J}(p)$ which is non-degenerate modulo isotropy, let $M = \text{Fix}_{\mathcal{M}}(\Gamma_p)$ and let L, Λ, N_0 and N_1 be as in (3.3), (3.4). Then every relative equilibrium close to Γp with isotropy G_p corresponds to a solution of (3.6) – (3.8). In particular, if $\nu(\tau), \tau \in \mathbb{R}^r, \nu(0) = 0$, is an r -dimensional manifold of solutions to (3.6) then there is a manifold $M_{\text{RE}} \subset M$ of relative equilibria with isotropy conjugate to L_p of dimension*

$$\dim(M_{\text{RE}}) = \dim \Lambda + r = \dim N(\Gamma_p) - \dim \Gamma_p + r$$

which is formed by the r -dimensional family of relative equilibria $Lp(\tau), p(\tau) \equiv (\text{id}, \nu(\tau), w(\nu(\tau)))$, with momentum $\mathbf{J}_M(p(\tau)) = \mu(\tau) = \mu_L + \nu(\tau)$ and drift velocity $\xi(\nu(\tau))$. This gives a manifold $\mathcal{M}_{\text{RE}}$ of relative equilibria $\Gamma p(\tau)$ with isotropy conjugate to G_p in $\mathcal{M}$ of dimension

$$\dim(\mathcal{M}_{\text{RE}}) = \dim \Gamma + r - \dim \Gamma_p$$

with $M_{\text{RE}} = \text{Fix}_{\mathcal{M}_{\text{RE}}}(\Gamma_p)$. The relative equilibria are reversible if $L_p \simeq \mathbb{Z}_2$, and non-reversible if L_p is trivial.

Because of this proposition in the following sections we can restrict to an L -invariant neighbourhood U of Lp in the manifold $M = \text{Fix}_{\mathcal{M}}(\Gamma_p)$ where the symmetry group Λ acts freely. Moreover we see that in order to obtain persistence results for nondegenerate relative equilibria we need to understand the solution manifold of (3.6). This is determined by the structure of the velocity-momentum pair space $(\mathbf{l} \oplus \mathbf{l}^*)^c$. In Section 4 we will study the structure of the velocity-momentum pair space near regular points. In Section 5 we will solve (3.6) under a genericity assumption on the velocity-momentum pair without taking reversibility into account, ie we will consider the case $G = \Gamma$ so that L_p is trivial. This yields persistence results for general relative equilibria. In Section 6 we will deal with the case $L_p \simeq \mathbb{Z}_2$ and prove a result on the persistence of reversible relative equilibria to nearby reversible relative equilibria.

4 Regular velocity-momentum pairs

In this section we investigate the variety $V = (\mathbf{g} \oplus \mathbf{g}^*)^c$ of velocity-momentum pairs introduced in (3.9). We introduce the notion of a regular velocity-momentum pair, describe the variety locally using a section transverse to the group orbit and give sufficient conditions for a velocity-momentum pair to be regular. We do not consider reversing symmetries in this section. They are treated in Section 6 below.

The following concepts generalize the notions of [32, 34] to non-compact groups.

Definition 4.1

- a) We call $\mu \in \mathbf{g}^*$ regular (or minimal [37]) if its isotropy subgroup G_μ for the coadjoint action of G on $\mathbf{g}^*$ has minimal dimension $r_\mu(G)$.*

- b) We call $\xi \in \mathfrak{g}$ regular (or minimal) if its isotropy G_ξ for the adjoint action of G on $\mathfrak{g}$ has minimal dimension $r_\xi(G)$.
- c) We call a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ regular if $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is a manifold near (ξ, μ) . In this case let $r = r_{(\xi, \mu)}(G)$ be such that $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ has dimension $\dim_{(\xi, \mu)}(\mathfrak{g} \oplus \mathfrak{g}^*)^c = \dim G + r$ near (ξ, μ) .
- d) Define a G -action on $\mathfrak{g} \oplus \mathfrak{g}^*$ by

$$g(\xi, \mu) = (\text{Ad}_g \xi, g\mu), \quad \xi \in \mathfrak{g}, \mu \in \mathfrak{g}^*, g \in G,$$

so that $G_{(\xi, \mu)} = G_\xi \cap G_\mu$. We say that a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is minimal if $\dim G_{(\xi, \mu)}$ is locally minimal in $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$.

Remark 4.2 We occasionally call the elements of $\mathfrak{g}^*$ momenta because they can be interpreted as values of a momentum map $\mathbf{J} : \mathcal{M} \rightarrow \mathfrak{g}^*$ on a symplectic manifold $\mathcal{M}$ with a symplectic G -action. But the notion of a regular momentum μ where μ is a regular element of $\mathfrak{g}^*$ in the above defined sense must not be confused with the notion of a regular value $\mu = \mathbf{J}(p)$ of a momentum map.

Patrick [32] considered compact groups G and called $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ or $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ regular if $\mathfrak{g}_\xi$, $\mathfrak{g}_\mu$ or $\mathfrak{g}_\xi \cap \mathfrak{g}_\mu$ is the Lie algebra of a maximal torus of G . Recall that a maximal torus $T \subset G$ of a compact group G is an abelian subgroup of G^0 of maximal dimension. For compact groups G every $\xi \in \mathfrak{g}$ lies in the Lie algebra $\mathfrak{t}$ of a maximal torus T of G , almost every $\xi \in \mathfrak{g}$ generates a maximal torus, and all maximal tori are conjugate [5]. We denote the dimension of the maximal tori of a compact group G by $\text{rank}(G)$. The following proposition states that for compact groups regularity in the sense of Patrick is equivalent to regularity in our sense.

Proposition 4.3 [34] *In the case of a compact group G a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is regular if and only if it is minimal. Moreover $\xi \in \mathfrak{g}$ ($\mu \in \mathfrak{g}^*$, $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$) are minimal if and only if $\mathfrak{g}_\xi$ ($\mathfrak{g}_\mu$, $\mathfrak{g}_{(\xi, \mu)}$) is the Lie algebra of a maximal torus in G , and in this case $r_\xi(G) = \text{rank}(G)$ ($r_\mu(G) = \text{rank}(G)$, $r_{(\xi, \mu)}(G) = \text{rank}(G)$).*

Proof. The theory of maximal tori [5] implies that the isotropy subalgebras of minimal velocities, momenta, velocity-momentum pairs of compact group actions are Lie algebras of maximal tori. To be more precise, let $\xi \in \mathfrak{g}$. Then there is a maximal torus T such that $\xi \in \mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of T , see [5]. So $\mathfrak{t} \subset \mathfrak{g}_\xi$, and $\mathfrak{g}_\xi$ has minimal dimension if $\mathfrak{t} = \mathfrak{g}_\xi$, or, equivalently, $\dim \mathfrak{g}_\xi = \text{rank}(G)$. Since for compact groups adjoint and coadjoint actions can be identified the same conclusions hold for the coadjoint actions. In particular for a given $\mu \in \mathfrak{g}^*$ there is a maximal torus T with $\mathfrak{t} \subseteq \mathfrak{g}_\mu$. Now let $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$. Then $\mathfrak{g}_{(\xi, \mu)}$ has minimal dimension if and only if ξ lies in the Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}_\mu$ of only one maximal torus $T \subseteq G_\mu$ of G_μ , so that $\mathfrak{t} = \mathfrak{g}_{(\xi, \mu)}$. The fact that for compact groups G a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is regular if and only if it is minimal follows from the stratification of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ given in [34], see also Proposition 4.9 c). ■

For noncompact groups the situation is much more complicated, as we will see in the sequel. Before continuing we recall some elementary real algebraic geometry. An (algebraic) variety $V \subset \mathbb{R}^m$ is an algebraic set, ie, a subset of $\mathbb{R}^m$ defined by polynomial equations $f_1(x) = \dots = f_n(x) = 0$ where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $f_i \in \mathbb{R}[x]$, $i = 1, \dots, n$. We write $V = V(f)$. Here $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_m]$ denotes the ring of polynomials in m variables.

An algebraic variety is called *irreducible* if it is not a union of two different nonempty algebraic varieties. Every algebraic variety can be decomposed into its irreducible components [3], but these irreducible components are often not disjoint. Consider for example the variety $V = \{x \in \mathbb{R}^2, x^2 - y^2 = 0\}$. Its irreducible components $x = y$ and $x = -y$ intersect in $(x, y) = 0$.

A point $p \in V$ is called *regular* or non-singular if p is a *smooth* point of V , that is, V is a manifold near p , otherwise p is called *singular*. Smooth points of algebraic varieties are open and dense [9, Thm. I.2.4].

Remarks 4.4

- a) The notion of regular points of varieties (as well as the notion of a variety itself) is used in different ways in the literature. The notion of regular points in Bochnak et al. [3] is stronger than the one we and Gibson et al. [9] use. Indeed, points of algebraic varieties which are regular in the sense of [3] are smooth, by [3, Prop. 3.3.9,10], but the converse is not true, cf. [3, Example 3.3.11 b)].
- b) We call a point p of an algebraic variety $V = V(f)$ for a vector of polynomials f *minimal* (with respect to f) if $\ker Df(p)$ has locally minimal dimension in V . If the components f_i , $i = 1, \dots, n$, of the vector $f = (f_1, \dots, f_n)$ of polynomials generate the ideal $I(V) \subseteq \mathbb{R}[x]$ of all polynomials which vanish on V then for every $p \in V$ the dimension of $\ker Df(p)$, with $\hat{f} \in I(V)$, has a minimum for $\hat{f} = f$, and in this case $\dim \ker Df(p)$ is called the *embedding dimension* $\text{edim}_p V$ of V at p [4]. For every smooth point p we have $\text{edim}_p V \geq \dim_p V$, but in general smooth points of varieties are not minimal, ie equality does not in general hold, even if f generates $I(V)$, cf. [3, Example 3.3.11 b)].
- c) The *stratification theorem for algebraic varieties* [9, Thm. I.2.7],[3] states that every algebraic variety can be decomposed in a canonical way into a minimal finite number of disjoint submanifolds, called *strata*, which are semi-algebraic sets, such that that this decomposition is locally finite and Whitney-regular. In this paper we restrict attention to persistence of Hamiltonian relative equilibria with regular velocity-momentum pairs. Near such velocity-momentum pairs the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is a manifold, and the general stratification result is not needed. To generalize the results of [34] on persistence of transversal relative equilibria with arbitrary velocity-momentum pairs to noncompact non-free group actions this stratification result will be essential. This paper is a first step towards this more general goal.

The following proposition shows that a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is minimal in the sense of Definition 4.1 if it is a minimal point of the variety $V(f) = (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with respect to f as defined in Remark 4.4 b).

Proposition 4.5 *For $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ we have $\dim G_{(\xi, \mu)} + \dim G = \dim \ker Df(\xi, \mu)$ with f from (3.8).*

Proof. Let $(\hat{\xi}, \hat{\mu}) \in \ker Df(\xi, \mu)$, ie $\text{ad}_{\hat{\xi}}^* \mu + \text{ad}_{\hat{\mu}}^* \mu = 0$. Then $\text{ad}_{\hat{\xi}}^* \hat{\mu} \in \mathfrak{g}\mu$. Write $\hat{\mu} = \text{ad}_{\eta}^* \mu + \hat{\nu}$ with $\hat{\nu} \in \text{ann}(\mathfrak{n}_{\mu})$ and $\eta \in \mathfrak{n}_{\mu}$. Since $\xi \in \mathfrak{g}_{\mu}$ the condition $\text{ad}_{\hat{\xi}}^* \hat{\mu} \in \mathfrak{g}\mu$ is equivalent to $\text{ad}_{\xi}^{*\mu} \hat{\nu} = 0$. From the facts that $\dim \ker \text{ad}_{\xi}^{*\mu} = \dim \mathfrak{g}_{(\xi, \mu)}$, that $\hat{\xi}$ is only determined modulo $\mathfrak{g}_{\mu}$ and that $\text{ad}_{\eta}^* \mu \in \text{ann}(\mathfrak{g}_{\mu})$ can be chosen arbitrarily the proof follows. ■

The following proposition shows that regular $\mu \in \mathfrak{g}^*$, regular $\xi \in \mathfrak{g}$ and regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ are generic.

