Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems

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Abstract

Spectra of nonlinear waves in infinite-dimensional Hamiltonian systems are investigated. We establish a connection via the Krein signature between the number of negative directions of the second variation of the energy and the number of potentially unstable eigenvalues of the linearization about a nonlinear wave. We apply our results to determine the effect of symmetry-breaking on the spectral stability of nonlinear waves in weakly coupled nonlinear Schrödinger equations.

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1 Introduction

We are interested in certain spectral-stability aspects of waves in Hamiltonian nonlinear partial differential equations (PDEs). The abstract framework that we shall work with consists of a Hamiltonian system

$$u_t = JE'(u), \quad u(0) = u_0 \quad (1.1)$$

on a Hilbert space X , where $J : X \rightarrow X$ is invertible and skew-symmetric, and $E : X \rightarrow \mathbb{R}$ is a $\mathcal{C}^2$ -functional. We assume that the energy functional E is invariant under the action of an abelian Lie group G with Lie algebra $\mathfrak{g}$ so that (1.1) is equivariant under G . Solutions of particular interest are relative equilibria ϕ_0 whose time evolution takes place on the group orbit $G\phi_0 = \{g\phi_0; g \in G\}$ so that $\phi(t) \in G\phi_0$ for all t . In fact, we have $\phi(t) = \exp(\omega t)\phi_0$ for an appropriate element $\omega \in \mathfrak{g}$, and the relative equilibrium ϕ_0 is a genuine equilibrium of

$$u_t = J[E'(u) - J^{-1}T_\omega u] \quad (1.2)$$

where T_ω is the generator of $\exp(\omega t)$. Spectral stability of the relative equilibrium ϕ_0 is therefore determined by the spectrum of the linearization $J\mathcal{L}$ of (1.2) about ϕ_0 , where the self-adjoint operator $\mathcal{L} = E''(\phi_0) - J^{-1}T_\omega$ is the Hessian of the energy calculated in the frame $u \mapsto \exp(\omega t)u$ that moves along group orbit.

To associate a concrete example with this abstract framework, consider the system

$$\partial_t q_j = i \left[d_j \partial_{xx} q_j + \frac{\partial g}{\partial |q_j|^2} (|q_1|^2, \dots, |q_N|^2) q_j \right], \quad x \in \mathbb{R}, \quad j = 1, \dots, N \quad (1.3)$$

of coupled nonlinear Schrödinger equations (NLS) which is of the form (1.1) with $X = H^1(\mathbb{R}, \mathbb{C}^N)$, $J = i$ and

$$E(q) = \int_{\mathbb{R}} \left[\sum_{j=1}^N d_j |\partial_x q_j|^2 - g(|q_1|^2, \dots, |q_N|^2) \right] dx.$$

Equation (1.3) is equivariant under the group $G = \mathbb{R} \times \mathbb{T}^N$, where $\mathbb{T}^N$ denotes the n -dimensional torus, which acts by translation in x and by the rotations $q = (q_j) \mapsto e^{i\alpha} q = (e^{i\alpha_j} q_j)$ for $\alpha = (\alpha_j) \in \mathbb{T}^N$. In particular, each solution of the form $q(x, t) = e^{i\omega t} \phi_0(x)$ is a relative equilibrium of (1.3). Note that $T_\omega = i\omega$ so that $\mathcal{L} = E''(\phi_0) - \omega$.

Investigating spectral stability of a relative equilibrium ϕ_0 amounts to calculating the spectrum of $J\mathcal{L}$ where $\mathcal{L}$ is the Hessian of the energy. Since the energy is invariant under G , both $J\mathcal{L}$ and $\mathcal{L}$ will always have a nontrivial null space of dimension at least $\dim G\phi_0$. We denote by X_1 the range of $J\mathcal{L}$. The main result of [7, 8] states that the relative equilibrium ϕ_0 will be nonlinearly orbitally stable if $\mathcal{L}|_{X_1}$ is positive definite which implies that ϕ_0 is a constrained minimizer.

Hence, let us consider the case where the reduced energy $\mathcal{L}|_{X_1}$ has a finite number of negative eigenvalues. Note that ϕ_0 is spectrally stable provided the spectrum of $J\mathcal{L}$ lies on the imaginary axis since eigenvalues of $J\mathcal{L}$ come in quadruples $\{\pm\lambda, \pm\bar{\lambda}\}$. If $J\mathcal{L}$ has an eigenvalue λ with positive real part, then the reduced energy restricted to the real invariant subspace I_λ associated with the quadruplet $\{\pm\lambda, \pm\bar{\lambda}\}$ is necessarily indefinite: indeed, if it were definite, its level sets would be spheres, and solutions in the subspace I_λ to the linearized system

$$v_t = J\mathcal{L}v,$$

which also conserves energy, are bounded, contradicting the assumption that λ has positive real part. A useful term is therefore the Krein signature of an eigenvalue which is positive/negative or zero if the reduced energy restricted to the associated eigenspace is positive/negative definite or indefinite. Note that the above argument implies that, when purely imaginary eigenvalues collide on the imaginary axis, they can only leave the imaginary axis as a quadruplet when they have opposite Krein signature. This fact (and its converse,

namely that colliding purely imaginary eigenvalues of opposite Krein signature will generically leave the imaginary axis) is more rigorously proved in [3, 6, 13] (see §2.2 for details).

In light of the above discussion, we might expect that the number $n(\mathcal{L}|_{X_1})$ of negative eigenvalues of the reduced energy is a bound for the number k_u of unstable eigenvalues of $J\mathcal{L}$. In fact, we may expect that the difference between these two indices is the number k_i^- of purely imaginary eigenvalues of $J\mathcal{L}$ with negative Krein signature. Our first result, Theorem 3.2, states that $k_u + k_i^- \leq n(\mathcal{L}|_{X_1})$, while we prove in Theorem 3.3 that indeed

$$k_u + k_i^- = n(\mathcal{L}|_{X_1}) \quad (1.4)$$

provided J is the canonical symplectic operator and $\mathcal{L}$ is diagonal so that

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.$$

An expression for $n(\mathcal{L}|_{X_1})$ in terms of $n(\mathcal{L})$ is given in Theorem 3.1. We emphasize that the proofs of these results are based on the works [5–8] by Grillakis and coworkers; our contribution consists of an interpretation of their results in terms of the Krein signature. Our motivation for writing this paper was to give a short proof of (1.4) which generalizes several special cases that appeared recently in the literature (see below).

One consequence of the above results is that the number $k_u + k_i^-$ can only change when the negative index of the energy changes. One common way for this to happen is when perturbations are added to the energy that break the symmetry G , a scenario that we refer to as *forced symmetry breaking*. Indeed, as outlined above, forced symmetry breaking reduces the dimension of the null space of $\mathcal{L}$, and of $J\mathcal{L}$, from $\dim G\phi_0$ to $\dim H\phi_0$, where H is the subgroup of symmetries that are not broken by the perturbation. In Theorem 4.4, we calculate the perturbed eigenvalues of both $\mathcal{L}$ and $J\mathcal{L}$ near the origin, and collect various implications of this perturbation analysis for the count of $k_u + k_i^-$ in §5. We then apply these results in §6 to forced symmetry breaking of weakly coupled nonlinear Schrödinger equations. It turns out that breaking the rotation symmetries induced by phase invariances will always lead to eigenvalues that are unstable or have negative Krein signature, while purely imaginary eigenvalues arising from breaking the translation symmetry will always have positive Krein signature.

Finally, we comment on related work. In simultaneous work, Pelinovsky and coworkers recently proved (1.4) for two specific PDEs under more restrictive hypotheses. Firstly, Pelinovsky established the count (1.4) in [17, Theorems 6–8] for the coupled NLS system (1.3) provided [17, Assumption 2.15] is met: this hypothesis requires that the linearization about the nonlinear wave has no eigenvalues that are embedded in the essential spectrum, that the algebraic multiplicity coincides with the geometric multiplicity for all non-zero eigenvalues and that the Evans function does not vanish at branch points. Theorems 3.1 and 3.3 and Remark 3.1 in our paper are applicable to (1.3) without these assumptions, while yielding the same conclusions. The authors of [2] proved (1.4) for linear NLS operators in dimension three or higher provided the branch points of the essential spectrum are not resonances. Again, this result, stated in [2, Theorem 2 and Corollary 2.7], follows directly from Theorem 3.3 in our paper without the hypothesis on the branch points. Forced symmetry breaking of a one-dimensional symmetry group in Hamiltonian systems has been studied previously by Skryabin [19]. Theorem 4.4 is a generalization to arbitrary symmetry groups. Kapitula [9] proved part of Theorem 4.4 using the Evans function but without computing the Krein signature.

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2 Formulation

2.1 General setup and hypotheses

Let U, V, X denote three real Hilbert spaces so that $U \subset X \subset V$ are densely and continuously embedded. Throughout this paper, we will only use the scalar product $\langle \cdot, \cdot \rangle$ on the space V . In particular, we have $U \subset X \subset V \subset X^*$ where X^* denotes the dual space of X . Adjoint operators will always be taken with respect to the scalar product on V .

Let G be a finite-dimensional abelian Lie group with Lie algebra $\mathfrak{g}$. We denote by $\exp(\omega) = e^\omega$ for $\omega \in \mathfrak{g}$ the exponential map from $\mathfrak{g}$ into G . We assume that $T : G \rightarrow L(V)$ is a unitary representation of G on V so that $T'(e)$ maps $\mathfrak{g}$ into the space of closed skew-symmetric operators on V with domain X . We use the notation $T'(e)\omega =: T_\omega$ for the generator of the semigroup $T(e^{\omega t})$. We assume that U is contained in the domain of T_ω^2 . Note that T_ω is linear in $\omega \in \mathfrak{g}$. The group orbit Gu of an element $u \in X$ is defined by $Gu := \{T(g)u; g \in G\}$. Lastly, we set $n = \dim G$.

