Information aggregation with a continuum of types

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Abstract

We study the problem of designing a voting rule which makes voting by cut-off strategies efficient for settings where voters have state-dependent common preferences over and vote on accepting or rejecting an issue but hold private information in the form of continuous types about the true state. We show that such rules only exist under a restrictive condition on the model parameters.

JEL Classification C70, D70, D71, D80, D82

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1 Introduction

Consider decision making situations where a group needs to accept or reject an issue. A policy is approved or rejected in a referendum, a defendant is convicted or acquitted by the jury in a court trial, or a job candidate is hired or not by a hiring committee. We model such situations as voting problems where individuals have state-dependent common preferences and private information in the form of types about the true state. We assume that types are distributed from a state-dependent continuous density. Our model is similar to that of Duggan and Martinelli (2001) and Meirowitz (2002), who derive a Condorcet jury type theorem for a fixed mechanism. We instead focus on the problem of designing a voting rule which makes voting by cut-off strategies efficient for any fixed number of voters. A cut-off strategy means that a voter votes ‘yes’ if and only if the type of the voter (a number between zero and one) exceeds a certain threshold, the cut-off. Efficiency means that the most likely correct outcome is chosen given the available information, i.e., types. Our paper can also be seen as an extension of Austen-Smith and Banks (1996) to a continuum of types.

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A cut-off strategy reflects what is often called ‘informative voting’. In models where the set of types and the set of possible votes are equal or have the same size, defining informative voting is straightforward. Cut-off strategies constitute the most obvious and natural form of informative voting in our model. We consider cut-off strategy profiles where each voter uses the same cut-off. This is a natural assumption, given that in our model all voters are *ex ante* completely symmetric.

We show that a voting rule which makes voting according to such cut-off strategy profiles efficient, exists under a specific and restrictive condition on the model parameters (see Theorem 1). This condition says that there exists a number $\tau \in (0, 1)$ such that whenever an issue is more likely to be true than false given the types, it should be more likely to be true than false based on the number of types exceeding τ . In that case, a specific quota rule, depending on τ , must be used in order to make voting by strategies with cut-off τ efficient. The theorem supports the intuition that when private information reflects a set of possibilities richer than indicating ‘yes’ or ‘no’ only, a binary voting rule usually cannot aggregate the whole available information efficiently.

Other related contributions focusing on information aggregation include Barelli et al (2017), who again study asymptotic efficiency. Azrieli and Kim (2014) and Schmitz and Tröger (2012) focus on mechanism design for collective choice problems with two alternatives and private values. In some sense our paper can be seen as a much more detailed version of the model in the latter paper, and therefore we are able to express our results on the basis of these details, namely the objective probabilities of the states and the type functions, as well as the binary voting method.

2 The model

There are $n \geq 2$ voters, and two possible states of the world 0 and 1. The prior probability of state 1 is equal to π , with $0 < \pi < 1$.

Voter i ’s type is denoted by $t_i \in [0, 1]$, and represents i ’s private information about the true state. Each t_i is distributed according to the density f_1 or f_0 , depending on whether the state is 1 or 0. We assume that f_1 and f_0 are piecewise continuous positive functions, and that f_0/f_1 is weakly decreasing on $[0, 1]$. The latter is the familiar monotone likelihood ratio property, ensuring that higher values of types are stronger indications of the state being 1. We write $\mathbf{t} = (t_1, \dots, t_n) \in [0, 1]^n$ for a vector of types. We assume that types are drawn independently, conditional on the state.

A strategy is a function $\sigma : [0, 1] \rightarrow \{0, 1\}$, mapping each type $t \in [0, 1]$ to a vote $\sigma(t)$. A strategy profile is a vector $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of voters’ strategies.

A voting rule is a function $g : \{0, 1\}^n \rightarrow \{0, 1\}$, assigning to each profile of votes

a decision 0 ('reject') or 1 ('accept'). The voters have common utility function u , defined on pairs (d, s) of a decision $d \in \{0, 1\}$ and a state $s \in \{0, 1\}$, by $u(d, s) = 1$ if $d = s$ and $u(d, s) = 0$ if $d \neq s$.

For a type profile $\mathbf{t}$, we call a decision $d \in \{0, 1\}$ *efficient* if it has maximal expected utility conditional on $\mathbf{t}$, which in our model means that it is equal to the most likely state given $\mathbf{t}$. To avoid distraction by special cases, we assume that for almost every type profile there is exactly one efficient decision. Given a voting rule g , a strategy profile σ is *efficient* if for every type profile $\mathbf{t}$ the resulting decision $d = g(\sigma_1(t_1), \dots, \sigma_n(t_n))$ is efficient.