Proposition 4.6

- a) *The set of regular elements $\mu \in \mathfrak{g}^*$ is open and dense in $\mathfrak{g}^*$.*
- b) *The set of regular elements $\xi \in \mathfrak{g}$ is open and dense in $\mathfrak{g}$.*
- c) *The set of regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is open and dense in $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$.*

Proof. Part a) was proved by Duflo and Vergne, see [25, Theorem 9.3.10]. Part b) can be shown in the same way: Let $\xi \in \mathfrak{g}$ be regular, $r = r_{\xi}(G)$. The condition $\dim(\mathfrak{g}_{\xi}) = \dim \ker \text{ad}_{\xi} = r$ is an algebraic condition on $\xi \in \mathfrak{g}$ since it is equivalent to the conditions that the determinants of all (r, r) -minors of the matrix ad_{ξ} vanish. Moreover these conditions define a proper subset of $\mathfrak{g}$. Therefore its complement is (Zariski-) open and dense in $\mathfrak{g}$. Part c) follows from the corresponding statement [9, Thm. I.2.4] for general varieties mentioned above. ■

Note that minimal pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ are also open and dense in $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$. The following technical lemma which we need in proofs later generalizes the results of [32, 34] on regular velocity-momentum pairs to noncompact groups.

Lemma 4.7

- a) *The space $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is G -invariant.*

- b) Let $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$, $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{n}_\mu$, and identify $\mathfrak{g}_\mu^* = \text{ann}(\mathfrak{n}_\mu)$. Using the section $\mu + \mathfrak{g}_\mu^*$ transverse to $G\mu$ at μ from Lemma 2.1 a), we get the following parametrization of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ in a neighbourhood $U_{(\mathfrak{g} \oplus \mathfrak{g}^*)^c}(\xi, \mu)$ of (ξ, μ) in $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$:

$$U_{(\mathfrak{g} \oplus \mathfrak{g}^*)^c}(\xi, \mu) = \{ \gamma(j_\mu(\chi)(\nu)\chi, \mu + \nu), \quad \gamma \in U_G(\text{id}), \\ (\chi, \nu) \in U_{\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*}(\xi, 0) \text{ satisfies } f_\mu(\chi, \nu) = 0 \}.$$

Here $U_G(\text{id})$, $U_{\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*}(\xi, 0)$ are neighbourhoods of id in G and $(\xi, 0)$ in $\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*$ respectively, f_μ is defined by

$$f_\mu : \mathfrak{g}_\mu \times \mathfrak{g}_\mu^* \rightarrow \mathfrak{g}_\mu^*, \quad f_\mu(\chi, \nu) = \text{ad}_{j_\mu(\nu)\chi}^*(\mu + \nu), \quad (4.1)$$

and (γ, χ, ν) are locally unique modulo the action of $U_G(\text{id}) \cap Z_{\mu, \nu}$ given by

$$\gamma_\mu(\gamma, \chi, \nu) := (\gamma\gamma_\mu^{-1}, \gamma_\mu \circ \nu, \gamma_\mu \circ \nu), \quad \gamma_\mu \in U_G(\text{id}) \cap Z_{\mu, \nu}, \quad (4.2)$$

where $\gamma_\mu \circ \nu \chi \in \mathfrak{g}_\mu$ is defined by the equation

$$\text{Ad}_{\gamma_\mu}(j_\mu(\nu)\chi) = j_\mu(\gamma_\mu \circ \nu)\gamma_\mu \circ \nu \chi,$$

$Z_{\mu, \nu}$ is as in (2.3), and $\gamma_\mu \circ \nu$ as in (2.5). Note that if μ is split and $\mathfrak{n}_\mu$ is chosen G_μ^0 -invariant then $\gamma_\mu \circ \nu = \gamma_\mu \nu$, $\gamma_\mu \circ \nu \chi = \gamma_\mu \chi$.

Moreover $f^{-1}(0) = (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is locally a manifold near $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ if and only if $f_\mu^{-1}(0)$ is locally a manifold near $(\xi, 0)$.

- c) Let $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ be regular and let $r = r_{(\xi, \mu)}(G)$ be such that $\dim_{(\xi, \mu)}(\mathfrak{g} \oplus \mathfrak{g}^*)^c = r + \dim G$. Then $r \leq \dim \mathfrak{g}_{(\xi, \mu)}$ and near $(\chi, \nu) = (\xi, 0)$ the set $f_\mu^{-1}(0)$ can be parametrized by $(\chi, \nu(\chi, \tau))$ where (χ, τ) lies in a neighbourhood $U_{\mathfrak{g}_\mu \oplus \mathbb{R}^r}(\xi, 0)$ of $(\xi, 0)$ in $\mathfrak{g}_\mu \oplus \mathbb{R}^r$, $\nu : U_{\mathfrak{g}_\mu \oplus \mathbb{R}^r}(\xi, 0) \rightarrow \mathfrak{g}_\mu^*$ is smooth, $\nu(\xi, 0) = 0$, $D_\chi \nu(\xi, 0) = 0$ and the columns of $D_\tau \nu(\xi, 0)$ span an r -dimensional subspace of $\ker(\text{ad}_\xi^{\mu, *})$. If μ is split then $\nu(\chi, \tau)$ is linear in τ .

Proof. a) is clear, b) follows from Lemma 2.1. To prove part c) let (ξ, μ) be a smooth point of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with $r = r_{(\xi, \mu)}(G)$. Then by part b) $f_\mu^{-1}(0) \subseteq \mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*$ is a manifold near $(\xi, 0)$. Moreover, $\dim_{(\xi, 0)} f_\mu^{-1}(0) = r + \dim \mathfrak{g}_\mu$. Since $f_\mu^{-1}(0)$ clearly contains the vectorspace $\mathfrak{g}_\mu \oplus \{0\} \subseteq \mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*$ we can locally parametrize $f_\mu^{-1}(0)$ as $(\chi, \nu(\chi, \tau))$ where $\chi \in \mathfrak{g}_\mu$ is close to ξ , $\tau \in \mathbb{R}^r$ is small, $\nu(\xi, 0) = 0$, $D_\chi \nu(\xi, 0) = 0$ and the rank of $D_\tau \nu(\xi, 0)$ is r . Differentiating $f_\mu(\chi, \nu(\chi, \tau)) = 0$ at $(\xi, 0)$ we get $\text{ad}_\xi^{\mu, *} D_\tau \nu(\xi, 0) = 0$. As a consequence $r \leq \dim \ker \text{ad}_\xi^{\mu, *} = \dim \mathfrak{g}_{(\xi, \mu)}$. The last statement follows from the fact that for μ split $j_\mu(\nu) \equiv \text{id}$ by Lemma 2.1 c). ■

Remark 4.8 In general it is unclear when $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$. If the polynomial vector f defining the algebraic variety $V = (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ generates the ideal $I(V)$ then $\dim G + \dim \mathfrak{g}_{(\xi, \mu)}$ is the embedding dimension at (ξ, μ) by Proposition 4.5, and then the equality $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$ holds for points which are regular in the sense of [3]. When this is satisfied, and whether the above equality holds for any velocity-momentum pair which is regular in our sense, is a non-trivial problem of algebraic geometry. Similarly, it is unclear how minimal and regular velocity-momentum pairs are related in general, cf. Remark 4.4.

The questions raised in Remark 4.8 can be answered positively for special cases as the following proposition shows.

Proposition 4.9

- a) If $\mu \in \mathfrak{g}^*$ is regular then (ξ, μ) is regular and minimal for every $\xi \in \mathfrak{g}_\mu$, and $\dim \mathfrak{g}_\mu = r_\mu(G) = r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$. Moreover

$$f_\mu(\chi, \nu) \equiv 0 \quad \text{for } \nu \in \mathfrak{g}_\mu^*, \nu \approx 0, \chi \in \mathfrak{g}_\mu. \quad (4.3)$$

- b) If $\xi \in \mathfrak{g}$ is regular in $\mathfrak{g}$ then (ξ, μ) is regular and minimal for every $\mu \in \mathfrak{g}^*$ with $\text{ad}_\xi^* \mu = 0$, and $r_\xi(G) = r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$.

c) If μ is split then the following conditions are equivalent:

- (i) ξ is regular in $\mathfrak{g}_\mu$;
- (ii) $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is regular;
- (iii) $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is minimal.

Moreover, if μ is split and (ξ, μ) regular then $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$.

Note that for G compact every μ is split so that a velocity-momentum pair is regular if and only if it is minimal, as stated in Proposition 4.3.

Proof.

a) If μ is minimal then $G_{\hat{\mu}}$ has locally constant dimension for $\hat{\mu}$ in a small neighbourhood of μ in $\mathfrak{g}^*$. Let $\hat{\mu} = \mu + \nu$ with $\nu \in \text{ann}(\mathfrak{n}_\mu)$ small. Clearly $\mathfrak{g}_{\mu+\nu} \subseteq T_{\text{id}}Z_{\mu, \nu}$ which means, by Lemma 2.1 b), that $\mathfrak{g}_{\mu+\nu} \subseteq j_\mu(\nu)\mathfrak{g}_\mu$. Since $j_\mu(\nu)$ is injective for ν small and $\dim \mathfrak{g}_{\mu+\nu} = \dim \mathfrak{g}_\mu$ we conclude that $\mathfrak{g}_{\mu+\nu} = j_\mu(\nu)\mathfrak{g}_\mu$ so that (4.3) holds. Moreover, close to the set $\mathfrak{g}_\mu \oplus \{\mu\} \subseteq (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is a trivial bundle with fibre $\mathfrak{g}_{\hat{\mu}}$ isomorphic to $\mathfrak{g}_\mu$ over a neighbourhood $\hat{\mu} \approx \mu$ of μ in $\mathfrak{g}^*$. The dimension of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ near $\mathfrak{g}_\mu \oplus \{\mu\}$ is therefore $\dim G + \dim G_\mu$. This shows that for every $\xi \in \mathfrak{g}_\mu$ the velocity-momentum pair (ξ, μ) is a smooth point of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with $r_{(\xi, \mu)}(G) = r_\mu(G) = \dim G_\mu$. Since $r_{(\xi, \mu)}(G) \leq \dim \mathfrak{g}_{(\xi, \mu)}$ (as we saw in Lemma 4.7 c)) and $\dim \mathfrak{g}_{(\xi, \mu)} \leq \dim \mathfrak{g}_\mu = r_\mu(G)$, we have $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)} = r_\mu(G)$. Since regular momenta are open by Proposition 4.6 this dimension is locally constant when (ξ, μ) is varied in $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ so that (ξ, μ) is minimal.

b) If ξ is minimal then $G_{\hat{\xi}}$ has locally constant dimension for $\hat{\xi}$ in a small neighbourhood of ξ in $\mathfrak{g}$. So, as in a), close to $\{\hat{\xi}\} \oplus \ker \text{ad}_\xi^* \subseteq (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is a trivial bundle with fibre $\ker \text{ad}_{\hat{\xi}}^*$ isomorphic to $\ker \text{ad}_\xi^*$ over a neighbourhood $\hat{\xi} \approx \xi$ of ξ in $\mathfrak{g}$, and the dimension of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ near $\{\hat{\xi}\} \oplus \ker \text{ad}_\xi^*$ is therefore $\dim G + \dim G_\xi$. This shows that (ξ, μ) is a smooth point of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with $r = r_\xi(G)$ for every $\mu \in \mathfrak{g}^*$ with $\text{ad}_\xi^* \mu = 0$. The rest follows as in a).

c) If μ is split then $j_\mu = \text{id}$ and $f_\mu(\chi, \nu) = \text{ad}_\chi^* \nu$ where $\nu \in \mathfrak{g}_\mu^*$, $\chi \in \mathfrak{g}_\mu$.