We assume that $J : V \rightarrow V$ is a bounded, invertible and skew-symmetric linear operator so that $J^* = -J$. Furthermore, we assume that $E : X \rightarrow \mathbb{R}$ is a $\mathcal{C}^2$ functional such that the gradient $E'(u)$ and the Hessian $E''(u)$, defined using the scalar product on V , are bounded as operators from U into $\mathbb{R}$ and V , respectively, for each $u \in U$. We will frequently identify the Hessian $E''(u)$ on U and the associated bilinear form on X .

Next, we introduce two compatibility assumptions. First, we assume that E is invariant under G so that

$$E(T(g)u) = E(u)$$

for all $u \in X$ and all $g \in G$. We also assume that

$$T(g)J = JT(g^{-1})^*$$

for all $g \in G$. Lastly, for each $\omega \in \mathfrak{g}$, we define the bounded functional $Q_\omega : X \rightarrow \mathbb{R}$ by $Q_\omega(u) := \frac{1}{2} \langle J^{-1}T_\omega u, u \rangle$. We see that the second derivative $Q''_\omega(u) = J^{-1}T_\omega$ is a bounded symmetric linear operator from X to V . Note also that Q_ω is invariant¹ under G for each $\omega \in \mathfrak{g}$.

We are interested in the abstract Hamiltonian system

$$\frac{du}{dt} = JE'(u). \quad (2.1)$$

Specifically, we are interested in relative equilibria of this system. These are solutions $u(t)$ whose time orbit is contained in the group orbit Gu_0 so that $u(t) \in Gu(0)$ for all t . Thus, $\phi \in X$ is a relative equilibrium if, and only if, there is a $\omega \in \mathfrak{g}$ so that $u(t) = T(e^{\omega t})\phi$ satisfies (2.1). In this context, relative equilibria are also called bound states or solitary waves. Substituting the ansatz $u(t) = T(e^{\omega t})\phi$ into (2.1), we see that $\phi \in X$ is a relative equilibrium if, and only if,

$$T_\omega \phi = JE'(\phi), \quad (2.2)$$

for some $\omega \in \mathfrak{g}$, that is, if and only if ϕ is a critical point of the functional $H_\omega := E - Q_\omega$ for some $\omega \in \mathfrak{g}$. We assume that there exists a smooth family of bound states:

Hypothesis 2.1 (Relative equilibria) *There is a non-empty open set $\Omega \subset \mathfrak{g}$ and a $\mathcal{C}^1$ function $\phi : \Omega \rightarrow U$, $\omega \mapsto \phi_\omega$ such that ϕ_ω is a relative equilibrium, i.e., a critical point of $H_\omega = E - Q_\omega$ for each $\omega \in \Omega$. We assume that the isotropy subgroups $\{g \in G; T(g)\phi_\omega = \phi_\omega\}$ are discrete for all ω so that the group orbits $G\phi_\omega$ have dimension n .*

¹This has been established in [8, Remark on p. 312] whose assumption is satisfied due to [18, Lemma 4.1]

Since G is abelian, the entire group orbit $T(g)\phi_\omega$ with $g \in G$ consists of relative equilibria with time evolution $T(e^{\omega t})$.

The Hessian of the energy at the relative equilibrium ϕ_ω is defined by

$$\mathcal{L}_\omega := H''_\omega(\phi_\omega) = E''(\phi_\omega) - J^{-1}T_\omega \quad (2.3)$$

which maps its domain U into V . Note that the linearization of (2.1) about the relative equilibrium ϕ_ω in the co-moving frame is given by $J\mathcal{L}_\omega$. As outlined in the introduction, we shall focus on the relation of the spectra of $\mathcal{L}_\omega$ and $J\mathcal{L}_\omega$. Due to the invariance of H_ω under the abelian group G , it follows from [8, p. 314] that the space

$$Z = \text{span}\{T_\sigma\phi_\omega; \sigma \in \mathfrak{g}\}, \quad (2.4)$$

which is the tangent space of the group orbit $G\phi_\omega$ at ϕ_ω , is contained in the null space of $\mathcal{L}_\omega$. Hypothesis 2.1 implies that $\dim Z = \dim G = n$.

Hypothesis 2.2 (Spectral decomposition) *We assume that the space U is decomposed as a direct sum, $U = U_{\text{neg}} \oplus Z \oplus U_{\text{pos}}$, where U_{neg} is a finite-dimensional subspace such that $\langle u, \mathcal{L}_\omega u \rangle < 0$ for all $u \in U_{\text{neg}}$ with $u \neq 0$, and U_{pos} is a closed subspace so that there is a $\delta > 0$ such that $\langle u, \mathcal{L}_\omega u \rangle \geq \delta|u|_X^2$ for all $u \in U_{\text{pos}}$.*

For any bounded, self-adjoint linear operator A from U to V , we denote by $z(A)$ the dimension of its null space and by $n(A)$ and $p(A)$ the dimensions of the maximal subspaces of U on which A is negative and positive definite, respectively. In particular, if $n(A)$ is finite, then it coincides with the number of negative eigenvalues of A . Hence, as outlined in [8, p. 314], the null space of $\mathcal{L}_\omega$ is equal to Z , and the number of negative eigenvalues of $\mathcal{L}_\omega$ is equal to $\dim U_{\text{neg}}$.

We shall often consider the following, more special situation that arises frequently in applications.

Hypothesis 2.3 (Canonical symplectic structure) *We assume that each of the Hilbert spaces U, X, V is the direct product of two closed isomorphic subspaces. Writing operators as 2×2 blocks that respect this decomposition, we assume that*

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{L}_\omega = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

and that $Z = Z_+ \times Z_-$ is a direct product. Furthermore, we assume that there is a self-adjoint, strictly positive operator $\tilde{L}$ so that both L_+ and L_- are relatively compact perturbations of $\tilde{L}$.

Throughout this paper, we shall always assume that Hypotheses 2.1 and 2.2 are met. Whenever we make use of Hypothesis 2.3, we will indicate so.

2.2 The Krein signature

As already mentioned, we are interested in the relation of the spectra of $\mathcal{L}_\omega$ and $J\mathcal{L}_\omega$. Specifically, we wish to establish a connection between the number $n(\mathcal{L}_\omega)$ of negative eigenvalues of the energy $\mathcal{L}_\omega$ and the number of eigenvalues of the linearization $J\mathcal{L}_\omega$ that have negative Krein signature. Roughly speaking, the Krein signature of an eigenvalue λ of $J\mathcal{L}_\omega$ measures whether the energy $\mathcal{L}_\omega$ is positive definite, negative definite or indefinite when restricted to the generalized real eigenspace associated with the eigenvalues $\pm\lambda$ and $\pm\bar{\lambda}$.

To be precise, assume that $\lambda \in \mathbb{C}$ is an eigenvalue of $J\mathcal{L}_\omega$. Since J is skew-symmetric and $\mathcal{L}_\omega$ is symmetric, it follows that $\pm\lambda$ and $\pm\bar{\lambda}$ are also eigenvalues of $J\mathcal{L}_\omega$. Let I_λ denote the real invariant subspace of U that is the sum of the generalized eigenspaces associated with the eigenvalues $\pm\lambda$ and $\pm\bar{\lambda}$. Following [13, p. 144–145],

we say that λ has positive or negative Krein signature if $\mathcal{L}_\omega$ restricted to I_λ is positive or negative definite and that it has zero Krein signature if $\mathcal{L}_\omega$ restricted to I_λ is indefinite.

It is a consequence of [13, Lemma 5(ii)] that the Krein signature of λ is zero if λ has non-zero real part. The Krein signature is also zero if the geometric and algebraic multiplicity of λ as an eigenvalue of $J\mathcal{L}_\omega$ are not equal, i.e., if λ has a non-trivial Jordan block.

If $\lambda \in i\mathbb{R}$ is a simple, non-zero, purely imaginary eigenvalue of $J\mathcal{L}_\omega$ with complex eigenfunction $v = v_1 + iv_2$, then we have $I_\lambda = \text{span}\{v_1, v_2\}$. In this case, the Krein signature of λ is non-zero and is, in fact, given by $\text{sign}\langle v, \mathcal{L}_\omega v \rangle = \text{sign}\langle v_1, \mathcal{L}_\omega v_1 \rangle$ (see [13, Proof of Lemma 5(i)]). More generally, the results stated in [13, Lemmas 4 and 5(i)] imply that the energy $\mathcal{L}_\omega$, considered as a bilinear form on I_λ , is non-degenerate and has an even number of positive and negative eigenvalues whenever $\lambda \in i\mathbb{R}$ is a non-zero purely imaginary eigenvalue of $J\mathcal{L}_\omega$. In this case, we denote by $2n_\lambda^-$ the number of negative squares of $\mathcal{L}_\omega$ restricted to I_λ .

Definition 2.4 *Consider the eigenvalues of $J\mathcal{L}_\omega$ on X . Counting multiplicity, we denote by k_r the number of positive real eigenvalues of $J\mathcal{L}_\omega$ and by k_c the number of quadruplets $\{\pm\lambda, \pm\bar{\lambda}\}$ of eigenvalues λ of $J\mathcal{L}_\omega$ that are neither in $\mathbb{R}$ nor in $i\mathbb{R}$. Lastly, we denote by k_i^- the sum of the numbers n_λ^- where λ runs through all purely imaginary eigenvalues with positive imaginary part.*

The relevance of the Krein signature is that non-zero eigenvalues of $J\mathcal{L}_\omega$ on the imaginary axis can leave the imaginary axis only when eigenvalues with opposite Krein signature collide (see [13, Theorem on p. 146] for systems without symmetries and [3, §4] for equivariant equations) or if an eigenvalue with negative sign is originally embedded in the continuous spectrum (see [6, Theorem 2.4]). We refer to [13] for more detailed background information about the material reviewed in this section.