A strategy σ is a *cut-off strategy* if there exists a $\hat{t} \in (0, 1)$ such that $\sigma(t) = 1$ if $t \geq \hat{t}$ and $\sigma(t) = 0$ if $t < \hat{t}$, for all $t \in [0, 1]$. The number $\hat{t}$ is called the *cut-off* and $\sigma_i^{\hat{t}}$ is voter i 's cut-off strategy with cut-off $\hat{t}$. As mentioned, we only consider the case where all voters use the same cut-off. We write $\sigma^{\hat{t}} = (\sigma_1^{\hat{t}}, \dots, \sigma_n^{\hat{t}})$.

3 When is voting by cut-off strategies efficient?

Our goal is to design a voting rule such that there is an efficient symmetric cut-off strategy profile. Obviously, any efficient strategy profile is a (Bayesian Nash) equilibrium (see also McLennan, 1998), but not conversely. If such a voting rule exists, we say that *efficient information aggregation is feasible*.

In the case where private information is binary, there always exist voting rules which efficiently aggregate private information (Austen-Smith and Banks, 1996; Bozday et al, 2014), so efficient information aggregation is feasible. Our main result implies that this is no longer true under continuous types, and characterizes all parameter combinations for which such rules exist. Moreover, it shows that if such a rule exists, it is a so-called quota rule. A *quota rule* is defined by a threshold $m \in \{0, 1, \dots, n+1\}$: for each voting profile it results in $d = 1$ if and only if at least m voters vote 1. In order to state the theorem, for each type profile $\mathbf{t}$ and a type τ , we denote by $n_\tau^{\mathbf{t}} = |\{i : t_i \geq \tau\}|$ the number of types in $\mathbf{t}$ (weakly) exceeding τ . Also, F_0 and F_1 denote the cdf's associated with f_0 and f_1 , respectively.

Theorem 1 (a) *Efficient information aggregation is feasible if and only if there exists $\tau \in (0, 1)$ which satisfies*

$$\frac{\pi}{1-\pi} > \prod_{i \in \{1, \dots, n\}} \frac{f_0(t_i)}{f_1(t_i)} \iff \frac{\pi}{1-\pi} > \left(\frac{1-F_0(\tau)}{1-F_1(\tau)} \right)^{n_\tau^{\mathbf{t}}} \left(\frac{F_0(\tau)}{F_1(\tau)} \right)^{n-n_\tau^{\mathbf{t}}} \quad (1)$$

for each type profile $\mathbf{t}$.¹

¹Note that due to our assumptions the denominators at the right-hand side are unequal to zero.

(b) If (1) is satisfied for some τ , the voting rule at which efficient information aggregation is feasible is the quota rule with the threshold m^* , where

$$m^* = \min \left\{ n+1, k : k \in \{0, \dots, n\}, \frac{\pi}{1-\pi} > \left(\frac{1-F_0(\tau)}{1-F_1(\tau)} \right)^k \left(\frac{F_0(\tau)}{F_1(\tau)} \right)^{n-k} \right\}, \quad (2)$$

and σ^τ is an efficient strategy profile.

Thus, if (1) holds for some τ , then whenever state 1 is more likely than state 0 given the available information (this is the left-hand side of (1)), it should be more likely based on the information about the number of types exceeding τ (the right-hand side). Then by (2), this τ also determines the quota rule that makes the cut-off strategy profile with cut-off τ efficient.²

4 Examples and discussion

In the first example condition (1) is satisfied, and in the second example it is not.³

Example 1 This example is by Duggan and Martinelli (2001). Let $n = 2$ and $\pi = 0.6$. The density functions are

$$f_0(t) = \begin{cases} \frac{3}{2} & \text{if } 0 < t < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \quad \text{and } f_1(t) = \begin{cases} \frac{1}{2} & \text{if } 0 < t < \frac{1}{2} \\ \frac{3}{2} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

In this case, (1) is satisfied for $\tau = \frac{1}{2}$, and $m^* = 1$ in Theorem 1. In fact, it is not hard to see that this generates the same efficient decisions as the binary information model where a voter receives signal 1 with probability $\frac{3}{4}$ if the true state is 1, signal 0 with probability $\frac{3}{4}$ if the true state is 0, votes in accordance with the received signal, and the same quota rule is used.

Example 2 Let $n = 2$ and $\pi = 0.6028$. The density functions are

$$f_0(t) = \frac{e^{1-t}}{e-1} \text{ and } f_1(t) = 1$$

²Note that using a quota rule in this model is anyway very natural. For instance, under anonymity and monotonicity – if more voters vote 1 then the aggregate vote cannot switch from 1 to 0 – a quota rule has to be used, see Bozbay et al (2014). Also, if a quota rule is used, then it is without loss of generality to assume that any best reply of a voter is a cut-off strategy (see Bozbay and Peters, 2017).