We first show that (i) implies (ii). Let ξ be minimal in $\mathfrak{g}_\mu$. From part b) we see that $(\xi, 0)$ is a regular point of $(\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*)^c$ with $r_{(\xi, 0)}(G_\mu) = \dim \mathfrak{g}_{(\xi, \mu)}$. From Lemma 4.7 c) we conclude that (ξ, μ) is a regular point of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$.

Now we show that (ii) implies (iii). If μ is split and (ξ, μ) a regular point of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ with $\dim_{(\xi, \mu)}(\mathfrak{g} \oplus \mathfrak{g}^*)^c = \dim G + r$ then by Lemma 4.7 c) $(\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*)^c$ is a manifold near $(\xi, 0) \in (\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*)^c$ of dimension $r + \dim G_\mu$, and this manifold is locally parametrized by $(\chi, \nu(\chi, \tau))$ where $\chi \in \mathfrak{g}_\mu$ is close to ξ and $\tau \in \mathbb{R}^r$ is small. This implies that for every χ close to ξ we have $\dim \mathfrak{g}_{(\chi, \mu)} = \dim \ker \text{ad}_\chi^* = r$, so that ξ is minimal in $\mathfrak{g}_\mu$ which by b) means that $(\xi, 0)$ is minimal in $(\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*)^c$.

Now let $(\hat{\xi}, \hat{\mu}) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ be close to (ξ, μ) . Then by Lemma 4.7 b) there are $g \in G$, $\chi \in \mathfrak{g}_\mu$ and $\nu \in \mathfrak{g}_\mu^*$ with $g \approx \text{id}$, $\nu \approx 0$, $\chi \approx \xi$ such that $(\hat{\xi}, \hat{\mu}) = g(\chi, \mu + \nu)$. So $\dim \mathfrak{g}_{(\hat{\xi}, \hat{\mu})} = \dim \mathfrak{g}_{(\chi, \mu + \nu)}$, and since μ is split we have $\mathfrak{g}_{\mu + \nu} \subseteq \mathfrak{g}_\mu$. For $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ to be minimal it is therefore sufficient to show that $(\xi, 0)$ is a minimal point of $(\mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^*)^c$ which is true as we just saw.

Finally we show that (iii) implies (i). If $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is minimal then $\dim \mathfrak{g}_{(\chi, \mu)} = \dim \mathfrak{g}_{(\xi, \mu)}$ for $\chi \in \mathfrak{g}_\mu$ close to ξ , so that ξ is minimal in $\mathfrak{g}_\mu$. ■

The equation $\dim \mathfrak{g}_\mu = r_\mu(G) = r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}$ for $\mu \in \mathfrak{g}^*$ minimal, $\xi \in \mathfrak{g}_\mu$ arbitrary, implies the result of Duflo and Vergne that G_μ^0 is abelian for minimal μ [25, Theorem 9.3.10]. As a consequence isotropy groups of regular $\mu \in \mathfrak{g}^*$ are products of tori and lines.

Examples 4.10

- a) In the case $G = \text{SO}(3)$ the identity components of isotropy subgroups of regular $\mu \in \text{so}(3)^*$, regular $\xi \in \text{so}(3)$ and regular pairs $(\xi, \mu) \in (\text{so}(3) \oplus \text{so}(3)^*)^c$ are copies of $\text{SO}(2)$.

- b) The group $G = \text{SL}(2, \mathbb{R})$ is semisimple and therefore there is an isomorphism between the adjoint and coadjoint actions given by the Killing form. One computes that the identity components of minimal momentum and minimal velocity isotropy subgroups are either diffeomorphic to $G_\xi^0 = \mathbb{R}$ (if $\xi \in \mathfrak{sl}(2, \mathbb{R})$ is a semisimple matrix with real non-zero eigenvalues or a nilpotent matrix) or to $G_\xi^0 = \text{SO}(2)$ (if ξ is a semisimple matrix with non-zero purely imaginary eigenvalues). Note that μ is minimal but not split if μ is a nilpotent matrix [37].
- c) In contrast to compact and semisimple groups for general noncompact groups minimal momentum isotropy subgroups and minimal velocity isotropy subgroups can be quite different. Consider for example the special Euclidean group of the plane $\text{SE}(2) = \text{SO}(2) \ltimes \mathbb{R}^2$. The semidirect product $K \ltimes \mathbb{R}^N$ of a compact group K acting on $\mathbb{R}^N$ with the additive group $\mathbb{R}^N$ is defined by

$$(R_1, a_1)(R_2, a_2) = (R_1 R_2, a_1 + R_1 a_2), \quad \text{where } R_i \in K, a_i \in \mathbb{R}^N, i = 1, 2.$$

Write $\xi \in \mathfrak{se}(2)$ as $\xi = (\xi^r, \xi^a)$ where $\xi^r \in \mathfrak{so}(2)$, $\xi^a = (\xi_1^a, \xi_2^a) \in \mathbb{R}^2$, and similarly $\mu \in \mathfrak{se}(2)^* = \mathfrak{so}(2)^* \oplus \mathbb{R}^{2,*}$ as $\mu = (\mu^r, \mu^a)$. We compute that the adjoint action of $\text{SE}(2)$ is given by

$$\text{Ad}_{(R,a)}(\chi^r, \chi^a) = (\chi^r, \text{Ad}_R \chi^a - \chi^r a), \quad (4.4)$$

and the coadjoint action is

$$\text{Ad}_{(R,a)}^*(\mu^r, \mu^a) = (\mu^r - a_1 \mu_2^a + a_2 \mu_1^a, \text{Ad}_R^* \mu^a). \quad (4.5)$$

We see that the minimal momentum isotropy subalgebras $\mathfrak{g}_\mu$ are copies of $\mathbb{R}$ whereas minimal velocity momentum isotropy subalgebras $\mathfrak{g}_\xi$ are copies of $\mathfrak{so}(2)$. Moreover a pair $(\xi, \mu) \in (\mathfrak{se}(2) \oplus \mathfrak{se}(2)^*)^c$ is regular if and only if either ξ is regular or μ is regular.

Note that $(\xi, \mu) \in (\mathfrak{se}(2) \oplus \mathfrak{se}(2)^*)^c$ if either $\xi^r = 0$ and $\xi_1^a \mu_2^a - \xi_2^a \mu_1^a = 0$ or $\mu^a = 0$ and ξ is arbitrary which shows that the variety $(\mathfrak{se}(2) \oplus \mathfrak{se}(2)^*)^c$ has two irreducible components. The regular stratum of the first irreducible component consists of velocity-momentum pairs with regular momentum, the regular stratum of the second irreducible component consists of velocity-momentum pairs with regular velocity.

- d) Consider the special Euclidean group of three space $G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$. The adjoint and coadjoint actions of $\text{SE}(3)$ on $\mathfrak{se}(3)$ and $\mathfrak{se}(3)^*$ are given [37] by

$$\begin{aligned} \text{Ad}_{(R,a)} \chi &= (R \chi^r, R \chi^a - R \chi^r \times a), \\ \text{Ad}_{(R,a)}^* \mu &= (R^{-1}(\mu^r - a \times \mu^a), R^{-1} \mu^a), \quad R \in \text{SO}(3), a \in \mathbb{R}^3. \end{aligned} \quad (4.6)$$

One computes that $\mu \in \mathfrak{se}(3)^*$ is regular if and only if $\mu^a \neq 0$, that a velocity $\xi \in \mathfrak{se}(3)$ is regular if and only if $\xi^r \neq 0$, that a pair $(\xi, \mu) \in (\mathfrak{se}(3) \oplus \mathfrak{se}(3)^*)^c$ is regular if and only if either ξ or μ is regular, and that the identity components of isotropy subgroups of regular $\xi \in \mathfrak{se}(3)$, regular $\mu \in \mathfrak{se}(3)^*$ and regular pairs $(\xi, \mu) \in (\mathfrak{se}(3) \oplus \mathfrak{se}(3)^*)^c$ are all of the form $\text{SO}(2) \times \mathbb{R}$.

- e) Let $G = \text{SO}(2) \ltimes \mathbb{R}^4$ with the action of $\text{SO}(2)$ on $\mathbb{R}^4$ given by $Ra = (Ra_1, Ra_2)$, where $a = (a_1, a_2)$, $a_i \in \mathbb{R}^2$, $i = 1, 2$, and R_ϕ acts via its standard action on $\mathbb{R}^2$. Writing $\xi \in \mathfrak{so}(2) \oplus \mathbb{R}^4$ as $\xi = (\xi^r, \xi^a)$ with $\xi^r \in \mathfrak{so}(2)$, $\xi^a = (\xi^{a_1}, \xi^{a_2}) \in \mathbb{R}^4$, $\xi^{a_i} = (\xi_1^{a_i}, \xi_2^{a_i}) \in \mathbb{R}^2$, $i = 1, 2$, and similarly for $\mu \in \mathfrak{g}^*$, the adjoint action is given by the formula (4.4), and the infinitesimal coadjoint action is

$$\text{ad}_{(\xi^r, \xi^a)}^*(\mu^r, \mu^a) = (\xi_2^{a_1} \mu_1^{a_1} - \xi_1^{a_1} \mu_2^{a_1} + \xi_2^{a_2} \mu_1^{a_2} - \xi_1^{a_2} \mu_2^{a_2}, \text{ad}_{\xi^r}^* \mu^a).$$

Hence, if $\mu^a = 0$ then $G_\mu = G$, otherwise $G_\mu \cong \mathbb{R}^3$, so that for a minimal μ we have $r_\mu(G) = 3$. On the other hand $\xi \in \mathfrak{g}$ is minimal if $\xi^r \neq 0$. In this case $r_\xi(G) = 1$ and $\mathfrak{g}_\xi$ is a copy of $\mathfrak{so}(2)$. As a consequence, $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ has two strata with regular points, one stratum of dimension 8 and another one with dimension 6.

The examples show that in the case of non-compact groups the isotropy subalgebras $\mathfrak{g}_{(\xi, \mu)}$ of regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ may be non-isomorphic (Example 4.10 b), c), e)), that $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ may have several irreducible components (Example 4.10 c)), and that these irreducible components may even have different dimensions (Example 4.10 e)). The situation is much simpler in the case of compact symmetry groups as Proposition 4.3 and the following result show.

Proposition 4.11 *For compact groups G the algebraic variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is irreducible.*

Proof. The isotropy subalgebra $\mathfrak{g}_{(\xi, \mu)}$ of every pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ contains the Lie algebra $\mathfrak{t}$ of a maximal torus. Within the subspace $\mathfrak{t} \oplus \mathfrak{t}^*$ of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ the non-regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ have codimension ≥ 2 since both $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ have to be non-regular for (ξ, μ) to be non-regular. Hence the set of regular points of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ intersects $\mathfrak{t} \oplus \mathfrak{t}^*$ in a connected open dense subset. Since all maximal tori are conjugate we have $G^0(\mathfrak{t} \oplus \mathfrak{t}^*) = (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ which proves that the set of regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is a connected subset of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$. If (ξ, μ) is regular then $\text{edim}_p V = \dim_p V$ with $p = (\xi, \mu)$, $V = (\mathfrak{g} \oplus \mathfrak{g}^*)^c$. By [3, Prop. 3.3.9] points p with this property lie in only one irreducible component of V . Since the set of regular pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ forms a dense, connected subset of $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ and since every irreducible component of an algebraic variety is clearly closed in the Euclidean topology this shows that $(\mathfrak{g} \oplus \mathfrak{g}^*)^c$ is irreducible for compact groups G . ■

5 Persistence of general relative equilibria

In this section we present a result on persistence of non-degenerate relative equilibria with velocity-momentum pairs which are regular modulo isotropy. This extends results of [32, 34, 30] to the case of noncompact group actions. We restrict the dynamics onto a Λ -invariant neighbourhood U of Λp in $M = \text{Fix}_{\mathcal{M}}(\Gamma_p)$ where $\Lambda = (N(\Gamma_p) \cap \Gamma)/\Gamma_p$ acts freely near p and choose the drift velocity ξ of the relative equilibrium Γp to lie in $\mathfrak{l}$ considered as subspace of $\mathfrak{g}$ and thus identify $\xi \in \mathfrak{g}$ with $\xi_L \in \mathfrak{l}$, see Section 3 above. Moreover, as in Section 3, we consider $\mathfrak{l}^*$ as subspace of $\mathfrak{g}^*$ and denote $\mu_L = \mu|_{\mathfrak{l}}$. We do not consider time-reversible symmetries in this section, ie., we set $G = \Gamma$ and $L = \Lambda$.