3 Counting eigenvalues with negative Krein signature

Throughout this section, we assume that Hypotheses 2.1 and 2.2 are satisfied.

3.1 The reduced Hamiltonian

Recall that the null space of the operator $\mathcal{L}_\omega$ is given by $Z = \text{span}\{T_\sigma \phi_\omega; \sigma \in \mathfrak{g}\}$ which, by assumption, has dimension $n = \dim G$. We fix a basis $\{\sigma_1, \dots, \sigma_n\}$ of $\mathfrak{g}$, and denote by $D \in \mathbb{R}^{n \times n}$ the symmetric matrix with entries

$$D_{ij} = \langle \partial_{\sigma_i} \phi_\omega, \mathcal{L}_\omega \partial_{\sigma_j} \phi_\omega \rangle. \quad (3.1)$$

Since J is invertible by assumption, we see that $\lambda = 0$ is an eigenvalue of $J\mathcal{L}_\omega$ with geometric multiplicity n . We now have

$$J\mathcal{L}_\omega T_{\sigma_j} \phi_\omega = 0, \quad J\mathcal{L}_\omega \partial_{\sigma_j} \phi_\omega = T_{\sigma_j} \phi_\omega, \quad \forall j \quad (3.2)$$

where the left equation follows from [8, p. 314] upon using that G is abelian, while the right equation follows by differentiating (2.2). Thus, the algebraic multiplicity of $\lambda = 0$ as an eigenvalue of $J\mathcal{L}_\omega$ is at least $2n$. Using that the null space of the adjoint of $J\mathcal{L}_\omega$ is spanned by

$$J^{-1} T_{\sigma_j} \phi_\omega = \mathcal{L}_\omega \partial_{\sigma_j} \phi_\omega, \quad \forall j, \quad (3.3)$$

it is not hard to prove that the algebraic multiplicity of $\lambda = 0$ is equal to $2n$ if, and only if, the matrix D is invertible. We denote by X_1 ,

$$X_1 = \{u \in X; \langle u, \mathcal{L}_\omega \partial_{\sigma_i} \phi_\omega \rangle = 0\},$$

the range of $J\mathcal{L}_\omega$.

Theorem 3.1 ([8, Theorem 3.1]) *Assume that Hypotheses 2.1 and 2.2 are met. The operator $\mathcal{L}_\omega$ restricted to X_1 has*

$$n(\mathcal{L}_\omega|_{X_1}) = n(\mathcal{L}_\omega) - n(D) - z(D)$$

negative eigenvalues, and its null space has dimension

$$z(\mathcal{L}_\omega|_{X_1}) = z(D) + \dim G.$$

Finally, if $n(\mathcal{L}_\omega|_{X_1}) = 0$ and $z(D) = 0$, then ϕ_ω is a local minimizer of H_ω .

Proof. [8, Theorem 3.1] gives our theorem upon using that, in the notation of [8], we have $z(\mathcal{L}_\omega) = \dim G$, $d'' = -D$, and $z_0 := \dim\{\sigma \in \mathfrak{g}; T_\sigma \phi_\omega = 0\} = 0$ is equal to the dimension of the isotropy subgroup of ϕ_ω which we assumed to be zero. ■

3.2 General setup

Theorem 3.1 gives the number of unstable eigenvalues of the reduced energy $\mathcal{L}_\omega|_{X_1}$. In this section, we show how this result can be used to estimate the number of eigenvalues of the linearization $J\mathcal{L}_\omega$ about the wave ϕ_ω that have either negative Krein signature or are unstable.

Theorem 3.2 *If Hypotheses 2.1 and 2.2 are met, then*

$$k_r + 2k_c + 2k_i^- \leq n(\mathcal{L}_\omega) - n(D) - z(D), \quad (3.4)$$

where the left-hand side has been introduced in Definition 2.4. In particular, the left-hand side is finite.

Note that the preceding theorem is sharper than [8, Theorem 5.8]. We remark that [8, Theorem 5.1] asserts that $k_r \geq 1$ whenever $z(D) = 0$ and $n(\mathcal{L}_\omega) - n(D)$ is odd.

Proof. Let $\{\lambda_j; j = 1, \dots, k\}$ denote the eigenvalues of $J\mathcal{L}_\omega$ that are either positive and real, or non-zero and purely imaginary with negative Krein signature, or lie neither in $\mathbb{R}$ nor in $i\mathbb{R}$. For each of these eigenvalues, we obtain the real invariant subspace I_{λ_j} that we defined in §2.2. Consider the space I_λ for an eigenvalue λ with non-zero real part. In this case, [13, Lemma 5(ii)] implies that there is a subspace I_λ^- of dimension $(\dim I_\lambda)/2$ so that $\mathcal{L}_\omega$ restricted to I_λ^- is negative definite. If, on the other hand, λ is a non-zero purely imaginary eigenvalue with negative Krein signature, then $\mathcal{L}_\omega$ restricted to $I_\lambda^- := I_\lambda$ is negative definite.

We now denote by I^- the direct sum of the subspaces $I_{\lambda_j}^-$ that we obtained in the preceding discussion. It is a consequence of [13, Lemma 3] that $\mathcal{L}_\omega$ restricted to I^- is also negative definite. Furthermore, since $\lambda_j \neq 0$ for all j , we have $I_{\lambda_j} \subset X_1$ for all j and therefore also $I^- \subset X_1$. This implies that $\dim I^- \leq n(\mathcal{L}_\omega|_{X_1})$. Counting the dimension of I^- and using Theorem 3.1 to get an expression for $n(\mathcal{L}_\omega|_{X_1})$ completes the proof. ■

3.3 Canonical symplectic structure

Theorem 3.2 is not entirely satisfactory as it gives only an upper bound for $k_r + 2k_c + 2k_i^-$ instead of an exact count. In this section, we establish such an exact count provided Hypothesis 2.3 is satisfied. Thus, throughout this section, we shall assume that Hypotheses 2.1, 2.2 and 2.3 are met.

We then have $X = \check{X} \oplus \check{X}$,

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{L}_\omega = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad J\mathcal{L}_\omega = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \quad (3.5)$$

and $Z = Z_+ \times Z_- \subset \check{X} \times \check{X}$ with $\dim Z_\pm = z_\pm$ where $z_+ + z_- = n$. After adjusting the basis $\{\sigma_i\}$ of $\mathfrak{g}$ and using (3.2) and (3.5), we see that there are elements $\phi_i^+, \phi_j^-, \psi_i^+, \psi_j^- \in \check{X}$, $\check{\phi}_i^+ \in Z_+$ and $\check{\phi}_j^- \in Z_-$, defined for $i = 1, \dots, z_+$ and $j = 1, \dots, z_-$, such that

$$T_{\sigma_i} \phi_\omega = \begin{pmatrix} \phi_i^+ \\ 0 \end{pmatrix}, \quad T_{\sigma_{z_++j}} \phi_\omega = \begin{pmatrix} 0 \\ \phi_j^- \end{pmatrix}, \quad \partial_{\sigma_i} \phi_\omega = \begin{pmatrix} \check{\phi}_i^+ \\ \psi_i^+ \end{pmatrix}, \quad \partial_{\sigma_{z_++j}} \phi_\omega = \begin{pmatrix} \psi_j^- \\ \check{\phi}_j^- \end{pmatrix} \quad (3.6)$$

where $i = 1, \dots, z_+$ and $j = 1, \dots, z_-$. It is then straightforward to check that the matrix D , defined in (3.1)

$$D_{ij} = \langle \partial_{\sigma_i} \phi_\omega, \mathcal{L}_\omega \partial_{\sigma_j} \phi_\omega \rangle,$$

is of the form

$$D = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}, \quad D_- \in \mathbb{R}^{z_+ \times z_+}, \quad D_+ \in \mathbb{R}^{z_- \times z_-},$$

where the entries of D_+ and D_- are given by

$$[D_-]_{ij} = \langle \psi_i^+, L_- \psi_j^+ \rangle = \langle \psi_i^+, \phi_j^+ \rangle, \quad [D_+]_{ij} = \langle \psi_i^-, L_+ \psi_j^- \rangle = \langle \psi_i^-, \phi_j^- \rangle. \quad (3.7)$$

Theorem 3.3 *Assume that Hypotheses 2.1, 2.2 and 2.3 are satisfied and that the matrix D is invertible. We then have*

$$k_r + 2k_c + 2k_i^- = n(\mathcal{L}_\omega) - n(D) = n(L_+) + n(L_-) - n(D_+) - n(D_-), \quad (3.8)$$

where the left-hand side has been introduced in Definition 2.4.

Thus, if all non-zero eigenvalues are simple, Theorem 3.3 states that the number of unstable eigenvalues in the open right half-plane plus the number of non-zero, purely imaginary eigenvalues with negative Krein signature is equal to the number of negative directions of the reduced energy. The following remark gives a lower bound on the number of unstable eigenvalues using [5].

Remark 3.1 *Under the hypotheses of Theorem 3.3, we have the lower bound*

$$k_r \geq |n(L_+) - n(L_-) - n(D_+) + n(D_-)| \quad (3.9)$$

for the number of unstable real eigenvalues. Furthermore, we have

$$k_r = |n(L_+) - n(D_+)|, \quad k_c = k_i^- = 0, \quad \text{provided} \quad n(L_-) = n(D_-). \quad (3.10)$$

The bound (3.9) follows immediately from [5, Theorem 2.1] and Lemma 3.2 below, while (3.10) follows from [5, Corollary 1.1] together with Lemma 3.2. [We remark that [5, Corollary 1.1] is true only if, in the notation of [5], R and S are both invertible; we prove this for our operators in Lemma 3.2 below].

