On Single-look Multivariate $\mathcal{G}$ Distribution for PolSAR Data

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Abstract

For many applications where High Resolution (HR) Synthetic Aperture Radar (SAR) images are required, like urban structures detection, road map detection, marine structures and ship detection etc., single-look processing of SAR images may be desirable. The $\mathcal{G}$ family of distributions have been known to fit homogeneous to extremely heterogeneous Polarimetric SAR (PolSAR) data very well and can be very useful for processing single-look images. The multi-look polarimetric $\mathcal{G}$ distribution has a limitation that it does not reduce to single-look form for (multivariate) PolSAR data. This paper presents the new single-look polarimetric $\mathcal{G}$ distribution, which reduces to its two well-known special forms, the single-look $\mathcal{K}_p$ and $\mathcal{G}_p^0$ distributions, when the domain of its parameters are restricted. The significance of this distribution becomes evident as it fits X- & S-band sub-meter resolution ($< 1\text{m}^2$) PolSAR data (acquired over the same scene at the same time in X- & S-bands) better than the $\mathcal{G}_p^0$ & $\mathcal{K}_p$ distributions, while it fits the X-band decameter resolution ($\approx 10\text{m}^2$) PolSAR data as good as the $\mathcal{G}_p^0$ distribution. Numerical Maximum Likelihood Estimation (MLE) method for parameter estimation of multivariate $\mathcal{G}$, $\mathcal{G}_p^0$, and $\mathcal{K}_p$ distributions is proposed. Simulated PolSAR data has been generated to validate the convergence and accuracy of maximum likelihood parameter estimates to values corresponding to globally maximum likelihood. A new iterative algorithm for accurate estimation of speckle covariance matrix is also proposed.

Index Terms

radar polarimetry, synthetic aperture radar, data models

I. INTRODUCTION

Statistical models have been widely used for Synthetic Aperture Radar (SAR) data analysis as these data are inherently probabilistic. These models offer a wide variety of applications

including image classification, segmentation, filtering, and physical feature extraction.

It is well known in literature that, under certain assumptions, the complex return from a single-look SAR image follows a zero-mean complex Gaussian distribution, which results in a Rayleigh distributed amplitude, and an exponential distributed intensity [1]. With the recent advent of High Resolution (HR) space-borne SARs, the complex Gaussian assumption of the return is not always true. In particular, regions in the SAR image with high degree of heterogeneity (e.g. urban areas) deviate from, while the homogeneous areas adhere to, the complex Gaussian assumption. As a result new methods and techniques need to be identified for modeling such HR data.

SAR images have been consistently analysed and processed using the product model [2]. This model proposes that, under certain assumptions, the complex return from a single-look SAR image can be modeled as the product of zero-mean complex Gaussian distributed speckle and a texture random variable [2]. In the case of homogeneous areas the texture random variable is considered a constant which leads to a Rayleigh distributed amplitude and an exponentially distributed intensity. In contrast to this, for heterogeneous areas the texture can be modeled by some Probability Density Function (PDF). The choice of PDF depends on the degree of heterogeneity of the image area, and various PDFs have thus been proposed to model texture. These include Gamma, inverse Gaussian, Generalized Inverse Gaussian (GIG), inverse Gamma, Beta, and Fisher distributions among others [3]–[6]. When the texture variable is Gamma, reciprocal of Gamma, Fisher distributed the observed signal follows the $\mathcal{K}$, $\mathcal{G}$, Kummer-U distributions, respectively [3]–[6]. This paper deals with a particular family of distributions of the returned signal called the $\mathcal{G}$ family of distributions, which are obtained by assuming a GIG distributed texture [5], [6]. The univariate-complex, -amplitude and -intensity $\mathcal{G}$ distributions have been derived and analysed in [5]. It is important to note here that the $\mathcal{G}_I^0$ distribution, derived in [5], is the same as the Fisher distribution, except that the former was derived for the return [5] while the latter was proposed for modeling the backscatter [4]. The multivariate (polarimetric) extension of the $\mathcal{G}$ distribution for multi-look PolSAR images has been presented in [6]. A detailed bibliography of texture PDFs and the resulting distributions of the return signal can be obtained in [7].