Definition 5.1 *We say that a relative equilibrium Γp of (2.1) with momentum $\mu = \mathbf{J}(p)$ and drift velocity $\xi \in \mathfrak{g}_\mu$, has regular momentum (regular velocity, regular velocity-momentum pair) modulo isotropy if μ_L is regular in $\mathfrak{l}^*$ (ξ_L is regular in $\mathfrak{l}$, (ξ_L, μ_L) is regular in $(\mathfrak{l} \oplus \mathfrak{l}^*)^c$) where $\mathfrak{l}$ is the Lie algebra of $L = N(\Gamma_p)/\Gamma_p$.*

Note that $\mathbf{J}_M : M \rightarrow \mathfrak{l}^*$ is surjective from a Λ -invariant neighbourhood U_M of Λp in M onto a Λ -invariant neighbourhood $U_{\mathfrak{l}^*}$ of μ_L in $\mathfrak{l}^*$ since the Λ -action is locally free near p in M . Therefore Proposition 4.6 applied onto $\mathfrak{l}^*$ implies that the set of regular $\mu \in \mathfrak{l}^*$ is open and dense in $U_{\mathfrak{l}^*}$.

Now we can formulate a persistence result for relative equilibria with regular velocity-momentum pairs modulo isotropy:

Theorem 5.2 *Let Γp be a relative equilibrium of (2.1) which is non-degenerate and has regular velocity-momentum pair (ξ, μ) modulo isotropy and let $\Lambda = N(\Gamma_p)/\Gamma_p$. Then:*

- a) *The solutions of (3.6)-(3.8) form an r -dimensional manifold $\nu(\tau) \in \mathfrak{l}_\mu^* \subseteq \mathfrak{g}_\mu^*$, $\tau \in \mathbb{R}^r$, $r = r_{(\xi_L, \mu_L)}(\Lambda)$ with $\nu(0) = 0$ which, by Proposition 3.2, gives an r -dimensional family of relative equilibria $\Gamma p(\tau)$ with $p(0) = p$, with isotropy Γ_p , drift velocity $\xi(\tau)$ close to ξ and momentum $\mathbf{J}(p(\tau)) = \mu + \nu(\tau)$ close to μ ; moreover there are no other relative equilibria with the same isotropy near Γp .*
- b) *If μ is regular modulo isotropy, then a) holds, $r = r_{\mu_L}(\Lambda)$, and $\Lambda p \in M$ persists to every nearby momentum $\hat{\mu}_L \in \mathfrak{l}^*$.*
- c) *If ξ is regular modulo isotropy, then a) holds, and $r = r_{\xi_L}(\Lambda)$.*
- d) *Let $\mathcal{M}_{\text{RE}}$ be the submanifold formed by these relative equilibria and let $M_{\text{RE}} = \text{Fix}_{\mathcal{M}_{\text{RE}}}(\Gamma_p)$. Then M_{RE} is a symplectic manifold if and only if μ is regular modulo isotropy. In this case $M_{\text{RE}} \equiv \Lambda \times \mathfrak{l}_\mu^*$.*

Proof.

- a) We solve $F(\nu) = f_\mu(D_\nu h(\nu, w(\nu)), \nu) = 0$ where $\nu \in \mathfrak{l}_\mu^*$, $\xi_L = D_\nu h(0) \in \mathfrak{l}_\mu$, F is from (3.6) and f_μ from (4.1). Lemma 4.7 c), applied to the group L , implies that close to $(\chi, \nu) = (\xi_L, 0)$ there is an r -dimensional manifold $(\chi, \nu(\chi, \tau))$, with $\nu(\xi_L, 0) = 0$, of solutions to $f_\mu(\chi, \nu) = 0$. The equation

$$\chi = D_\nu h(\nu(\chi, \tau), w(\nu(\chi, \tau))) \quad (5.1)$$

can be solved for $\chi = \chi(\tau)$, τ small, by the implicit function theorem because it is satisfied for $\tau = 0$ with $\chi(0) = \xi_L$ and because $D_\chi \nu(\xi_L, 0) = 0$ by Lemma 4.7 c). This gives an r -dimensional family $\nu(\tau) := \nu(\chi(\tau), \tau)$ of solutions of (3.6).

- b) follows from the proof of a) and Proposition 4.9 a).
- c) follows from the proof of a) and Proposition 4.9 b).
- d) By the Witt decomposition (see Section 2), applied onto $M = \text{Fix}_{\Gamma_p}(\mathcal{M})$, we have $\mathcal{T}_p M = T_0 \oplus T_1 \oplus N_0 \oplus N_1$, where $T_0 + T_1 \simeq \mathbf{l}$, $T_0 \simeq \mathbf{l}_\mu$, $T_1 \simeq \mathbf{l}/\mathbf{l}_\mu$, $N_0 \simeq \mathbf{l}_\mu^*$, T_1 , $T_0 \oplus N_0$ and N_1 are symplectic spaces, and $\omega|_{\mathcal{T}_p M} = \omega|_{T_0 \oplus N_0} + \omega|_{N_1} + \omega|_{T_1}$. We have

$$\mathcal{T}_p M_{\text{RE}} \simeq T_0 \oplus T_1 \oplus \left(\begin{array}{c} \text{id} \\ Dw(0) \end{array} \right) \text{im } D\nu(0)$$

where we used bundle coordinates on the right hand side and $\text{im } D\nu(0)$ denotes the image of $D\nu(0)$. Since $\text{im } D\nu(0) \subset \mathbf{l}_\mu^*$ the space $\mathcal{T}_p M_{\text{RE}}$ is symplectic if and only if $\mathbf{l}_\mu^* = \text{im } D\nu(0)$, ie, if μ_L is minimal in $\mathbf{l}^*$. ■

Propositions 4.3 and 4.9 can be used with $\mathbf{g}$ replaced by $\mathbf{l}$ to compute the quantities $r_{\xi_L}(\Lambda)$, $r_{\mu_L}(\Lambda)$ and $r_{(\xi_L, \mu_L)}(\Lambda)$. Most statements of parts b) and d) occur in [37, Corollary 4.3]. For compact groups this theorem is contained in [34].

Examples 5.3

- a) Consider a Hamiltonian system with a free action of the symmetry group $\Gamma = \text{SE}(3)$. Then the isotropy subalgebras of regular velocity-momentum pairs are isomorphic to $\mathfrak{so}(2) \oplus \mathbb{R}$, as Example 4.10 d) shows. Note that any velocity $\xi \in \mathfrak{se}(3)$ corresponds to an infinitesimal rotation around some axis and/or translation about the same axis.

Let Γp be a non-degenerate relative equilibrium with regular velocity-momentum pair $(\xi, \mu) \in (\mathfrak{se}(3) \oplus \mathfrak{se}(3)^*)^c$. By Theorem 5.2 there is a two-parameter family $\Gamma p(\tau)$, $\tau \in \mathbb{R}^2$, of relative equilibria containing Γp , and no other relative equilibria closeby.

As an example consider a relative equilibrium with regular momentum $\mu = (\mu^r, \mu^a)$, ie with non-zero linear momentum μ^a (cf Example 4.10 d)). Then by Theorem 5.2 b) the relative equilibrium persists to every nearby momentum. Here we use that by (4.6) on every coadjoint orbit $G\mu$ there is some $\hat{\mu} \in G\mu$ such that $\hat{\mu}^r$ and $\hat{\mu}^a$ are parallel.

As another example consider a relative equilibrium with zero linear momentum $\mu = (\mu^r, 0)$, where $\Gamma_\mu^0 = \text{SO}(2) \ltimes \mathbb{R}^3$. Then μ is non-split [37]. Let $\mu^r \in \mathfrak{so}_3(2)^*$ where $\mathfrak{so}_i(2)^*$ denotes the μ_i^r -axis, $i = 1, 2, 3$, and let $\mathbf{n}_\mu = \mathfrak{so}_1(2)^* \oplus \mathfrak{so}_2(2)^*$. Similarly, let $\mathbb{R}_i^*$, $i = 1, 2, 3$, denote the μ_i^a -axis in $\mathbb{R}^*$. If the rotation frequency of the relative equilibrium is non-zero, $\xi^r \neq 0$, then Theorem 5.2 applies. Notice that the momenta $\mathbf{J}(p(\tau)) \in \text{ann}(\mathbf{n}_\mu) \simeq \mathfrak{g}_\mu^*$ of the points $\{p(\tau), \tau \in U_{\mathbb{R}^2}(0)\}$ on the family of relative equilibria form a neighbourhood of 0 in the subspace $\mathfrak{so}_3(2)^* \oplus \mathbb{R}_3^* \subset \mathfrak{g}_\mu^*$, and therefore typically have a non-zero μ^a -component which means that they are typically regular.

A concrete example of a Hamiltonian system where this symmetry group arises is the Kirchhoff model of a neutrally buoyant rigid body in an ideal irrotational fluid [17, 18]. The configuration space is the special Euclidean group of three-space $Q = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$, with $(R, a) \in \text{SE}(3)$ describing the rotation angle and the position of the body. The phase space is $\mathcal{M} = T^*\text{SE}(3) = \text{SE}(3) \times \mathfrak{se}(3)^*$ where $\nu = (\nu^r, \nu^a) \in \mathfrak{se}(3)^* = \mathfrak{so}(3)^* \oplus (\mathbb{R}^3)^*$ are angular and linear impulse of the body. Assume that the body has coincident centres of gravity and buoyancy. Then the body does not feel gravity, and so the symmetry group is $\Gamma = \text{SE}(3) = Q$. The Hamiltonian for this system in the case of an ellipsoidal rigid body with center of mass on a principal axis of the body is given by

$$H(\nu^r, \nu^a) = \frac{1}{2} \left((\nu^r)^T I^{-1} \nu^r + (\nu^a)^T M^{-1} \nu^a \right),$$

where the diagonal positive definite matrices I and M are the inertia tensor and mass matrix respectively. A two-parameter family of relative equilibria is given by $(R, a) = \text{id}$, $\nu^a = (0, 0, \nu_3^r)$, $\nu^r = (0, 0, \nu_3^a)$ with ν_3^r and ν_3^a arbitrary [17, Section 3.2], [18, Section 4]. In this concrete example a relative equilibrium with zero linear momentum and non-vanishing angular momentum spins about some axis, but does not translate, and persists to a family of relative equilibria which spin and drift along the same fixed axis. It is easy to check the nondegeneracy condition required for the persistence result of Theorem 5.2. Since the momentum map $\mathbf{J}$ on $\mathcal{M} = T^*\text{SE}(3)$ is given by $\mathbf{J}(R, a, \nu^r, \nu^a) = \text{Ad}_{(R,a)}^*(\nu^r, \nu^a)$ and is therefore linear on the (global) slice N given by $(R, a) = \text{id}$, and since D^2H is constant and positive definite, we see that $D^2H^\xi(p)|_{N_1} = D^2H(p)|_{N_1}$ is definite for all relative equilibria $p \in \mathcal{M}$ with drift velocity ξ in the Kirchhoff model, and so the non-degeneracy condition of Theorem 5.2 is always satisfied.