3.4 Proof of Theorem 3.3

We prove Theorem 3.3 and Remark 3.1 in the following, slightly more general setting: We do not assume equivariance with respect to any symmetry but require only that the null spaces $Z_\pm$ of $L_\pm$ satisfy $Z_+ \subset \text{Rg}(L_-)$ and $Z_- \subset \text{Rg}(L_+)$; note that equation (3.6) shows that this is true in the equivariant context. We can therefore find finite-dimensional spaces $Y_\pm \in \check{X}$ with $\dim Y_\pm = z_\pm$ so that $Z_+ = L_-(Y_+ \oplus Z_-)$ and $Z_- = L_+(Y_- \oplus Z_+)$. Next, choose bases $\{y_1^\pm, \dots, y_{z_\pm}^\pm\}$ of $Y_\pm$ and define the matrices $D_\pm$ via

$$[D_+]_{ij} = \langle y_i^-, L_+ y_j^- \rangle, \quad [D_-]_{ij} = \langle y_i^+, L_- y_j^+ \rangle.$$

Note that this definition of D coincides with the one given in (3.7) in the situation considered in §3.3.

We set $\check{Z} = Z_+ + Z_-$ and denote by P and $P_\pm$ the orthogonal projections in $\check{X}$ onto $\check{Z}^\perp$ and $\check{Z}_\pm^\perp$, respectively.

Lemma 3.2 Assume that Hypotheses 2.1, 2.2 and 2.3 are satisfied. We then have $Z_+ \perp Z_-$ which implies $\check{Z} = Z_+ \oplus Z_-$ and $P = P_+P_- = P_-P_+$. Furthermore, if the matrix D is invertible, then the operators PL_+P and PL_-P are invertible on $\check{Z}^\perp$ and

$$n(PL_+P) = n(L_+) - n(D_-), \quad n(PL_-P) = n(L_-) - n(D_+). \quad (3.11)$$

Proof. By definition, the vectors $\phi_i^\pm = L_\mp y_i^\pm$ form a basis of $Z_\pm$. The identity

$$\langle \phi_i^+, \phi_j^- \rangle = \langle L_- y_i^+, \phi_j^- \rangle = \langle y_i^+, L_- \phi_j^- \rangle = 0$$

proves that $Z_+ \perp Z_-$. Next, we remark that L_+ and L_- are Fredholm operators with index zero by Hypothesis 2.2. Thus, PL_+P and PL_-P are invertible if, and only if, their null spaces are trivial. We prove that PL_+P has trivial null space on $\check{Z}^\perp$ if, and only if, D_- is invertible. Thus, assume that $PL_+Pu = 0$ for some $u \in \check{Z}^\perp$, which means that $L_+u \in Z_-$. The preimage $L_+^{-1}Z_-$ of Z_- under L_+ is given by $Y_- + Z_+$ where $Y_- = \text{span}\{\psi_j^-; j = 1, \dots, z_-\}$. Thus, the null space of PL_+P is trivial if, and only if, $(Y_- + Z_+) \cap \check{Z}^\perp$ is trivial. On the other hand, if the matrix D_- is invertible, then (3.7) implies that $Y_- \cap Z_-^\perp$ is trivial, which in turn shows that $(Y_- + Z_+) \cap \check{Z}^\perp$ is trivial. This proves the statement concerning the invertibility of PL_+P and PL_-P . Lastly, (3.5) implies that the range X_1 of $J\mathcal{L}_\omega$ is given by $X_1 = \text{Rg}(L_-) \times \text{Rg}(L_+)$. Furthermore, we have

$$\mathcal{L}_\omega|_{X_1} = \begin{pmatrix} L_+|_{\text{Rg}(L_-)} & 0 \\ 0 & L_-|_{\text{Rg}(L_+)} \end{pmatrix} = \begin{pmatrix} L_+|_{Z_-^\perp} & 0 \\ 0 & L_-|_{Z_+^\perp} \end{pmatrix}.$$

Using these expressions together with [15, Lemma 6 and Theorem 2] for each of the operators on the diagonal proves the index formulas (3.11). $\blacksquare$

From now on, we assume that D is invertible. Thus, PL_+P and PL_-P are both invertible as operators from $\check{Z}^\perp$ into itself. We define $R := PL_+P$ and $S := (PL_-P)^{-1}$ both considered on $\check{Z}^\perp$. Of interest is then the generalized eigenvalue problem

$$[R - zS]u = 0 \quad (3.12)$$

for $u \in \check{Z}^\perp$ and $z \in \mathbb{C}$.

Lemma 3.3 Assume that Hypotheses 2.1, 2.2 and 2.3 are satisfied and that the matrix D is invertible. In this situation, non-zero eigenvalues z of (3.12) on $\check{Z}^\perp$ and non-zero eigenvalues λ of $J\mathcal{L}_\omega$ on X are in 1:1 correspondence (including their geometric and algebraic multiplicity) via $z = -\lambda^2$.

Proof. We prove that non-zero eigenvalues of (3.12) on $\check{Z}^\perp$ and $J\mathcal{L}_\omega$ on X are in 1:1 correspondence. Thus, suppose that $(u, v) \in \check{X} \times \check{X}$ satisfies

$$L_-v = \lambda u, \quad -L_+u = \lambda v \quad (3.13)$$

for some $\lambda \neq 0$. We conclude that $u \in Z_-^\perp$ and $v \in Z_+^\perp$ so that (3.13) and

$$P_-L_-P_-v = \lambda P_-u, \quad -P_+L_+P_+u = \lambda P_+v \quad (3.14)$$

with $u \in Z_-^\perp$ and $v \in Z_+^\perp$ are equivalent. Due to these restrictions on u and v , (3.13) is in fact equivalent to

$$P_-L_-Pv = \lambda P_-u, \quad -P_+L_+Pu = \lambda P_+v \quad (3.15)$$

with $u \in Z_-^\perp$ and $v \in Z_+^\perp$. Lastly, upon writing $u = \check{u} + \phi^+$ and $v = \check{v} + \phi^-$ with $\check{u}, \check{v} \in \check{Z}^\perp$ and $\phi^\pm \in Z_\pm$, and projecting the two equations in (3.15) using P_+ and P_- , we see that (3.13) is equivalent to

$$PL_-P\check{v} = \lambda\check{u}, \quad -PL_+P\check{u} = \lambda\check{v} \quad (3.16)$$

and

$$(1 - P_+)P_-L_-P\check{v} = \lambda\phi^+, \quad -(1 - P_-)P_+L_+P\check{u} = \lambda\phi^-. \quad (3.17)$$

Note that we used $Z_+ \perp Z_-$. Using the definition of R and S , we see that (3.16) is equivalent to $[R + \lambda^2 S]\check{u} = 0$ which is exactly (3.12), while (3.17) uniquely determines ϕ^+ and ϕ^- . The claim regarding the algebraic multiplicity can be proved in the same fashion. $\blacksquare$

Suppose now that $(p, q) \in \check{X} \times \check{X}$ is an eigenfunction associated with the non-zero purely imaginary eigenvalue $\lambda = i\mu$ of $J\mathcal{L}_\omega$ so that

$$J\mathcal{L}_\omega \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = i\mu \begin{pmatrix} p \\ q \end{pmatrix}$$

and $[R - \mu^2 S]p = 0$. We may assume that p is real and q is purely imaginary. A straightforward calculation shows that the signature of $\mathcal{L}_\omega$ restricted to the real invariant eigenspace associated with (p, q) is then given by

$$\left\langle \mathcal{L}_\omega \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle = 2\mu^2 \langle Sp, p \rangle = 2\mu^2 \langle R^{-1}q, q \rangle. \quad (3.18)$$

Thus, we shall concentrate on the generalized eigenvalue problem

$$[R - zS]p = 0 \quad (3.19)$$

$$\langle Sp, p \rangle \leq 0. \quad (3.20)$$

In [6, Theorem 2.3] it is stated that (3.19)-(3.20) has precisely $n(S)$ pairs of solutions (z_i, p_i) and $(\bar{z}_i, \bar{p}_i)$. Furthermore, if $z \in \mathbb{R}$ corresponds to a Jordan block, or if $\text{Im } z \neq 0$, then $\langle Sp, \bar{p} \rangle = 0$. Alternatively, we can consider

$$[S^{-1} - zR^{-1}]q = 0 \quad (3.21)$$

$$\langle R^{-1}q, q \rangle \leq 0. \quad (3.22)$$

Applying the same theorem, we see that (3.21)-(3.22) has precisely $n(R) = n(R^{-1})$ pairs of solutions (z_j, q_j) and $(\bar{z}_j, \bar{q}_j)$.

Note that (z, p) satisfies $[R - zS]p = 0$ if, and only if, (z, q) satisfies $[S^{-1} - zR^{-1}]q = 0$ where $q = Sp$. In this situation, we therefore have

$$\langle Sp, p \rangle = \langle q, S^{-1}q \rangle = \langle S^{-1}q, q \rangle = z \langle R^{-1}q, q \rangle.$$

Since the scalar products are real, we have shown that

$$\langle Sp, p \rangle = \langle R^{-1}q, q \rangle = 0 \quad (3.23)$$

or $z \in \mathbb{R}$ (or both). In summary, solutions (z, p) and (z, q) of (3.19) and (3.21) are in 1:1 correspondence via $q = Sp$, and we have (3.23) whenever $z \notin \mathbb{R}$.

We now account for all the possibilities of (z, p) and (z, q) not only solving (3.19) and (3.21) but also (3.20) and (3.22). As in [6], we will denote the set of equations (3.19)-(3.20) by (S) and equations (3.21)-(3.22) by (R). Also, p and q are always related via $q = Sp$.