³See Bozbay and Peters (2017) for the derivations of the claims in the examples.

for each $t \in [0, 1]$. The likelihood ratio is $\frac{f_0(t)}{f_1(t)} = \frac{e^{1-t}}{e-1}$ for each $t \in [0, 1]$. One can easily see that under this combination of parameters there exists no $\tau \in (0, 1)$ for which (1) holds. Although, according to Theorem 1, we cannot *always* reach efficiency, we may specify a voting rule and a cut-off which lead to efficient decisions *most often*. In panel a) of the figure below the area in grey (white) shows all type profiles for which $d = 1$ ($d = 0$) is efficient. Suppose the cut-off $\hat{t}$ is chosen as in panel b), and consider the quota rule with $m = 1$. This rule leads to the ‘wrong’ decision – the decision that is not efficient given the type profile – under cut-off voting whenever the type profile falls in the dotted area. If we instead choose $m = 2$, the type profiles which lead to the inefficient decision under cut-off voting fall in the dotted area given in panel c).

It is easy to see that the probability of a wrong (inefficient) decision is much less in a setting such as in panel b). The probability that a given type profile $\mathbf{t} = (t_1, t_2)$ belongs to the dotted area in panel b), assuming that the threshold $\hat{t}$ takes a value as shown in the figure, is minimal for $\hat{t} = 0.3031$. In other words, voting with this cut-off is most efficient. Similarly as in Example 1, the equivalent model with binary signals is the one where a voter receives signal 1 with probability 0.6969 (namely, $1 - \hat{t}$) if the true state is 1 and signal 0 with probability 0.417 (namely, $F_0(\hat{t})$) if the true state is 0. If this voter votes in accordance with the signal, and if the quota rule with $m = 1$ is used, then this results in efficient decisions conditional on the binary signals.

These examples illustrate that it is possible to always have efficient decisions under cut-off strategies if the areas at which the decisions 0 and 1 are efficient, are

(unions of) rectangles. This is the case in Example 1, where $d = 0$ is efficient in the square between $(0, 0)$ and $(.5, .5)$, and $d = 1$ is efficient elsewhere.

Thus, in contrast to previous results (where informative voting is always and only efficient with quota rules), efficient information aggregation in our model is feasible only under a restrictive condition, basically mimicking the binary signal model. Nevertheless, in practice private information of a voter cannot always be summarized as indicating only ‘yes’ or ‘no’. The evidence regarding the job candidate’s skills in teaching may include the student evaluations, which take values in a given interval. The belief of a juror about a defendant may be less conclusive than ‘guilty’ or ‘innocent’. Even if efficient information aggregation is not possible, one may still try and use the second best voting rule, as in Example 2 above.

5 Proof of Theorem 1

The probabilities of the states conditional on the full information $\mathbf{t} \in [0, 1]^n$ are derived as:

$$\Pr(1|\mathbf{t}) = \frac{\pi \prod_{i \in \{1, \dots, n\}} f_1(t_i)}{\prod_{i \in \{1, \dots, n\}} (\pi f_1(t_i) + (1 - \pi) f_0(t_i))} \quad (3)$$

$$\Pr(0|\mathbf{t}) = \frac{(1 - \pi) \prod_{i \in \{1, \dots, n\}} f_0(t_i)}{\prod_{i \in \{1, \dots, n\}} (\pi f_1(t_i) + (1 - \pi) f_0(t_i))}. \quad (4)$$

(a) Suppose efficient information aggregation is feasible. Hence, a voting rule g makes some cut-off strategy-profile $\sigma^{\hat{t}}$ efficient. We show that $\tau = \hat{t}$ satisfies (1) for every $\mathbf{t}$.

First, consider a type profile $\mathbf{t}$ and suppose that a decision d^* is efficient for $\mathbf{t}$. This means

$$\Pr(d^*|\mathbf{t}) > \Pr(d|\mathbf{t}) \quad (5)$$

for $d \neq d^*$. (Note that efficiency ties occur with probability zero and can be neglected.) Let $N_{\hat{t}}^{\mathbf{t}} := \{i : t_i \geq \hat{t}\}$. Voting with cut-off $\hat{t}$ being efficient implies that (5) is equivalent to $\Pr(d^*|N_{\hat{t}}^{\mathbf{t}'}) > \Pr(d|N_{\hat{t}}^{\mathbf{t}'})$ for every $\mathbf{t}'$ with $\sigma^{\hat{t}}(\mathbf{t}') = \sigma^{\hat{t}}(\mathbf{t})$, where $\Pr(d^*|N_{\hat{t}}^{\mathbf{t}'})$ is the probability that d^* is true given $N_{\hat{t}}^{\mathbf{t}'}$. Now suppose that $\frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})}\right)^{n_{\hat{t}}^{\mathbf{t}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})}\right)^{n-n_{\hat{t}}^{\mathbf{t}}}$. As $|N_{\hat{t}}^{\mathbf{t}}| = n_{\hat{t}}^{\mathbf{t}}$, this can be written as