- b) Consider a Hamiltonian system (2.1) with free action of the symmetry group $\Gamma = \text{SE}(2) \times \mathbb{R} = \text{SO}(2) \ltimes \mathbb{R}^3$ where $\text{SE}(2)$ is the special Euclidean group of motions of the plane. As in Example 4.10 c) we write $\xi = (\xi^r, \xi^a)$ where $\xi^a \in \mathbb{R}^3$ and $\xi^r \in \text{so}(2)$ acts on $\xi^a \in \mathbb{R}^3$ as infinitesimal rotation around the ξ_3^a -axis.

We conclude from Example 4.10 c) that (ξ, μ) is a regular velocity-momentum pair if and only if either μ or ξ are minimal. Moreover a momentum $\mu = (\mu^r, \mu^a)$ is minimal if $(\mu_1^a, \mu_2^a) \neq 0$ in which case $\Gamma_\mu = \mathbb{R}^2$. A regular velocity satisfies $\xi^r \neq 0$ and then Γ_ξ is isomorphic to $\text{SO}(2) \times \mathbb{R}$.

Let Γp be a non-degenerate relative equilibrium of (2.1) with regular velocity-momentum pair (ξ, μ) . Then Theorem 5.2 gives a two-parameter family of relative equilibria near Γp with velocity-momentum pairs close to (ξ, μ) .

If μ is regular this family provides one relative equilibrium for each momentum $\hat{\mu}$ close to μ , by Theorem 5.2 b). Since $\xi \in \mathfrak{g}_\mu$ and $\Gamma_\mu = \mathbb{R}^2$ these relative equilibria translate, but do not rotate. If ξ is regular then $\xi^r \neq 0$, and the corresponding linear momentum points along the μ_3^a -axis, since $(\mu_1^a, \mu_2^a) = 0$. The relative equilibria of this family rotate and typically drift along the axis of rotation.

A concrete example where this symmetry group arises is a neutrally buoyant rigid body, as in a), now with non-coincident centres of gravity and buoyancy aligned on the e_3 -axis. Then gravity is acting on the body, and so the symmetry group is $\Gamma = \text{SE}(2) \times \mathbb{R}$. The relative equilibria with regular momentum correspond to rigid bodies which translate in any non-vertical direction and do not spin, see also [17, Section 3.2], [18, Section 4.3.2]. The relative equilibria with regular velocity have linear momentum parallel to the axis of gravity (since $(\mu_1^a, \mu_2^a) = 0$) and translate and spin along the axis of gravity, cf. also [17, Section 3.2], [18, Section 4.4.2]. Concrete formula of these relative equilibria and stability computations by Hessian tests which imply the nondegeneracy condition required in Theorem 5.2 can be found in these references.

Note that our persistence results do not require the rigid body in the Kirchhoff model to be ellipsoidal, and indeed the results hold for any non-degenerate relative equilibrium with such a momentum value in a Hamiltonian system with this symmetry group.

6 Persistence of reversible relative equilibria

In this section we extend the results of Section 5 to the reversible case. We present a result on persistence of non-degenerate reversible relative equilibria with regular reversible velocity-momentum pairs.

Let $\rho \in G_p \backslash \Gamma_p$ be a reversing symmetry of a reversible relative equilibrium $\Gamma p \subseteq \mathcal{M}$ of (2.1). Since we are only interested in relative equilibria nearby with the same isotropy G_p we can restrict (2.1) to the flow-invariant manifold $M = \text{Fix}_{\mathcal{M}}(\Gamma_p)$ for which the (reversing) symmetry group of (2.1) is $L = N(\Gamma_p)/\Gamma_p$. Then the symmetry group $\Lambda = (N(\Gamma_p) \cap \Gamma)/\Gamma_p$ acts freely on an L -invariant neighbourhood U of Lp in M , and, since $\rho^2 \in \Gamma_p$, the reversing symmetry ρ is an order two element of L , $L_p = \mathbb{Z}_2^\rho := \{\text{id}, \rho\}$ and $L = \Lambda \rtimes \mathbb{Z}_2^\rho$. Since the momentum mapping is equivariant we have $\rho\mu = \mu$ where $\mu = \mathbf{J}(p) \in \mathfrak{l}^*$ and $\rho\mu = -\text{Ad}_\rho^*\mu$, cf. (2.2). We call such momenta reversible. Similarly the drift velocity $\xi \in \mathfrak{l}$ of the relative equilibrium is reversible, ie, satisfies $\text{Ad}_\rho\xi = -\xi$, see [16, 37].

We first generalize the theory of regular velocity-momentum pairs developed in Section 4 to the reversible case.

For any Lie group Γ with Lie algebra $\mathfrak{g}$ and index two extension $G = \Gamma \rtimes \mathbb{Z}_2^\rho$ generated by Γ and some involution $\rho \in G$ define

$$\mathfrak{g}^- = \{\xi \in \mathfrak{g} \mid \text{Ad}_\rho \xi = -\xi\}, \quad \mathfrak{g}^+ = \{\xi \in \mathfrak{g} \mid \text{Ad}_\rho \xi = \xi\}, \quad (6.1)$$

and

$$\begin{aligned} \mathfrak{g}^{+,*} &= (\mathfrak{g}^*)^+ = \{\nu \in \mathfrak{g}^*, \text{Ad}_\rho^* \nu = \nu\} = \text{ann}(\mathfrak{g}^-), \\ \mathfrak{g}^{-,*} &= (\mathfrak{g}^*)^- = \{\nu \in \mathfrak{g}^*, \text{Ad}_\rho^* \nu = -\nu\} = \text{ann}(\mathfrak{g}^+). \end{aligned}$$

We have $\mathfrak{g}^+ = \text{Fix}_{\mathfrak{g}}(\rho)$ where ρ acts on $\mathfrak{g}$ by the usual adjoint action and $\mathfrak{g}^{-,*} = \text{Fix}_{\mathfrak{g}^*}(\rho)$ where ρ acts on $\mathfrak{g}^*$ by the s -coadjoint action as introduced in Section 2. As mentioned above we call $\xi \in \mathfrak{g}$ (ρ -)reversible if $\xi \in \mathfrak{g}^-$, and similarly $\mu \in \mathfrak{g}^*$ (ρ -)reversible if $\mu \in \mathfrak{g}^{-,*}$.

Note that $\mathfrak{g}^+$ is the Lie algebra of the Lie group $G^+ = Z_G(\rho)$, the centralizer of ρ in G , and that

$$[\mathfrak{g}^-, \mathfrak{g}^-] \subseteq \mathfrak{g}^+, \quad [\mathfrak{g}^-, \mathfrak{g}^+] \subseteq \mathfrak{g}^-. \quad (6.2)$$

So in general $\mathfrak{g}^-$ is not a Lie algebra. If it is then it is abelian because for $\xi_1, \xi_2 \in \mathfrak{g}^-$ we have

$$-[\xi_1, \xi_2] = \text{Ad}_\rho[\xi_1, \xi_2] = [\text{Ad}_\rho \xi_1, \text{Ad}_\rho \xi_2] = [\xi_1, \xi_2].$$

From (6.2) we get

$$\text{ad}_{\mathfrak{g}^-}^* \mathfrak{g}^{-,*} \subseteq \mathfrak{g}^{+,*}, \quad \text{ad}_{\mathfrak{g}^+}^* \mathfrak{g}^{-,*} \subseteq \mathfrak{g}^{-,*}, \quad (6.3)$$

so that $Z_G(\rho)$ acts on $\mathfrak{g}^{-,*}$. We now extend the notion of regular momenta, velocities and velocity-momentum pairs that we introduced in Section 4 to the reversible case.

Definition 6.1 Let $G = \Gamma \rtimes \mathbb{Z}_2^\rho$ be a Lie group and let $\mathbb{Z}_2^\rho$ be generated by the involution $\rho \in G$.

- a) We say that $\mu \in \mathfrak{g}^{-,*}$ is regular (or minimal) reversible if $\dim(\mathfrak{g}_\mu^-)$ is minimal and denote this dimension by $r_\mu^-(G)$.
- b) We call $\xi \in \mathfrak{g}^-$ regular (or minimal) reversible if the Lie algebra $\mathfrak{g}_\xi^+$ of $Z_{G_\xi}(\rho)$ has minimal dimension and define $r_\xi^-(G) = \dim \ker \text{ad}_\xi^*|_{\mathfrak{g}^{-,*}}$.
- c) We denote by f^- the restriction of f to $\mathfrak{g}^- \oplus \mathfrak{g}^{-,*}$, ie

$$f^- : \mathfrak{g}^- \oplus \mathfrak{g}^{-,*} \rightarrow \mathfrak{g}^{+,*}, \quad f^-(\xi, \mu) = \text{ad}_\xi^*(\mu), \quad (6.4)$$

and call a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^c$ reversible if $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ where

$$(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c} = (f^-)^{-1}(0) = \{(\xi, \mu) \in \mathfrak{g}^- \oplus \mathfrak{g}^{-,*} : \text{ad}_\xi^* \mu = 0\}.$$

We say that $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is

- (i) regular if (ξ, μ) is a smooth point of the variety $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$; in this case let $r_{(\xi, \mu)}^-(G)$ be such that $\dim_{(\xi, \mu)}(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c} = \dim \mathfrak{g}^- + r_{(\xi, \mu)}^-(G)$;
- (ii) minimal if the dimension $\dim \mathfrak{g}_{(\xi, \mu)}^+$ of the centralizer $G_{(\xi, \mu)}^+ = Z_{G_{(\xi, \mu)}}(\rho)$ of ρ in $G_{(\xi, \mu)}$ is minimal in a neighbourhood of (ξ, μ) in $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$.

Remarks 6.2

- a) Note that the condition of minimality of $\xi \in \mathfrak{g}^-$ is equivalent to the condition that $\text{ad}_\xi^*|_{\mathfrak{g}^{-,*}}$ has a kernel of minimal dimension $r_\xi^-(G)$. This is due to the fact that $\mathfrak{g}_\xi^+ = \ker \text{ad}_\xi|_{\mathfrak{g}^+}$, $[\xi, \mathfrak{g}^+] \subseteq \mathfrak{g}^-$ and that $\text{ad}_\xi^*|_{\mathfrak{g}^{-,*}} = (\text{ad}_\xi|_{\mathfrak{g}^+})^T$. If we define $A \in \text{Mat}(\mathfrak{g})$ such that $A|_{\mathfrak{g}^+} = \text{ad}_\xi$, $A|_{\mathfrak{g}^-} = 0$ and use that $\dim \ker A = \dim \ker A^T$ we get

$$r_\xi^-(G) = \dim \ker \text{ad}_\xi^*|_{\mathfrak{g}^{-,*}} = \dim \mathfrak{g}_\xi^+ + \dim \mathfrak{g}^- - \dim \mathfrak{g}^+.$$

- b) Because of (6.5) below reversible minimality of μ is equivalent to minimality of $\dim \mathfrak{g}_\mu$ for $\mu \in \mathfrak{g}^{-,*}$ which again is equivalent to minimality of $\dim \mathfrak{g}_\mu^+$ for $\mu \in \mathfrak{g}^{-,*}$.
- c) For every $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ we have

$$\dim \ker Df^-(\xi, \mu) = \dim \mathfrak{g}^- + \dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}.$$

This can be shown as in the proof of Proposition 4.5. Applying part a) to $\mathfrak{g}_\mu$ and using (6.5) we get the relation

$$\dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}} = \dim \mathfrak{g}_{(\xi,\mu)}^+ + \dim \mathfrak{g}^- - \dim \mathfrak{g}^+.$$

So we see that $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is minimal reversible if and only if it is a minimal point of the variety $V(f^-) = (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ with respect to f^- , as in the non-reversible situation.