- If $z > 0$, then (z, p, q) satisfies (S) and (R), and we have $\langle Sp, p \rangle = \langle R^{-1}q, q \rangle \leq 0$. In particular, the eigenvalues have zero or negative signature.
- If $\text{Im } z \neq 0$, or $z \in \mathbb{R}$ corresponds to a Jordan block, then (z, p, q) satisfies (S) and (R), and we have $\langle Sp, p \rangle = \langle R^{-1}q, q \rangle = 0$ (see above).

- If $z < 0$, then we have the following exclusive alternative: Either $\langle Sp, p \rangle = \langle R^{-1}q, q \rangle = 0$, in which case (z, p, q) satisfies (S) and (R), or else $\text{sign}\langle Sp, p \rangle = -\text{sign}\langle R^{-1}q, q \rangle \neq 0$ in which case (z, p, q) solves either (S) or (R) but not both.

We count the solutions in each category. To summarize, if

- $z > 0$ with negative signature (ℓ_1 solutions),
- $\text{Im } z \neq 0$ (ℓ_2 solutions)
- $z \in \mathbb{R}$ corresponds to a Jordan block (ℓ_3 solutions), or
- $z < 0$ with vanishing signature (ℓ_4 solutions),

then (z, p, q) satisfies both (S) and (R). On the other hand, if $z < 0$ has non-zero signature, then (z, p, q) satisfies either (S) or (R) but not both (so that we have ℓ_5 solutions with $\ell_5 = \ell_5^R + \ell_5^S$ with the obvious notation). Recall that we have precisely $n(R)$ solutions to (R) and precisely $n(S)$ solutions to (S). Hence, we have

$$n(R) = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5^R, \quad n(S) = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5^S,$$

which gives the count

$$n(R) + n(S) = 2\ell_1 + 2\ell_2 + 2\ell_3 + 2\ell_4 + \ell_5. \quad (3.24)$$

Lastly, Lemma 3.3 shows that the right-hand side counts, with multiplicity, $k_r + 2k_c + 2k_i^-$. Using Lemma 3.2 for the left-hand side of (3.24) then completes the proof of Theorem 3.3.

4 Forced symmetry breaking: perturbation analysis

The results presented in the previous sections provide counts of unstable eigenvalues and purely imaginary eigenvalues with negative Krein signature of the linearization $J\mathcal{L}_\omega$ about relative equilibria in Hamiltonian nonlinear PDEs that are equivariant under the abelian Lie group G . In addition to these eigenvalues, there are always $2n$ eigenvalues at $\lambda = 0$ that are enforced by the symmetry group G where $n = \dim G$. If the symmetry G is broken, some of the eigenvalues at $\lambda = 0$ may move away from the origin and possibly generate additional unstable eigenvalues or eigenvalues with negative Krein signature. In this section, we investigate this scenario.

Hypothesis 4.1 *Suppose that $H \leq G$ is a subgroup of G with Lie algebra $\mathfrak{h}$ and dimension $\dim H = h$. We assume that the energy is of the form $E = E_0 + \epsilon E_1$ for $0 < \epsilon \ll 1$ where E_0 is invariant under G and E_1 is invariant under H . Furthermore, we assume that Hypotheses 2.1 and 2.2 are met for $\epsilon = 0$.*

We denote by D_G the matrix defined in (3.1) and choose bases $\{\tau_1, \dots, \tau_h\}$ of $\mathfrak{h}$ and $\{\tau_1^\perp, \dots, \tau_{n-h}^\perp\}$ of a complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathfrak{g}$.

4.1 Persistence of relative equilibria

First, we consider the persistence of relative equilibria for $\epsilon > 0$. Pick a relative equilibrium ϕ_ω^0 for $\epsilon = 0$ with $\omega \in \mathfrak{h}$ and define

$$E_{\text{red}} : \mathfrak{h}^\perp \longrightarrow \mathbb{R}, \quad \tau \longmapsto E_1(T(\exp(\tau))\phi_\omega^0) \quad (4.1)$$

which evaluates the perturbation E_1 along the directions in $G\phi_\omega^0$ that are transverse to $H\phi_\omega^0$. We can now formulate the following persistence result that can be proved easily using equivariant Lyapunov–Schmidt reduction.

Proposition 4.2 *Assume that Hypothesis 4.1 is satisfied and D_G is invertible. Let ϕ_ω^0 be a relative equilibrium defined for $\epsilon = 0$ with $\omega \in \mathfrak{h}$. If $\tau = 0$ is a nondegenerate critical point of the associated reduced energy E_{red} , then there exists a smooth family $\phi_\omega(\epsilon)$ of relative equilibria for $0 < \epsilon \ll 1$ with $\phi_\omega(0) = \phi_\omega^0$.*

Note that, by definition, $\tau = 0$ is a nondegenerate critical point of E_{red} if

$$\langle T_{\tau_j^\perp} \phi_\omega^0, E_1'(\phi_\omega^0) \rangle = 0, \quad j = 1, \dots, n - h$$

and the matrix $M_1^\perp$ with entries

$$\langle T_{\tau_i^\perp} \phi_\omega^0, E_1''(\phi_\omega^0) T_{\tau_j^\perp} \phi_\omega^0 \rangle, \quad i, j = 1, \dots, n - h$$

is invertible. Since E_1 is invariant under H , the symmetric bilinear form $E_1''(\phi_\omega^0)$ restricted to the tangent space of the group orbit $G\phi_\omega^0$ is represented by the symmetric matrix

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & M_1^\perp \end{pmatrix} : \mathfrak{h} \times \mathfrak{h}^\perp \longrightarrow \mathfrak{h} \times \mathfrak{h}^\perp. \quad (4.2)$$

We denote by

$$\begin{aligned} \mathcal{L}_\omega(\epsilon) &= E_0''(\phi_\omega(\epsilon)) + \epsilon E_1''(\phi_\omega(\epsilon)) - Q_\omega'' \\ &= [E_0''(\phi_\omega^0) - J^{-1}T_\omega] + \epsilon [E_0'''(\phi_\omega^0)[\partial_\epsilon \phi_\omega^0] + E_1''(\phi_\omega^0)] + O(\epsilon^2) =: \mathcal{L}_0 + \epsilon \mathcal{L}_1 + O(\epsilon^2) \end{aligned} \quad (4.3)$$

the Hessian of the energy $E_0 + \epsilon E_1 - Q_\omega$ evaluated at the relative equilibrium $\phi_\omega(\epsilon)$.

For $\epsilon > 0$ the relevant symmetry group is H . Thus, the relevant matrix D which we used in §3 will now be denoted by $D_H \in \mathbb{R}^{h \times h}$. The entries of the symmetric matrix D_H are given by

$$\langle \partial_{\tau_i} \phi_\omega(\epsilon), \mathcal{L}_\omega(\epsilon) \partial_{\tau_j} \phi_\omega(\epsilon) \rangle = \langle \partial_{\tau_i} \phi_\omega^0, \mathcal{L}_0 \partial_{\tau_j} \phi_\omega^0 \rangle + O(\epsilon), \quad i, j = 1, \dots, h. \quad (4.4)$$

4.2 Location of small eigenvalues

Since we assumed that D_G is invertible, $J\mathcal{L}_\omega(\epsilon)$ will have $2n$ small eigenvalues near $\lambda = 0$ for $0 < \epsilon \ll 1$. We now locate these eigenvalues and determine their Krein signature.

Theorem 4.4 *Assume that the hypotheses in Proposition 4.2 are satisfied.*

(i) *The $2n$ eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ near $\lambda = 0$ and the associated eigenvectors are given by*

$$\lambda = \sqrt{\epsilon} \lambda_1 + O(\epsilon), \quad u = \sum_{i=1}^n [T_{\sigma_i} \phi_\omega^0 + \sqrt{\epsilon} \lambda_1 \partial_{\sigma_i} \phi_\omega^0] v_i + O(\epsilon) \quad (4.5)$$

where λ_1 and $v \in \mathbb{C}^n$ satisfy the reduced generalized eigenvalue problem

$$[M_1 + \lambda_1^2 D_G] v = 0. \quad (4.6)$$

Due to (4.2), i.e. the equivariance under H , there will be at least $2h$ eigenvalues $\lambda = 0$.

(ii) *The Krein signature of a simple, non-zero, purely imaginary eigenvalue $\lambda = \sqrt{\epsilon} \lambda_1 + O(\epsilon)$ is given by*

$$\text{sign} \langle v, M_1 v \rangle = \text{sign} \langle v, D_G v \rangle$$

where v satisfies (4.6).

(iii) If D_H is invertible, then

$$k_r + 2k_c + 2k_1^- = n(M_1) + n(D_G) - n(D_H) \geq n(M_1), \quad k_r \geq |n(D_G) - n(D_H) - n(M_1)|, \quad (4.7)$$

where the left-hand sides (introduced in Definition 2.4) refer to the $2n$ eigenvalues near $\lambda = 0$ only.

Note that the reduced eigenvalue problem depends only on the unperturbed wave profile ϕ_ω^0 . This is in sharp contrast to the situation when dissipative perturbations are added to a Hamiltonian PDE. In this case, knowledge of the perturbed profile $\phi_\omega(\epsilon)$ is typically needed to compute the perturbed eigenvalues (see, for instance, [10, §4–5] for an example).