$$\pi(1 - F_1(\hat{t}))^{|N_{\hat{t}}^{\mathbf{t}}|} F_1(\hat{t})^{(n-|N_{\hat{t}}^{\mathbf{t}}|)} > (1 - \pi)(1 - F_0(\hat{t}))^{|N_{\hat{t}}^{\mathbf{t}}|} F_0(\hat{t})^{(n-|N_{\hat{t}}^{\mathbf{t}}|)},$$

which is equivalent to $\Pr(1|N_{\hat{t}}^{\mathbf{t}}) > \Pr(0|N_{\hat{t}}^{\mathbf{t}})$. By the first part of the proof, this is equivalent to $\Pr(1|\mathbf{t}) > \Pr(0|\mathbf{t})$, which, in turn, implies $\frac{\pi}{1-\pi} > \prod_{i \in \{1, \dots, n\}} \frac{f_0(t_i)}{f_1(t_i)}$

by (3) and (4). Thus, we have proved that the right-hand side of (1) for $\tau = \hat{t}$ implies the left-hand side. By converting this argument the converse implication of (1) follows.

Second, for the converse implication assume that there is a number τ which satisfies (1) for each type profile. We show that efficient information aggregation is feasible, i.e., that there is a voting rule g and an efficient cut-off strategy-profile $\sigma^{\hat{t}}$. Take $\hat{t} = \tau$. Consider voting rule g defined as follows for every voting profile $\mathbf{v}$:

$$g(\mathbf{v}) = 1 \iff \frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})} \right)^{n^{\mathbf{v}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})} \right)^{n-n^{\mathbf{v}}}, \quad (6)$$

where $n^{\mathbf{v}}$ is the number of 1-votes in $\mathbf{v}$. We show that g makes $\sigma^{\hat{t}}$ efficient. Consider type profile $\mathbf{t}$ and let $\mathbf{v} = \sigma^{\hat{t}}(\mathbf{t})$. Suppose that $g(\mathbf{v}) = 1$. By (6) and $n^{\mathbf{v}} = n_i^{\mathbf{t}}$, we have $\frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})} \right)^{n_i^{\mathbf{t}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})} \right)^{n-n_i^{\mathbf{t}}}$. By (1), we have $\frac{\pi}{1-\pi} > \prod_{i \in \{1, \dots, n\}} \frac{f_0(t_i)}{f_1(t_i)}$. Hence, by (3) and (4), $d = 1$ is efficient for $\mathbf{t}$. The case where $g(\mathbf{v}) = 0$ can be shown similarly.

(b) Consider a voting rule g and suppose g makes $\sigma^{\hat{t}}$ efficient. By (a), $\hat{t}$ satisfies (1) for every type profile $\mathbf{t}$. We show that g is the quota rule with threshold m^* .

Consider any type profile $\mathbf{t} \in [0, 1]^n$ and let $\mathbf{v} = \sigma^{\hat{t}}(\mathbf{t})$. Suppose $g(\mathbf{v}) = 1$. By cut-off voting being efficient, $d = 1$ is efficient; hence, $\Pr(1|\mathbf{t})$ is maximal. From $\Pr(1|\mathbf{t}) > \Pr(0|\mathbf{t})$, we have $\frac{\pi}{1-\pi} > \prod_{i \in \{1, \dots, n\}} \frac{f_0(t_i)}{f_1(t_i)}$ by (3) and (4). By (a), this is equivalent to $\frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})} \right)^{n_i^{\mathbf{t}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})} \right)^{n-n_i^{\mathbf{t}}}$. If we denote the number of 1-votes in $\mathbf{v}$ by $n^{\mathbf{v}}$, this expression can be written as $\frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})} \right)^{n^{\mathbf{v}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})} \right)^{n-n^{\mathbf{v}}}$ since $n^{\mathbf{v}} = n_i^{\mathbf{t}}$ under cut-off voting. Hence,

$$g(\mathbf{v}) = 1 \iff \frac{\pi}{1-\pi} > \left(\frac{1-F_0(\hat{t})}{1-F_1(\hat{t})} \right)^{n^{\mathbf{v}}} \left(\frac{F_0(\hat{t})}{F_1(\hat{t})} \right)^{n-n^{\mathbf{v}}},$$

which, by (2) and by the right hand side of the inequality being decreasing in $n^{\mathbf{v}}$ implies $g(\mathbf{v}) = 1 \iff n^{\mathbf{v}} \geq m^*$. $\square$

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