- d) If G is compact then adjoint and coadjoint actions coincide so that the condition that $\xi \in \mathfrak{g}^-$ be minimal reversible is equivalent to the condition that $\mathfrak{g}_\xi^-$ has minimal dimension for $\xi \in \mathfrak{g}^-$ which by definition of minimal reversible velocities is equivalent to $\mathfrak{g}_\xi$ having minimal dimension for $\xi \in \mathfrak{g}^-$. Moreover in this case $r_\xi^-(G) = \dim \mathfrak{g}_\xi^-$. The same argument applied to G_μ rather than G shows that for compact G a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ minimal reversible if and only if $\dim \mathfrak{g}_{(\xi,\mu)}^-$ is minimal and in this case $r_{(\xi,\mu)}^-(G) = \dim \mathfrak{g}_{(\xi,\mu)}^-$.

Proposition 6.3 *Let $G = \Gamma \ltimes \mathbb{Z}_2^\rho$. Then the sets of minimal reversible elements $\xi \in \mathfrak{g}^-$, minimal reversible elements $\mu \in \mathfrak{g}^{-,*}$, minimal reversible pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ and regular reversible pairs $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ are open and dense in $\mathfrak{g}^-$, $\mathfrak{g}^{-,*}$, and $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ respectively.*

The proof is analogous to the proof of Proposition 4.6.

Definition 6.4 *We call $\mu \in \mathfrak{g}^{-,*}$ reversible split if there is a choice of $\mathbb{Z}_2^\rho$ -invariant complement $\mathfrak{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ such that $[\mathfrak{g}_\mu^-, \mathfrak{n}_\mu^+] \subseteq \mathfrak{n}_\mu$.*

Note that $\mu \in \mathfrak{g}^{-,*}$ is reversible split if and only if $\mathfrak{g}_\mu^-$ maps $\text{ann}(\mathfrak{g}^+ + \mathfrak{n}_\mu) \simeq \mathfrak{g}_\mu^{-,*}$ into $\text{ann}(\mathfrak{n}_\mu) \simeq \mathfrak{g}_\mu^*$. To see this let $\mathfrak{n}_\mu$ be $\mathbb{Z}_2^\rho$ -invariant as before and let $\nu \in \text{ann}(\mathfrak{g}^+ + \mathfrak{n}_\mu)$, $\xi \in \mathfrak{g}_\mu^-$. Then $\text{ad}_\xi^* \nu \in \text{ann}(\mathfrak{n}_\mu)$ if and only if $[\xi, \chi] \in \mathfrak{g}^+ + \mathfrak{n}_\mu$ for all $\chi \in \mathfrak{n}_\mu$. From (6.2) we obtain the stated equivalence.

In the sequel we will need the following reversible version of Lemma 2.1:

Lemma 6.5 *Let $G = \Gamma \ltimes \mathbb{Z}_2^\rho$ and let $\mu \in \mathfrak{g}^{-,*}$. Choose a $\mathbb{Z}_2^\rho$ -invariant complement $\mathfrak{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$. Then:*

- a) *The space $\mu + \mathfrak{g}_\mu^{-,*}$, where we identify $\mathfrak{g}_\mu^{-,*} \simeq \text{ann}(\mathfrak{n}_\mu) \cap \mathfrak{g}^{-,*}$, is a local ‘reversible’ section transverse to $G^+ \mu$ at μ in $\mathfrak{g}^{-,*}$, ie, there is a neighbourhood U^- of μ in $\mathfrak{g}^{-,*}$ such that*

$$\text{for all } \hat{\mu} \in U^- \text{ there are } \gamma \in G^+, \nu \in \mathfrak{g}_\mu^{-,*} \text{ such that } \hat{\mu} = \gamma(\mu + \nu)$$

where $\gamma = \exp(\chi)$ and $(\chi, \nu) \approx 0$ are locally unique in $\mathfrak{n}_\mu^+ \oplus \mathfrak{g}_\mu^{-,}$. If we allow $\gamma = \exp \chi$ to vary arbitrarily in a small neighbourhood $U_{G^+}(\text{id})$ of G^+ then the above decomposition $\hat{\mu} = \gamma(\mu + \nu)$ is locally unique modulo $Z_{\mu,\nu}^+ := Z_{\mu,\nu} \cap G^+$, with $Z_{\mu,\nu}^+$ acting as in (2.5).*

- b) *In a neighbourhood of id the set $Z_{\mu,\nu}^+$ is a manifold with $\dim Z_{\mu,\nu}^+ = \dim \mathfrak{g}_\mu^+$, and*

$$\dim \mathfrak{g}^- - \dim \mathfrak{g}^+ = \dim \mathfrak{g}_\mu^- - \dim \mathfrak{g}_\mu^+ \quad \forall \mu \in \mathfrak{g}^{-,*}. \quad (6.5)$$

- c) *If μ is reversible split then $j_\mu(\nu)|_{\mathfrak{g}_\mu^-} = \text{id}|_{\mathfrak{g}_\mu^-}$ for $\nu \in \mathfrak{g}_\mu^{-,*}$.*

Proof. By Lemma 2.1 a) we have $\hat{\mu} = \exp(\chi)(\mu + \nu)$ where $\chi \in \mathfrak{n}_\mu$ and $\nu \in \mathfrak{g}_\mu^*$ are locally unique. Since

$$\hat{\mu} = -\text{Ad}_\rho^* \hat{\mu} = \exp(\text{Ad}_\rho \chi)(\mu - \text{Ad}_\rho^* \nu)$$

and χ, ν are locally unique we get $\chi \in \mathfrak{n}_\mu^+$, $\nu \in \mathfrak{g}_\mu^{-,*}$. By Lemma 2.1 a) the decomposition $\hat{\mu} = \gamma(\mu + \nu)$ with $\gamma \approx \text{id}$ arbitrary in Γ is locally unique modulo $Z_{\mu,\nu}$ which proves a). The first statement of b) follows from the implicit function theorem applied to (2.3) with g varying in G^+ . Using the local reversible section transverse to $G^+ \mu$ at μ in $\mathfrak{g}^{-,*}$ and counting dimensions we obtain (6.5). Part c) follows from Definition 6.4. $\blacksquare$

The following technical lemma is the reversible analogue of Lemma 4.7.

Lemma 6.6 *Let $G = \Gamma \rtimes \mathbb{Z}_2^p$. Then:*

- a) $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is G^+ -invariant.
- b) Let $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ be reversible, let $\mathfrak{n}_\mu$ be a $\mathbb{Z}_2^p$ -invariant complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ and identify $\mathfrak{g}_\mu^{-,*} \simeq \text{ann}(\mathfrak{n}_\mu) \cap \mathfrak{g}^{-,*}$ as before. Using the section $\mu + \mathfrak{g}_\mu^{-,*}$ to $G^+\mu$ at μ in $\mathfrak{g}^*$ from Lemma 6.5 we get the following parametrization of $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ in a neighbourhood $U_{(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}}(\xi, \mu)$ of (ξ, μ) in $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$:

$$\begin{aligned} U_{(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}}(\xi, \mu) &= \{ \gamma(j_\mu(\nu)\chi, \mu + \nu), \quad \gamma \in U_{G^+}(\text{id}), \\ &\quad (\chi, \nu) \in U_{\mathfrak{g}_\mu^- \oplus \mathfrak{g}_\mu^{-,*}}(\xi, 0) \text{ satisfies } f_\mu^-(\chi, \nu) = 0 \}. \end{aligned}$$

Here $U_{G^+}(\text{id})$ and $U_{\mathfrak{g}_\mu^- \oplus \mathfrak{g}_\mu^{-,*}}(\xi, 0)$ are neighbourhoods of id in G^+ and $(\xi, 0)$ in $\mathfrak{g}_\mu^- \oplus \mathfrak{g}_\mu^{-,*}$, and $Z_{\mu, \nu}$ is as in (2.3). The map f_μ^- is the restriction of f_μ from (4.1) to $U_{\mathfrak{g}_\mu^- \oplus \mathfrak{g}_\mu^{-,*}}(\xi, 0)$, and (γ, χ, ν) are locally unique modulo the action of $U_G(\text{id}) \cap Z_{\mu, \nu}^+$ given by (4.2). Moreover $(\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is a manifold near $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ if and only if $(f_\mu^-)^{-1}(0)$ is a manifold near $(\xi, 0)$.

- c) Let $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ be regular reversible, $r = r_{(\xi, \mu)}^-(G)$. Then $r \leq \dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}$, and near $(\chi, \nu) = (\xi, 0)$ the set $(f_\mu^-)^{-1}(0)$ is a manifold of the form $(\chi, \nu(\chi, \tau))$ where $\nu : U_{\mathfrak{g}_\mu^- \oplus \mathfrak{g}_\mu^{-,*}}(\xi, 0) \rightarrow \mathfrak{g}_\mu^{-,*}$ is smooth with $\nu(\xi, 0) = 0$, $D_\chi \nu(\xi, 0) = 0$, and the columns of $D_\tau \nu(\xi, 0)$ form an r -dimensional subspace of $\ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}$. If μ is reversible split then $\nu(\chi, \tau)$ is linear in τ .

Proof. a) is clear, b) follows from a) and Lemma 6.5. Part c) follows from b), Lemma 6.5 and (6.5), similarly as in the proof of Lemma 4.7. $\blacksquare$

A torus $T \subset \Gamma$ is called a *maximal reversible torus* if T is generated by some $\xi \in \mathfrak{g}^-$ and if T is not contained in any reversible torus of higher dimension [16]. For compact groups G every reversible $\xi \in \mathfrak{g}^-$ lies in the Lie algebra $\mathfrak{t}^-$ of a maximal reversible torus T^- of G , almost every reversible $\xi \in \mathfrak{g}^-$ generates a maximal reversible torus, and all maximal reversible tori are conjugate [16]. In this case we denote the dimension of each of those maximal reversible tori by $\text{rank}^-(G)$.

In the following proposition, which is analogous to Propositions 4.9 and 4.3 in the non-reversible situation, we give some sufficient conditions for a velocity-momentum pair to be regular reversible.

Proposition 6.7 *Let $G = \Gamma \rtimes \mathbb{Z}_2^p$. Then:*

- a) If $\mu \in \mathfrak{g}^{-,*}$ is regular reversible then (ξ, μ) is regular reversible and minimal reversible for every $\xi \in \mathfrak{g}_\mu^-$, and $\dim \mathfrak{g}_\mu^- = r_\mu^-(G) = r_{(\xi, \mu)}^-(G) = \dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}$. As a consequence $\text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}} \equiv 0$ and $[\mathfrak{g}_\mu^+, \mathfrak{g}_\mu^-] = 0$. Moreover

$$f_\mu^-(\chi, \nu) \equiv 0 \quad \text{for } \nu \in \mathfrak{g}_\mu^{-,*}, \nu \approx 0, \chi \in \mathfrak{g}_\mu^-.$$

- b) If $\xi \in \mathfrak{g}^-$ is regular reversible then (ξ, μ) is regular reversible and minimal reversible for all $\mu \in \mathfrak{g}^-$ with $\xi \in \mathfrak{g}_\mu^-$ and $r_\xi^-(G) = r_{(\xi, \mu)}^-(G) = \dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}$.
- c) If μ is reversible split then the following conditions are equivalent:

- (i) ξ is regular reversible in $\mathfrak{g}_\mu^-$;
- (ii) $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is regular reversible;
- (iii) $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is minimal reversible.