Proof. To find the $2n$ eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ near the origin, we consider the generalized eigenvalue problem

$$[\mathcal{L}_\omega(\epsilon) - \lambda J^{-1}]u = 0, \quad (4.8)$$

which we solve for λ close to zero using a standard Lyapunov–Schmidt reduction (see [4, § VII.1]). The null space of the self-adjoint operator $\mathcal{L}_\omega(0) = \mathcal{L}_0$ is given by the space Z defined in (2.4). We denote by P the orthogonal projection of X onto Z so that $1 - P$ projects onto the range of $\mathcal{L}_0$. We also write $\lambda = \sqrt{\epsilon}\lambda_1$ and $u = u_0 + u_1 \in Z \oplus Z^\perp$. The idea is now to project (4.8) into its Z and $Z^\perp$ components. The equation in $Z^\perp$ can be solved uniquely for u_1 . Upon substituting the expression for u_1 into the Z -component of (4.8) and using the identity (3.3), it is straightforward, but tedious, to prove that (λ, u) satisfies (4.8) if, and only if, (λ_1, u_0) with $u_0 \in Z$ satisfies the reduced eigenvalue problem

$$\langle w_0, [\mathcal{L}_1 - \lambda_1^2 J^{-1} \mathcal{L}_0^{-1} J^{-1} + O(\sqrt{\epsilon})]u_0 \rangle = 0, \quad \forall w_0 \in Z \quad (4.9)$$

where $\mathcal{L}_0$ and $\mathcal{L}_1$ have been defined in (4.3), $\mathcal{L}_0^{-1} : Z^\perp \rightarrow Z^\perp$ and where u and u_0 are related via $u = [1 + \lambda \mathcal{L}_0^{-1} J^{-1} + O(\epsilon)]u_0$. Using (3.3) and the fact that J is skew-symmetric, it is then easy to show that the matrix representation of (4.9) with respect to the basis $T_{\sigma_j} \phi_\omega^0$ of Z coincides with (4.6) provided that

$$\langle T_\sigma \phi_\omega^0, \mathcal{L}_1 T_\tau \phi_\omega^0 \rangle = \langle T_\sigma \phi_\omega^0, E_1''(\phi_\omega^0) T_\tau \phi_\omega^0 \rangle \quad (4.10)$$

for all $\sigma, \tau \in \mathfrak{g}$. To establish (4.10), fix $\sigma, \tau \in \mathfrak{g}$ and note that

$$\langle T_\sigma \phi_\omega(\epsilon), H_\omega''(\phi_\omega(\epsilon)) T_\tau \phi_\omega(\epsilon) \rangle \equiv 0 \quad (4.11)$$

in ϵ since $H_\omega = E_0 - Q_\omega$ is invariant under G (the above expression is the second variation of H_ω restricted to the group orbit $G\phi_\omega(\epsilon)$). Taking the derivative of (4.11) at $\epsilon = 0$ gives

$$\langle T_\sigma \partial_\epsilon \phi_\omega^0, \mathcal{L}_0 T_\tau \phi_\omega^0 \rangle + \langle T_\sigma \phi_\omega^0, \mathcal{L}_0 T_\tau \partial_\epsilon \phi_\omega^0 \rangle + \langle T_\sigma \phi_\omega^0, E_0'''(\phi_\omega^0) [\partial_\epsilon \phi_\omega^0] T_\tau \phi_\omega^0 \rangle = 0. \quad (4.12)$$

The first two terms vanish since $T_\sigma \phi_\omega^0$ lies in the null space of $\mathcal{L}_0 = H_\omega''(\phi_\omega^0)$. This completes the proof of the first part of the theorem.

Using the results recalled in §2.2 and the expression (4.5), we see that the Krein signature of the simple, non-zero, purely imaginary eigenvalue $\lambda = i\sqrt{\epsilon}\mu + O(\epsilon)$ is given by

$$\begin{aligned} \langle u, \mathcal{L}_\omega(\epsilon)u \rangle &= \epsilon \left\langle \sum_{j=1}^n v_j T_{\sigma_j} \phi_\omega^0, \mathcal{L}_1 \sum_{j=1}^n v_j T_{\sigma_j} \phi_\omega^0 \right\rangle + \epsilon \mu^2 \left\langle \sum_{j=1}^n v_j \partial_{\sigma_j} \phi_\omega^0, \mathcal{L}_0 \sum_{j=1}^n v_j \partial_{\sigma_j} \phi_\omega^0 \right\rangle + O(\epsilon^{3/2}) \\ &= \epsilon \langle v, [M_1 + \mu^2 D_G]v \rangle + O(\epsilon^{3/2}) \end{aligned}$$

which gives the desired formula upon using the identity (4.6).

The last part of the theorem is a consequence of Theorem 3.3, in the version proved in §3.4, applied to

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} M_1 & 0 \\ 0 & D_G^{-1} \end{pmatrix},$$

where H is the underlying symmetry group. The estimate for k_r is a consequence of Remark 3.1 and §3.4. ■

5 Forced symmetry breaking: global count of eigenvalues

We continue our discussion of forced symmetry breaking. Throughout this section, we shall assume that we are in the setup of §4. Specifically, we assume that the assumptions of Proposition 4.2 are satisfied and that D_H is invertible. The goal of this section is to calculate the number of eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ that are unstable or have negative Krein signature for $\epsilon > 0$.

5.1 General setup

Without any additional structure, we have the following bound for the number of eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ that are unstable or have negative Krein signature for $\epsilon > 0$.

Corollary 5.1 *Assume that the assumptions in Proposition 4.2 are met and that D_H defined in (4.4) is invertible, then we have*

$$k_r + 2k_c + 2k_i^- \leq n(\mathcal{L}_0) + n(M_1) - n(D_H)$$

for $0 < \epsilon \ll 1$ where $\mathcal{L}_0$ and M_1 are defined in (4.3) and (4.2), respectively.

Proof. The corollary is a consequence of the counts provided in Theorem 3.2, applied to $J\mathcal{L}_\omega(\epsilon)$ with $\epsilon > 0$, and Theorem 4.4(iii). ■

5.2 Canonical symplectic structure

Next, we discuss the situation where Hypothesis 2.3 is satisfied for $E = E_0 + \epsilon E_1$. In this case, Theorem 3.3 provides an exact count for eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ that are unstable or have negative Krein signature and that are bounded away from the origin uniformly for $0 \leq \epsilon \ll 1$. On the other hand, Theorem 4.4(iii) provides such a count for the $2n$ eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ that are close to the origin $0 < \epsilon \ll 1$. Combining these counts gives the following corollary.

Corollary 5.2 *If the assumptions in Proposition 4.2 and Hypothesis 2.3 are met and if D_H defined in (4.4) is invertible, then we have*

$$k_r + 2k_c + 2k_i^- = n(\mathcal{L}_0) + n(M_1) - n(D_H)$$

for $0 < \epsilon \ll 1$ where $\mathcal{L}_0$ and M_1 are defined in (4.3) and (4.2), respectively.

5.3 Unperturbed constrained minimizers

On account of [8, Theorem 4.1], the unperturbed wave is nonlinearly stable for $\epsilon = 0$ provided $n(\mathcal{L}_0) = n(D_G)$. In this situation, there are therefore no unstable eigenvalues or eigenvalues with negative Krein signature for the unperturbed problem. Thus, the $2n$ eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ found in Theorem 4.4(iii) account for all eigenvalues that are unstable or have negative Krein signature for $0 < \epsilon \ll 1$.

Corollary 5.3 *Assume that the assumptions in Proposition 4.2 are met, that D_H defined in (4.4) is invertible, and that $n(\mathcal{L}_0) = n(D_G)$. We then have*

$$k_r + 2k_c + 2k_i^- = n(M_1) + n(D_G) - n(D_H), \quad k_r \geq |n(D_G) - n(D_H) - n(M_1)|$$

for $0 < \epsilon \ll 1$.

5.4 Edge bifurcation eigenvalues

Lastly, we address the existence of eigenvalues that emerge from the essential spectrum—in other words, the associated eigenfunctions do not have well-defined limits in the underlying function space as $\epsilon \rightarrow 0$. We refer to such bifurcations as *edge bifurcations* since they can, at least for integrable PDEs, occur only at branch points of the linear dispersion relation, with one of them being located at the edge of the essential spectrum [12]. Of interest here is the question whether edge bifurcation can lead to eigenvalues that are unstable or have negative Krein signature.

First suppose that Hypothesis 2.3 is satisfied. In this case, Theorems 3.3 and 4.4 and Corollary 5.2 show that all unstable eigenvalues as well as eigenvalues with negative Krein signature are small perturbations of the eigenvalues at $\epsilon = 0$. Hence, any eigenvalues arising from an edge bifurcation will be purely imaginary and have positive Krein signature. Using Theorems 3.3 and 4.4 and Corollary 5.3 shows that the same conclusion applies to problems where the unperturbed wave is a constrained minimizer. Thus, we obtain the following result:

Corollary 5.4 *Assume that the hypotheses in Proposition 4.2 are met and that D_H , defined in (4.4), is invertible. If Hypothesis 2.3 is met, or if $\mathfrak{n}(\mathcal{L}_0) = \mathfrak{n}(D_G)$, then any eigenvalues arising from an edge bifurcation will be purely imaginary with positive Krein signature.*

We do not know whether Corollary 5.4 is true when Hypothesis 2.3 is not met and the unperturbed wave is not a constrained minimizer. In this situation, the assertion of Corollary 5.4 could potentially fail when, as ϵ crosses through zero, a pair of resonance poles for $\epsilon < 0$ becomes a pair of unstable eigenvalues for $\epsilon > 0$. At $\epsilon = 0$, the resonance pole would be embedded in the essential spectrum, for instance at a branch point, but would not correspond to an eigenvalue, i.e., the associated eigenfunction would not be contained in the underlying function space. Such resonance poles arise, and can be tracked, as roots of the Evans function [11, 16]. Unfortunately, we do not know of any examples where this scenario occurs. It was proved in [16], however, that for the generalized KdV equation, which does not fit in the framework of this article as J is not bijective, that a resonance pole goes through the origin and creates a positive real eigenvalue as a parameter is varied.