Moreover, if μ is reversible split and (ξ, μ) regular reversible then $r_{(\xi, \mu)}^-(G) = \dim \ker \text{ad}_\xi^{\mu,*}|_{\mathfrak{g}_\mu^{-,*}}$.

- d) Let G be compact. Then a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ is regular reversible if and only if it is minimal reversible. Moreover $\xi \in \mathfrak{g}^-$, $\mu \in \mathfrak{g}^{-,*}$ and $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ are minimal reversible if and only if $\mathfrak{g}_\xi^-, \mathfrak{g}_\mu^-$ and $\mathfrak{g}_{(\xi, \mu)}^-$ are the Lie algebras of a maximal reversible torus. In this case $r_\xi^-(G) = \text{rank}^-(G)$, $r_\mu^-(G) = \text{rank}^-(G)$ or $r_{(\xi, \mu)}^-(G) = \text{rank}^-(G)$ respectively.

Proof. The proofs of parts a), b), c) are analogous to the proof of Proposition 4.9. The first statement of d) follows from c). To prove the second statement of d) let G be compact. By Remark 6.2 d) an element $\xi \in \mathfrak{g}^-$, and a pair $(\xi, \mu) \in (\mathfrak{g} \oplus \mathfrak{g}^*)^{-,c}$ are minimal reversible if and only if $\mathfrak{g}_\xi^-$ or $\mathfrak{g}_{(\xi, \mu)}^-$ have minimal dimension, and then $r_\xi(G) = \dim \mathfrak{g}_\xi^-$, $r_{(\xi, \mu)}(G) = \dim \mathfrak{g}_{(\xi, \mu)}^-$ respectively. Now we argue similarly as in the proof of Proposition 4.3. ■

Before we can formulate our persistence result for reversible relative equilibria we need one more definition.

Definition 6.8 *A reversible relative equilibrium $\Gamma_p \subset \mathcal{M}$ of (2.1) is said to have regular reversible momentum $\mu = \mathbf{J}(p)$ (regular reversible velocity ξ , regular reversible velocity-momentum pair (ξ, μ)) modulo isotropy if $\mu_L = \mu|_{\mathfrak{l}} \in \mathfrak{l}^{-,*}$ is regular reversible ($\xi_L \in \mathfrak{l}^-$ is regular reversible, $(\xi_L, \mu_L) \in (\mathfrak{l} \oplus \mathfrak{l}^*)^{-,c}$ regular reversible) for the group $L = N(\Gamma_p)/\Gamma_p$.*

Here we identify $\xi_L \in \mathfrak{l}$ with $\xi \in \mathfrak{g}$ by considering $\mathfrak{l}$ as subspace of $\mathfrak{g}$ and choosing ξ in $\mathfrak{l} \subseteq \mathfrak{g}$, and let $\mu_L = \mu|_{\mathfrak{l}}$ as in Section 3.

The following theorem presents a persistence result for reversible relative equilibria with regular reversible velocity-momentum pairs modulo isotropy:

Theorem 6.9 *Let Γ_p be a reversible relative equilibrium of (2.1) which is non-degenerate and has regular reversible velocity-momentum pair (ξ, μ) modulo isotropy. Then:*

- a) *The solutions of (3.6) form an r -dimensional manifold $\nu(\tau) \in \mathfrak{l}_\mu^{-,*} \subseteq \mathfrak{g}_\mu^*$, $\tau \in \mathbb{R}^r$, $r = r_{(\xi_L, \mu_L)}^-(L)$, with $\nu(0) = 0$ which, by Proposition 3.2, gives an r -dimensional family of reversible relative equilibria $\Gamma_p(\tau)$ with isotropy G_p and $p(0) = p$, and there are no other relative equilibria with the same isotropy G_p close to Γ_p .*
- b) *If μ is regular reversible modulo isotropy, then a) holds, $r = r_{\mu_L}^-(L)$, and $\Lambda p \subseteq M$ persists to every nearby reversible momentum $\hat{\mu}_L \in \mathfrak{l}^{-,*}$.*
- c) *If ξ is regular reversible modulo isotropy, then a) holds, and $r = r_{\xi_L}^-(L)$.*

The proof is analogous to the proof of Theorem 5.2.

Example 6.10 We consider Examples 5.3 again, a system with symmetry $\Gamma = \text{SE}(3)$ (case a)) or $\Gamma = \text{SE}(2) \times \mathbb{R}$ (case b)), and the same concrete realization, a neutrally buoyant rigid body in an ideal fluid with coincident and non-coincident centres of mass and buoyancy respectively, but now we take reversibility into account. We assume that (2.1) has a reversing symmetry ρ which commutes with all elements of Γ so that the Hamiltonian of the problem is G -invariant with $G = \Gamma \times \mathbb{Z}_2^o$. This is frequently satisfied in applications, for example for the rigid body in the fluid. In this case the system has a reversing symmetry ρ which acts on phase space $\mathcal{M} = T^*\text{SE}(3) = \text{SE}(3) \times \mathfrak{se}(3)^*$ by reversing the signs of the impulses, more precisely, by trivial action on the configuration space $\text{SE}(3)$, and by the s -coadjoint action on impulse space $\mathfrak{se}(3)^*$, cf. [17, 18].

- a) First we revisit the case where $\Gamma = \text{SE}(3)$ with a neutrally buoyant body in a fluid with coincident centres of mass and buoyancy as a concrete realization as in Example 5.3 a).

As we saw in Example 4.10 d) a velocity-momentum (ξ, μ) is regular if and only if $\mu^a \neq 0$ or $\xi^r \neq 0$. From (4.6) we see that μ^a is parallel to ξ^r , and that a reversing symmetry of a regular velocity-momentum pair (ξ, μ) is $\hat{\rho} = \rho(R_\pi, a)$ where R_π is a rotation by π around an axis orthogonal to the axis of linear momentum μ^a and angular velocity ξ^r , and $a \in \mathbb{R}^3$, $a \perp \mu^a$, $a \perp \xi^r$, is chosen appropriately. So we see that any regular velocity-momentum pair is $\hat{\rho}$ -reversible for some reversing symmetry $\hat{\rho}$. Indeed, it can easily be seen that this is true for any velocity-momentum pair. Note that the non-split momentum $\mu = (\mu^r, 0)$ studied in Example 5.3 a) is not reversible split.

A relative equilibrium Γp is reversible if p is invariant under the reversing symmetry $\hat{\rho}$ of its velocity-momentum pair (ξ, μ) . Then the two-parameter family of relative equilibria $\Gamma p(\tau)$ containing Γp which we described in Example 5.3 a) is also $\hat{\rho}$ -reversible.

We conclude that a rigid body in a fluid which does not feel gravity and moves on an $\text{SE}(3)$ group orbit, that is, rotates and/or drifts along some fixed axis, is always reversible.

- b) Now we consider the case $\Gamma = \text{SE}(2) \times \mathbb{R}$, with a neutrally buoyant body in a fluid with non-coincident centres of mass and buoyancy as a concrete realization as in Example 5.3 b).

A translating non-spinning relative equilibrium Γp with regular momentum μ , ie $(\mu_1^a, \mu_2^a) \neq 0$, as considered in Example 5.3 b) has a reversible velocity-momentum pair (ξ, μ) for some reversing symmetry $\hat{\rho}$ if and only if angular momentum and the third component of linear momentum and drift velocity vanish, $\mu^r = 0$, $\mu_3^a = \xi_3^a = 0$. In this case the reversing symmetry of the velocity-momentum pair is $\hat{\rho} = R_\pi \rho$ where the element $R_\pi \in \text{SO}(2)$ is a rotation by π , as can be seen from (4.5). The relative equilibrium Γp is reversible if it has such a velocity-momentum pair and if $\hat{\rho}$ is a reversing symmetry of the relative equilibrium: $\hat{\rho}p = p$. In this case $\mathfrak{g}_\mu^- = \mathbb{R}$, so if the relative equilibrium is in addition non-degenerate modulo isotropy then by Theorem 6.9 there is a one-parameter family of $\hat{\rho}$ -reversible relative equilibria closeby.

We see from (4.5) that the rotating and translating relative equilibria with singular momentum and regular velocity ξ , ie $\xi^r \neq 0$, are non-reversible.

In the concrete example of a rigid body in a fluid which feels gravity this means that a rigid body which uniformly drifts horizontally is a reversible relative equilibrium, but rigid bodies that fall, rise or rotate are non-reversible relative equilibria.

7 Symmetry-breaking bifurcations

Finally note that we can also treat bifurcation from a relative equilibrium Γp to relative equilibria with *smaller isotropy group* $\hat{G}_p$ by restricting the dynamics to $\text{Fix}_{\mathcal{M}}(\hat{\Gamma}_p)$ and using Theorems 5.2, 6.9 above. To make this precise, we extend Definition 3.1.

Definition 7.1 *Let Γp be a relative equilibrium of (2.1) with velocity-momentum pair (ξ, μ) and let $\hat{G}_p$ be a subgroup of G_p . Let $L := N(\hat{\Gamma}_p)/\hat{\Gamma}_p$, identify $\xi_L \simeq \xi \in \mathfrak{l} \subseteq \mathfrak{g}$ and let $\mu_L = \mu|_{\mathfrak{l}}$.*

- a) *We call the subgroup $\hat{G}_p$ with Lie algebra $\hat{\mathfrak{g}}_p$ a regular subgroup of G_p if*

$$\{\chi \in \mathfrak{g}_p, \quad s(\hat{g}_p)\text{Ad}_{\hat{g}_p}\chi = \chi \quad \forall \hat{g}_p \in \hat{G}_p\} \subseteq \hat{\mathfrak{g}}_p.$$

- b) *We call the relative equilibrium non-degenerate modulo $\hat{G}_p$ if it is non-degenerate on $\text{Fix}_{\mathcal{M}}(\hat{G}_p)$, ie if $D^2H^\xi(p)|_{\text{Fix}_{\mathcal{N}_1}(\hat{G}_p)}$ is non-degenerate.*

- c) *We say that the relative equilibrium Γp has regular velocity-momentum pair (ξ, μ) modulo $\hat{G}_p$ if*

(i) *either $\hat{L}_p \simeq \mathbb{Z}_2$ and $(\xi_L, \mu_L) \in (\mathfrak{l} \oplus \mathfrak{l}^*)^{-\cdot c}$ is regular reversible for the action of L , in which case we define $r_{(\xi, \mu)}(\hat{G}_p, G) = r_{(\xi_L, \mu_L)}^-(L)$,*

(ii) *or $\hat{L}_p = \{\text{id}\}$ and $(\xi_L, \mu_L) \in (\mathfrak{l} \oplus \mathfrak{l}^*)^c$ is regular for the action of L in which case we define $r_{(\xi, \mu)}(\hat{G}_p, G) = r_{(\xi_L, \mu_L)}(L)$.*

Here $\hat{L}_p = \hat{G}_p/\hat{\Gamma}_p \subset L$.

- d) *The notions of a regular momentum μ and a regular velocity ξ modulo $\hat{G}_p$ are defined analogously.*

The regularity assumption on $\hat{G}_p$ implies that Λ_p is finite. As a consequence relative equilibria with isotropy containing $\hat{G}_p$ are equilibria (and not only relative equilibria) of the slice equations (3.1) on $N_0 \oplus N_1$. Therefore the equation determining relative equilibria near Γp in $M = \text{Fix}_{\mathcal{M}}(\hat{\Gamma}_p)$ has the form (3.5) although in this case the action is not locally free near p in M . Therefore the proofs of Theorems 5.2, 6.9 also apply to M , and we get the following result:

Theorem 7.2 *Let Γp be a relative equilibrium of (2.1) and let $\hat{G}_p$ be a regular subgroup of G_p . Assume that the relative equilibrium is non-degenerate modulo $\hat{G}_p$ and has a velocity-momentum pair which is regular modulo $\hat{G}_p$ and let $r = r_{(\xi, \mu)}(\hat{G}_p, G)$. Then:*

- a) *there is an r -dimensional family of relative equilibria $\Gamma p(\tau)$, $\tau \in \mathbb{R}^r$ with isotropy containing $\hat{G}_p$ and velocity-momentum pair close to (ξ, μ) near Γp and no other relative equilibria with the same isotropy nearby.*

- b) If μ is regular modulo $\hat{G}_p$ then a) holds, $r = r_\mu(\hat{G}_p, G)$, and $\Lambda p \subseteq M$ persists to every nearby momentum in $\mathbf{l}^*$ with isotropy $\hat{L}_p$.
- c) If ξ is regular modulo $\hat{G}_p$ then a) holds, and $r = r_\xi(\hat{G}_p, G)$.