6 Weakly coupled nonlinear Schrödinger equations

We now apply the results obtained in the previous section to coupled nonlinear Schrödinger equations. Rather than presenting the most general setup possible, we will focus instead on a single representative example. We emphasize, however, that Theorems 3.3 and 4.4 are applicable to a quite general class of coupled NLS equations.

6.1 Formulation

Consider the system

$$\partial_t q_j = i \left[\frac{1}{2} \partial_{xx} q_j + |q_j|^2 q_j \right], \quad x \in \mathbb{R}, \quad j = 1, \dots, N \quad (6.1)$$

of N uncoupled nonlinear Schrödinger equations. Writing $q_j = u_j + iv_j$, we see that (6.1) can be written as

$$\partial_t \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \partial_{xx} u_j - (u_j^2 + v_j^2) u_j \\ -\frac{1}{2} \partial_{xx} v_j - (u_j^2 + v_j^2) v_j \end{pmatrix}, \quad j = 1, \dots, N \quad (6.2)$$

which is Hamiltonian with energy

$$E_0(u, v) = \frac{1}{2} \sum_{j=1}^N \int_{-\infty}^{\infty} [(\partial_x u_j)^2 + (\partial_x v_j)^2 - (u_j^2 + v_j^2)^2] dx.$$

We work in the Hilbert spaces

$$U = H^2(\mathbb{R}, \mathbb{R}^N) \times H^2(\mathbb{R}, \mathbb{R}^N), \quad X = H^1(\mathbb{R}, \mathbb{R}^N) \times H^1(\mathbb{R}, \mathbb{R}^N), \quad V = L^2(\mathbb{R}, \mathbb{R}^N) \times L^2(\mathbb{R}, \mathbb{R}^N)$$

and shall often identify $H^k(\mathbb{R}, \mathbb{R}^N) \times H^k(\mathbb{R}, \mathbb{R}^N)$ with $H^k(\mathbb{R}, \mathbb{C}^N)$ via $q = u + iv$. Equation (6.1) is invariant under the Lie group $G = \mathbb{R}^N \times \mathbb{T}^N$, where $\mathbb{T} = S^1$, which acts on V according to

$$[T(g)q]_j(x) = e^{i\theta_j} q_j(x + \xi_j), \quad g = (\xi, \theta) \in G = \mathbb{R}^N \times \mathbb{T}^N$$

so that the generators are, in complex notation, given by

$$T_{(e_j, 0)} q = \partial_x q_j e_j, \quad T_{(0, e_j)} q = i q_j e_j, \quad j = 1, \dots, N$$

where $e_j \in \mathbb{R}^N$ is the j th canonical basis vector. Relative equilibria $\phi_{(c, \alpha)} \in U$ of (6.1) with $(c, \alpha) \in \mathbb{R}^N \times \mathbb{R}^N = \mathfrak{g}$ are given by

$$\phi_{(c, \alpha)}(x) = \left(e^{-ic_1 x} \Phi_{\alpha_1 - c_1^2/2}(x), \dots, e^{-ic_N x} \Phi_{\alpha_N - c_N^2/2}(x) \right) \in \mathbb{C}^N \quad (6.3)$$

where $\Phi_\omega(x) = \sqrt{2\omega} \operatorname{sech}(\sqrt{2\omega}x)$ for $\omega > 0$. The time evolution of $\phi_{(c, \alpha)}$ is simply $T(ct, \alpha t)\phi_{(c, \alpha)}$. We shall focus on the wave

$$\phi_\omega(x) := (\Phi_\omega(x), \dots, \Phi_\omega(x), 0, \dots, 0) \quad (6.4)$$

that is obtained by choosing $c = 0$ and $\alpha_j = \omega > 0$ for all j . Using (6.3), it is now easy to see that

$$T_{(e_j, 0)} \phi_\omega = \partial_x \Phi_\omega e_j, \quad T_{(0, e_j)} \phi_\omega = \Phi_\omega e_{j+N}, \quad \partial_{(e_j, 0)} \phi_\omega = -x \Phi_\omega e_{j+N}, \quad \partial_{(0, e_j)} \phi_\omega = \partial_\omega \Phi_\omega e_j.$$

In particular, we obtain

$$D_G = \begin{pmatrix} \langle \partial_x \Phi_\omega, -x \Phi_\omega \rangle_{L^2} \mathbb{1}_N & 0 \\ 0 & -\langle \Phi_\omega, \partial_\omega \Phi_\omega \rangle_{L^2} \mathbb{1}_N \end{pmatrix} = \begin{pmatrix} \sqrt{2\omega} \mathbb{1}_N & 0 \\ 0 & -\sqrt{2\omega}^{-1} \mathbb{1}_N \end{pmatrix} \quad (6.5)$$

and therefore $n(D_G) = N$. Note that the upper-left and lower-right blocks of D_G corresponds to translation symmetry and phase invariance, respectively. The linearization of (6.2) about the wave ϕ_ω is given by $J\mathcal{L}_\omega$ where

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{L}_\omega = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

with $L_+ = \operatorname{diag}(L_r, \dots, L_r)$, $L_- = \operatorname{diag}(L_i, \dots, L_i)$ and

$$L_r = -\frac{1}{2} \partial_{xx} + \omega - 3\Phi_\omega^2(x), \quad L_i = -\frac{1}{2} \partial_{xx} + \omega - \Phi_\omega^2(x).$$

In particular, Hypotheses 2.1, 2.2 and 2.3 are satisfied. Furthermore, by Sturm–Liouville theory, we have $n(L_+) = N$ and $n(L_-) = 0$ so that $n(\mathcal{L}_\omega) = N$. Thus, on account of Theorem 3.1, the unperturbed wave is a constrained minimizer.

6.2 Symmetry-breaking perturbations

We now add a small perturbation $\epsilon E_1(u, v)$ to the energy E_0 and consider the perturbed NLS equation

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} (E_0 + \epsilon E_1)'(u, v), \quad (6.6)$$

where we assume that $E_1 : X \rightarrow \mathbb{R}$ is smooth and invariant under a subgroup H of G . Prototypical examples are

$$E_1'(u, v) = f(|u|^2 + |v|^2) \begin{pmatrix} u \\ v \end{pmatrix}, \quad H = \mathbb{R} \times \mathbb{T}^N = \mathbb{R}e \times \mathbb{T}^N \leq G = \mathbb{R}^N \times \mathbb{T}^N, \quad (6.7)$$

and

$$E_1'(u, v) = \begin{pmatrix} Cu \\ Cv \end{pmatrix}, \quad H = \mathbb{R} \times \mathbb{R} = \mathbb{R}e \times \mathbb{R}e \leq G = \mathbb{R}^N \times \mathbb{T}^N \quad (6.8)$$

where $C \in \mathbb{R}^{N \times N}$ is a symmetric coupling matrix and $e = (1, \dots, 1) \in \mathbb{R}^N$. Thus, in (6.7), all rotational symmetries and simultaneous translations in x in all components are preserved², while in (6.8) only simultaneous translations and rotations are preserved.

We shall assume that the wave ϕ_ω persists as a smooth function $\phi_\omega(\epsilon)$ for $\epsilon > 0$ and refer to Proposition 4.2 for sufficient conditions for this to occur. We are then interested in the fate of the $4N$ small eigenvalues of the linearization $J\mathcal{L}_\omega(\epsilon)$, calculated in the co-rotating frame $(u, v) \mapsto e^{i\omega t}(u, v)$, of (6.6) about the wave $\phi_\omega(\epsilon)$. We emphasize that the results in §4 depend only on the unperturbed profile ϕ_ω given in (6.4).

Note first that the symmetry H will tie h small eigenvalues, each with algebraic multiplicity at least two, to the origin. In fact, the algebraic multiplicity will be exactly two if, and only if, the matrix D_H , defined in (4.4), is invertible. The matrix D_H can be calculated as follows. We observe from (4.4) that D_H is, to leading order in ϵ , the restriction of the bilinear form D_G to $\mathfrak{h} \times \mathfrak{h}$. Thus, we identify $\mathfrak{g} = \mathbb{R}^{2N}$, choose orthonormal bases $\tau_j \in \mathfrak{h} \subset \mathbb{R}^{2N}$, $j = 1, \dots, h$, of $\mathfrak{h}$ and $\tau_j^\perp \in \mathfrak{h}^\perp \subset \mathbb{R}^{2N}$, $j = 1, \dots, 2N - h$, of $\mathfrak{h}^\perp$, and define the matrices $T_{\mathfrak{h}} \in \mathbb{R}^{2N \times h}$ with columns τ_j and $T_{\mathfrak{h}^\perp} \in \mathbb{R}^{2N \times (2N-h)}$ with columns $\tau_j^\perp$. We then have

$$D_H = T^t D_G T \in \mathbb{R}^{h \times h}. \quad (6.9)$$

For instance, the matrices associated with the symmetry groups H in (6.7) and (6.8) are invertible.

To apply Theorem 4.4, we also need the symmetric matrix $M_1 \in \mathbb{R}^{2N \times 2N}$ which is given by

$$M_1 = \begin{pmatrix} (\langle \partial_x \Phi_\omega e_i, E_1''(\phi_\omega) \partial_x \Phi_\omega e_j \rangle_{L^2})_{ij} & (\langle \partial_x \Phi_\omega e_i, E_1''(\phi_\omega) \Phi_\omega e_{j+N} \rangle_{L^2})_{ij} \\ (\langle \Phi_\omega e_{i+N}, E_1''(\phi_\omega) \partial_x \Phi_\omega e_j \rangle_{L^2})_{ij} & (\langle \Phi_\omega e_{i+N}, E_1''(\phi_\omega) \Phi_\omega e_{j+N} \rangle_{L^2})_{ij} \end{pmatrix}, \quad M_1^\perp = T_{\mathfrak{h}^\perp}^t M_1 T_{\mathfrak{h}^\perp} \quad (6.10)$$

where $i, j = 1, \dots, N$ so that each block matrix in the above expression is in $\mathbb{R}^{N \times N}$. An application of Theorem 4.4 gives the following result.

Proposition 6.1 *Assume that the matrices D_H and $M_1^\perp$, defined in (4.4) and (6.10), are invertible.*

(i) *The $4N$ eigenvalues of $J\mathcal{L}_\omega(\epsilon)$ near the origin are given by $\lambda = \sqrt{\epsilon}\lambda_1 + O(\epsilon)$ where λ_1 satisfies*

$$[M_1 + \lambda_1^2 D_G]v = 0 \quad (6.11)$$

for some $v \in \mathbb{C}^{2N}$. There are at least $2h$ eigenvalues λ at zero.

(ii) *The Krein signature of a simple, non-zero, purely imaginary eigenvalue $\lambda = \sqrt{\epsilon}\lambda_1 + O(\epsilon)$ is given by $\text{sign}(2\omega|v_1|^2 - |v_2|^2)$ where $v = (v_1, v_2) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfies (6.11).*

(iii) *We have*

$$k_r + 2k_c + 2k_i^- = N + n(M_1) - n(D_H), \quad k_r \geq |N - n(D_H) - n(M_1)| \quad (6.12)$$

where the left-hand sides have been introduced in Definition 2.4. All of these eigenvalues have modulus of order $O(\sqrt{\epsilon})$ and are therefore captured by (6.11).

(iv) *Eigenvalues arising through edge bifurcations at $\lambda = \pm i\omega$ are purely imaginary with positive Krein signature.*

²We remark that the results in [17] are not applicable to (6.6) since the Evans function vanishes at the branchpoint $\lambda = \pm i\omega$ for $\epsilon = 0$ (see [11])

6.3 Reversible perturbations

Next, we consider the important situation when E_1 is reversible. Thus, we shall assume that the following assumption is met.

Hypothesis 6.2 *Assume that $E_1 : X \rightarrow \mathbb{R}$ is invariant under the $\mathbb{Z}_2$ -symmetry induced on X by the bounded operator $\mathcal{R} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R})$ with $[\mathcal{R}u](x) = u(-x)$ for $x \in \mathbb{R}$.*

Since $\Phi_\omega(x)$ is even in x , we see that $E_1''(\phi_\omega)$ commutes with $\mathcal{R}$ and therefore maps even functions into even functions and odd functions into odd functions. In fact, the same statement is true for the full linearization $J\mathcal{L}_\omega(\epsilon)$ about the perturbed wave. Thus, $J\mathcal{L}_\omega(\epsilon)$ maps both $\text{Fix}(\mathcal{R})$ and $\text{Fix}(-\mathcal{R})$ into themselves for all $0 \leq \epsilon \ll 1$, so that even and odd eigenfunctions (and the associated eigenvalues) do not interact with each other. As a consequence, we see that M_1 is, in fact, of block-diagonal form:

$$M_1 = \begin{pmatrix} M_{\text{trans}} & 0 \\ 0 & M_{\text{rot}} \end{pmatrix}$$

where both M_{trans} and M_{rot} are symmetric. In particular, the reduced eigenvalue problem (6.11) decouples:

$$\left[M_{\text{trans}} + \lambda_1^2 \sqrt{2\omega} \mathbf{1}_N \right] v_1 = 0, \quad \left[M_{\text{rot}} - \lambda_1^2 \sqrt{2\omega}^{-1} \mathbf{1}_N \right] v_2 = 0. \quad (6.13)$$

If the matrices D_H and $M_1^\perp$ are invertible, we can apply Proposition 6.1(ii) and Remark 3.1 separately to the eigenvalue problem posed on odd functions and on even functions. In the odd case, we see that M_{trans} yields $p(M_{\text{trans}})$ pairs of non-zero, purely imaginary eigenvalues with necessarily positive Krein signature and $n(M_{\text{trans}})$ unstable real eigenvalues. In the even case, we obtain $p(M_{\text{rot}})$ unstable real eigenvalues and $n(M_{\text{rot}})$ pairs of non-zero, purely imaginary eigenvalues with necessarily negative Krein signature. Formulating this statement slightly differently and using the remark after Hypothesis 6.2 which shows that the decomposition (6.13) is preserved for the full linearization, we obtain the following proposition.

Proposition 6.3 *If the matrices D_H and $M_1^\perp$ are invertible, and Hypothesis 6.2 is met, then*

$$k_r = p(M_{\text{rot}}) + n(M_{\text{trans}}), \quad k_1^- = n(M_{\text{rot}}), \quad k_c = 0.$$

6.4 Dissipative perturbations

Lastly, we comment briefly on the system

$$u_t = [J - \epsilon \mathbf{1}] E'(u), \quad 0 \leq \epsilon \ll 1 \quad (6.14)$$

where the Hamiltonian structure (but not the equivariance under G) is broken by the particular dissipative perturbation $E'(u)$, i.e., the energy itself. In this case, any relative equilibrium ϕ_ω persists with no change of its profile. Furthermore, the linearization of (6.14) about ϕ_ω is given by $\mathcal{A}(\epsilon) = [J - \epsilon \mathbf{1}] \mathcal{L}_\omega$.

It is not hard to show, see [6, p. 305] and [14], that a non-zero simple pair of purely imaginary eigenvalues of $\mathcal{A}$ moves, for $\epsilon > 0$, into the unstable right half-plane if it has negative Krein signature and into the left half-plane if it has positive Krein signature. We discuss here the effect that adding the dissipative perturbation (6.14) has on the eigenvalues at the origin that occur in Jordan blocks. Note that we have

$$\begin{aligned} \mathcal{A}(\epsilon) &= [J - \epsilon \mathbf{1}] \mathcal{L}_\omega, & \mathcal{A}_0 &= \mathcal{A}(0) = J \mathcal{L}_\omega, & \mathcal{A}'_0 &= -\mathcal{L}_\omega \\ u_0 &= T_\sigma \phi_\omega, & w_0 &= \partial_\sigma \phi_\omega, & w_0^* &= J^{-1} u_0 = \mathcal{L}_\omega w_0 \\ \mathcal{A}(\epsilon) u_0 &= 0, & \mathcal{A}_0 w_0 &= u_0, & \mathcal{A}_0^* w_0^* &= 0 \end{aligned} \quad (6.15)$$

where we used (3.2) and (3.3). Thus, the symmetry group G still enforces the eigenvalue $\lambda = 0$ with eigenfunction $u_0 = T_\sigma \phi_\omega$. On account of [10, (4.5)], the other eigenvalue of the Jordan block is given by

$$\lambda(\epsilon) = \frac{\langle w_0^*, \mathcal{A}'_0 w_0 - u'_0 \rangle}{\langle w_0^*, w_0 \rangle} \epsilon + O(\epsilon^2).$$

Substituting the relevant expressions from (6.15), we obtain

$$\lambda(\epsilon) = -\frac{|\mathcal{L}_\omega \partial_\sigma \phi_\omega|^2}{\langle \partial_\sigma \phi_\omega, \mathcal{L}_\omega \partial_\sigma \phi_\omega \rangle} \epsilon + O(\epsilon^2). \quad (6.16)$$

Note that the matrix D_G is formed exactly by the scalar products in the denominator. Thus, if we now specialize to the nonlinear Schrödinger equation (6.1), we infer from (6.5) that translational eigenvalues will move into the left half-plane, while rotational eigenvalues will move into the unstable right half-plane.

We may therefore expect that dissipative perturbations will stabilize the translational eigenvalues, while the eigenvalues arising from the phase invariance are amenable to a destabilization. This is indeed what has been observed in the perturbation from the NLS to complex Ginzburg–Landau equations. This observation also indicates that adding energy in the direction of eigenfunctions that correspond to broken phase invariances may help to stabilize waves.

7 Discussion

Non-abelian symmetry groups

Our abstract results in §3 are applicable to Hamiltonian PDEs that are equivariant under the action of an abelian Lie group G and to relative equilibria with trivial isotropy. Specifically, our results are therefore applicable to radially-symmetric waves of nonlinear Schrödinger equations posed on $\mathbb{R}^d$ with $d \geq 2$. They do, however, *not* apply to non-radially symmetric rotating waves (spiral waves or vortices) on $\mathbb{R}^d$ since G will then contain the Euclidean symmetry group $SE(d)$ of $\mathbb{R}^d$ which is not abelian. The issue is that we have

$$\mathcal{L}_\omega T_\sigma \phi_\omega = T_{[\sigma, \omega]} \phi_\omega,$$

so that non-abelian Lie groups enforce not only spectrum at the origin but also potentially on the imaginary axis away from zero (see, for instance, [18] for a discussion). We believe that it is possible to extend the results presented here to cover non-abelian Lie groups.

Krein signature and the Evans function

Theorem 3.3 provides an exact count of eigenvalues that are either unstable or have negative Krein signature. It would be desirable to locate these eigenvalues, specifically the purely imaginary eigenvalues with negative Krein signature (since the number of unstable eigenvalues can then be computed using the count provided in Theorem 3.3). One possible tool for locating and tracking eigenvalues is the Evans function [1, 11, 16]. In light of the above discussion, it would be interesting to see how the Krein signature of an eigenvalue is encoded in the Evans function. Given the fact that only eigenvalues of opposite signature can leave the imaginary axis upon collision, while eigenvalues of identical signature are confined to the imaginary axis, the Krein signature must be encoded somehow in the Evans function but, alas, we do not know in which way.

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