Example 7.3 Consider Example 6.10 a), a rigid body in a fluid, again, and now assume that we are given a non-degenerate ρ -reversible equilibrium p at $\mu = 0$. The zero-momentum $\mu = 0$ is regular ρ -reversible, but $(\xi, \mu) = (0, 0)$ is isolated in the space of ρ -reversible velocity-momentum pairs. To get persistence to nearby momentum values the reversing symmetry ρ has to be broken. Assume that the body has spherical symmetry. Then there is an additional action of $\text{SO}(3)$ on $T^*\text{SE}(3) = \text{SE}(3) \times \mathfrak{se}(3)^*$ acting by right multiplication on $\text{SE}(3)$. We denote this additional $\text{SO}(3)$ -symmetry by $\text{SO}(3)_R$, and the original $\text{SO}(3)$ -symmetry which acts on the left, by $\text{SO}(3)_L$. The isotropy group G_p of the spherical equilibrium then contains the diagonal embedding $\text{SO}(3)_D$ of $\text{SO}(3)$ into $\text{SO}(3)_L \times \text{SO}(3)_R$. In this situation we also have an additional reflection symmetry $\kappa = (-I, -I) \in \text{O}(3)_D$ acting by conjugation on $\text{SE}(3)$ and by its coadjoint action on $\mathfrak{se}(3)^*$ so that the isotropy of the spherical equilibrium becomes $G_p = \mathbb{Z}_2^{\rho} \times \text{O}(3)_D$ and the symmetry group of the system is $G = \mathbb{Z}_2^{\rho} \times \text{SO}(3)_R \times (\mathbb{Z}_2^{\kappa} \ltimes \text{SE}(3))$.

Let $\hat{G}_p \subset G_p$ be the subgroup of G_p defined by $\hat{G}_p = \text{SO}(2)_D \rtimes \mathbb{Z}_2^{\rho R\pi}$ where $\text{SO}(2)_D$ is a copy of $\text{SO}(2)$ in $\text{SO}(3)_D$ and $R_\pi \in \text{SO}(3)_D$ is a rotation by π around an axis orthogonal to the $\text{SO}(2)_D$ -axis. Let $\mathfrak{so}(2)_{AD}$ be the $\text{SO}(2)_D$ -invariant complement to the Lie algebra $\mathfrak{so}(2)_D$ in $\mathfrak{so}(2)_L \oplus \mathfrak{so}(2)_R$ (corresponding to an antidiagonal embedding of $\mathfrak{so}(2)$ in $\mathfrak{so}(2)_L \oplus \mathfrak{so}(2)_R$). Then the Lie algebra $\mathbf{l}$ of $L := N(\hat{G}_p)/\hat{G}_p$, given by $\mathbf{l} = \mathfrak{so}(2)_{AD} \oplus \mathbb{R}$, is abelian and $\hat{\rho}$ -reversible with $\hat{\rho} = \rho R_\pi$: $\mathbf{l} = \mathbf{l}^-$. Therefore every $\nu \in \mathbf{l}^{-,*}$ is regular reversible. Hence by Theorem 7.2, if Γp is non-degenerate modulo $\hat{G}_p$, there is a two-parameter family $\Gamma p(\tau)$, $\tau \in \mathbb{R}^2$, of reversible axisymmetric relative equilibria close to p with isotropy conjugate to $\hat{G}_p$. Using the fact that the drift velocity $\hat{\xi}$ of any $\hat{\rho}$ -relative equilibrium $\hat{\Gamma} p$ with momentum $\hat{\mu} = \mathbf{J}(\hat{p})$ satisfies $\hat{\xi} \in \mathfrak{g}_{\hat{\mu}}$, $\Gamma_{\hat{p}} \subseteq \Gamma_{\hat{\xi}}$, and is typically a generic $\hat{\rho}$ -reversible element of the subset of $\mathfrak{g}$ defined by these conditions we see that the relative equilibria of this family typically rotate and drift. The family contains two one-parameter subfamilies with higher isotropy: a family of spinning, non-translating reversible relative equilibria with isotropy conjugate to $(\text{SO}(2)_D \rtimes \mathbb{Z}_2^{\rho R\pi}) \times \mathbb{Z}_2^{\kappa}$, and a family of translating, non-spinning reversible relative equilibria with isotropy conjugate to $\text{SO}(2)_D \rtimes (\mathbb{Z}_2^{\kappa R\pi} \times \mathbb{Z}_2^{\rho R\pi})$.

Acknowledgements

I want to thank Mark Roberts and James Montaldi for many useful hints and fruitful discussions, and the University of Warwick for their hospitality during several visits when parts of this article were written. This work has been supported by the EU-TMR network ‘Mechanics and Symmetry in Europe’, contract number HPRN-CT-2000-0011, and by a Marie-Curie fellowship of the EU under contract number HPMF-CT-2000-00542.

References

- [1] R. Abraham and J.E. Marsden. *Foundations of Mechanics. Second edition. Revised, enlarged, reset.* The Benjamin/Cummings Publishing Company, Massachusetts, 1978.
- [2] L. Bates and E. Lerman. Proper group actions and symplectic stratified spaces. *Pac. J. Math.*, 181(2):201–229, 1997.
- [3] J. Bochnak, M. Coste and M.-F. Roy. *Géométrie algébrique réelle.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 12, Springer, New York, Berlin, Heidelberg, Tokyo, 1987.
- [4] M. Brodmann. *Algebraische Geometrie - Eine Einführung.* Basler Textbücher, Birkhäuser, 1989.
- [5] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups.* Graduate Texts in Mathematics, vol. 98, Springer, New York, Berlin, Heidelberg, Tokyo, 1985.
- [6] P. Chossat, D. Lewis, J.-P. Ortega and T. Ratiu. Bifurcation of relative equilibria in Hamiltonian systems. Preprint, Lausanne, 1999.

- [7] M. Dellnitz, I. Melbourne, and J.E. Marsden. Generic bifurcation of Hamiltonian vector fields with symmetry. *Nonlinearity*, 5(4):979–996, 1992.
- [8] B. Fiedler, B. Sandstedt, A. Scheel, and C. Wulff. Bifurcation from relative equilibria to non-compact group actions: Skew products, meanders, and drifts. *Doc. Math. J. DMV*, 1:479–505, 1996.
- [9] C.G. Gibson, K. Wirthmüller, A.A. du Plessis and E.J.N. Looijenga. *Topological Stability of Smooth Mappings*. Lecture Notes in Mathematics, vol. 552, Springer, New York, Heidelberg, Tokyo, 1976.
- [10] M. Golubitsky and I. Stewart. Generic bifurcation of Hamiltonian systems with symmetry. *Physica D*, 24:391 – 405, 1987.
- [11] V. Guillemin, E. Lerman and S. Sternberg. *Symplectic Fibrations and Multiplicity Diagrams*. Cambridge University Press, Cambridge, 1996.
- [12] V. Guillemin and S. Sternberg. *Symplectic Techniques in Physics*. Cambridge University Press, Cambridge, 1990.
- [13] I.N. Kozin, R.M. Roberts and J. Tennyson. Relative equilibria of D_2H^+ and H_2D^+ . *Mol. Phys.* 98(5):295–307, 2000.
- [14] M. Krupa. Bifurcations of relative equilibria. *SIAM J. Math. Anal.*, 21:1453–1486, 1990.
- [15] H.-P. Kruse and J. Scheurle. On the bifurcation and stability of rigidly rotating inviscid liquid bridges. *J. Nonlin. Sci.*, 8:215–232, 1998.
- [16] J.S.W. Lamb and C. Wulff. Reversible relative periodic orbits. *J. Differential Equations*, 178, 60–100, 2002.
- [17] N.E. Leonard. Stability of a bottom-heavy underwater vehicle. *Automatica*, 33(3): 331–346, 1997.
- [18] N.E. Leonard and J.E. Marsden. Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry. *Physica D*, 105: 130–162, 1997.
- [19] E. Lerman and S.F. Singer. Stability and persistence of relative equilibria at singular values of the moment map. *Nonlinearity*, 11:1637 – 1649, 1998.
- [20] D. Lewis. Nonlinear stability of a rotating planar liquid drop. *Arch. Rat. Mech. Anal.*, 106:287 – 333, 1989.
- [21] D. Lewis. Bifurcation of liquid drops. *Nonlinearity*, 6:491 – 522, 1993.
- [22] D. Lewis and J.C. Simo. Nonlinear stability of rotating pseudo-rigid bodies. *Proc. R. Soc. London*, A427:281 – 319, 1990.
- [23] C. Lim, J.A. Montaldi and R.M. Roberts. Relative equilibria of point vortices on the sphere. *Physica D*, 148:97 – 135, 2001.
- [24] C.-M. Marle. Modèle d'action Hamiltonienne d'un groupe de Lie sur une variété symplectique. *Rendiconti del Seminario Matematico, Università e Politecnico, Torino*, 43:227–251, 1985.
- [25] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry*. Springer-Verlag, New York, Berlin, Heidelberg, 1994.
- [26] A. Mielke. *Hamiltonian and Lagrangian flows on a center manifold with applications to elliptic variational problems*, volume 1489 of *Lect. Notes Math.* Springer-Verlag, Berlin, Heidelberg, New York, 1991.
- [27] J. Montaldi. Persistence and stability of relative equilibria. *Nonlinearity*, 10:449 – 466, 1997.
- [28] J. Montaldi and R.M. Roberts. Relative equilibria of molecules. *J. Nonlin. Sci.* 9:53–88, 1999.
- [29] J. Montaldi and T. Tokieda. Openness of momentum maps and persistence of extremal relative equilibria. Preprint, University of Manchester, 2002.
- [30] J.-P. Ortega and T.S. Ratiu. Persistance et différentiabilité de l'ensemble des éléments critiques relatifs dans les systèmes hamiltoniens symétriques. *C. R. Acad. Sci. Paris*, I 325:1107–1111, 1997.

- [31] R.S. Palais. On the existence of slices for actions of noncompact Lie groups. *Ann. of Math.*, 73:295 – 323, 1961.
- [32] G. Patrick. Relative equilibria of Hamiltonian systems with symmetry: Linearization, smoothness and drift. *J. Nonlinear Sc.*, 5(5):373 – 418, 1995.
- [33] G. Patrick. Dynamics of perturbed relative equilibria of point vortices on the sphere or plane. *J. Nonlin. Sci.*, 10:401 – 415, 2000.
- [34] G. Patrick and R.M. Roberts. The transversal relative equilibria of Hamiltonian systems with symmetry. *Nonlinearity*, 13:2089 – 2105, 2000.
- [35] S. Pekarsky and J.E. Marsden. Point vortices on a sphere: stability of relative equilibria. *J. Mathematical Physics*, 39:5894 – 5906, 1998.
- [36] R.M. Roberts and M.E.R. de Sousa Dias. Bifurcations from relative equilibria of Hamiltonian systems. *Nonlinearity*, 10(6):1719–1738, 1997.
- [37] R.M. Roberts, C. Wulff, and J. Lamb. Hamiltonian systems near relative equilibria. *J. Differential Equations*, 179, 562–604, 2002.