Many applications of SAR imagery require Very HR (VHR) data. This is usually dictated by the size of the target to be detected. Some of the most prominent application areas where VHR data are essential are urban structures detection [8], road map detection [9], marine structures

and ship detection [10] etc. For such applications single look complex data with the highest possible spatial resolution are essential and multi-looking is not an option. Single-look images, thus, need to be processed and analysed. Unfortunately, the mathematical basis on which the multi-look polarimetric $\mathcal{G}$ distribution has been derived does not hold for the special case of single-look PolSAR data. In this paper, the single-look polarimetric $\mathcal{G}$ distribution is presented to fill this gap. The single-look polarimetric $\mathcal{K}_p$ and $\mathcal{G}_p^0$ distributions, listed in Table I in [11], resulting from Gamma and reciprocal of Gamma distributed texture, are the two special forms of the single-look polarimetric $\mathcal{G}$ distribution, as shown in section IV.

The utility of $\mathcal{G}$ distribution is demonstrated by fitting $\mathcal{G}$, $\mathcal{G}_p^0$, & $\mathcal{K}_p$ to amplitude histograms of sub-meter resolution X- & S-band PolSAR data, and also to decameter resolution X-band PolSAR data for homogeneous, moderately heterogeneous, and extremely heterogeneous areas. The sub-meter resolution X- & S-band data have been acquired over the same scene at the same time. Further, a scheme to numerically estimate parameters of multivariate distributions using Maximum Likelihood Estimation (MLE) is proposed. This scheme is also compared to the estimation method proposed in [6], which averages MLEs of univariate (single channel) distributions to find multivariate MLEs. Further, PolSAR data has been simulated using GIG Gaussian texture and zero-mean complex Gaussian speckle. The convergence of numerical MLE to global maxima and the validation of Maximum Likelihood Parameter Estimates (MLEs) has been done using simulated PolSAR data. Also, a new iterative algorithm for accurate estimation of speckle covariance matrix is proposed.

The paper is organized as follows: Section II presents the speckle statistics of SAR images, elaborates the product model describing the modeling of speckle noise and texture, and also explains the estimation of speckle covariance matrix and the limitation of the multi-look model; Section III introduces the GIG distribution and proves its two special forms, the Gamma and reciprocal of Gamma distributions; Section IV presents the single-look polarimetric $\mathcal{G}$ distribution and its two special forms, $\mathcal{K}_p$ and $\mathcal{G}_p^0$; Section V explains parameter estimation of these distributions, the speckle covariance matrix estimation algorithm and the convergence/validation of MLEs using simulated PolSAR data; Section VI contains the application of the proposed distributions to X- & S-band sub-meter resolution and X-band decameter resolution real PolSAR data; finally, Section VII summarises the conclusions and future work.

II. POLARIMETRIC SAR SPECKLE STATISTICS

SAR images are characterized by a granular pattern called speckle. Speckle appears when electromagnetic waves, emitted by a coherent source, illuminate a surface with many elementary scatterers causing the reflected wavelets from each of these scatterers to reach back at the point of observation with different delays [12]. These de-phased wavelets interfere constructively at some points and destructively at others, depending on the surface and the geometry of observation, resulting in chaotic bright and dark spots, and also intermediate levels of brightness in the final SAR image.

Speckle appears very unordered with no obvious relationship with the macroscopic features of the surface and is best described by statistical methods. In order to understand the information content in SAR images, it is therefore essential to study their speckle characteristics.

A. Product Model

The product model suggests that the observed speckle is composed of a product of two statistically independent random variables; the square root of a positive random variable (scene backscattering intensity texture) X and the speckle noise Y . The former is generally considered to be a positive real number whereas the latter may either be complex if the SAR image is in complex format or positive real if the image is in amplitude format. For polarimetric SAR, the speckle noise random variable is a p -dimensional vector Y_p , where $p = 3$ for a mono-static SAR and $p = 4$ for a bi-static configuration. For the mono-static case the p -dimensional complex observation vector, Z_p , can be represented as:

$$Z_p = \begin{bmatrix} S_{hh} & \sqrt{2}S_{hv} & S_{vv} \end{bmatrix}^T \quad (1)$$

where S_{hh} , S_{hv} , and S_{vv} are the complex polarimetric channels and S_{xy} has x as transmit and y as receive electromagnetic polarization (h -horizontal, v -vertical). In terms of the product model the p -dimensional complex observation can be written as:

$$Z_p = \sqrt{X}Y_p \quad (2)$$

The product model for one-dimensional SAR data can be written as [13]:

$$S = \sqrt{\sigma}m \quad (3)$$

where S is the observed complex data, σ is the observed local Radar Cross Section (RCS), and m is a zero-mean and unit variance complex Gaussian random variable. For a monostatic polarimetric SAR, the p -dimensional complex observation can be written as:

$$\begin{aligned} \mathbf{Z}_p &= \begin{bmatrix} \sqrt{\sigma_{hh}}m_{hh} & \sqrt{2\sigma_{hv}}m_{hv} & \sqrt{\sigma_{vv}}m_{vv} \end{bmatrix}^T \\ &= \begin{bmatrix} \sqrt{\sigma_{hh}} & 0 & 0 \\ 0 & \sqrt{2\sigma_{hv}} & 0 \\ 0 & 0 & \sqrt{\sigma_{vv}} \end{bmatrix} \begin{bmatrix} m_{hh} \\ m_{hv} \\ m_{vv} \end{bmatrix} \end{aligned} \quad (4)$$

which shows that the backscattering texture should be modeled as the square root of a matrix, with RCSs of the channels along its diagonal.

It must be noted that in this paper, contrary to (4), the backscattering texture has been modeled as the square root of a positive random variable. Consequently, this analysis is based on the assumptions that the scene presents the same texture in all channels, the texture is a function of the backscattering power only, and it is spatially uncorrelated to speckle.

B. Modeling Speckle Noise

The speckle noise, $\mathbf{Y}_p$, is a p -tuple of complex random variables whose real and imaginary parts are $2p$ -variate zero mean Gaussian distributed. It has been shown in literature that, under certain assumptions [1], such a p -tuple vector follows the zero mean complex Gaussian distribution [14]:

$$f_{\mathbf{Y}_p}(\mathbf{y}) = \frac{1}{\pi^p |\mathbf{C}|} \exp(-\mathbf{y}^{*t} \mathbf{C}^{-1} \mathbf{y}) \quad (5)$$

where $\mathbf{C}$ is a $p \times p$ Hermitian covariance matrix given by $\mathbf{C} = \mathbf{E}[\mathbf{Y}_p \mathbf{Y}_p^{*t}]$, with ' $\mathbf{y}^{*t}$ ' representing the transposed complex conjugate of $\mathbf{y}$, while $|\cdot|$ is a symbol for calculating the determinant. The covariance matrix $\mathbf{C}$ contains useful information about the covariance between different polarimetric channels and represents their second order moments of fluctuations. It must be noted that:

$$\mathbf{E}[\mathbf{Z}_p \mathbf{Z}_p^{*t}] = \mathbf{E}[X] \mathbf{E}[\mathbf{Y}_p \mathbf{Y}_p^{*t}] \quad (6)$$

$$\neq \mathbf{E}[\mathbf{Y}_p \mathbf{Y}_p^{*t}] \quad (7)$$

as $\mathbf{E}[X]$ is not always unity. However, the $\mathbf{C}$ matrix can be indirectly estimated using the observation vector, $\mathbf{Z}_p$. If the texture is considered deterministic, the Approximate Maximum

Likelihood (AML) estimator of the normalized covariance matrix is the Fixed Point (FP) solution of the following recursive equation [11], [15]–[17]:

$$\hat{\mathbf{C}}_{\text{AML}} = f_{\text{AML}}(\hat{\mathbf{C}}_{\text{AML}}) = \frac{p}{N} \sum_{i=1}^N \frac{\mathbf{z}_i \mathbf{z}_i^{*t}}{\mathbf{z}_i^{*t} \hat{\mathbf{C}}_{\text{AML}}^{-1} \mathbf{z}_i} \quad (8)$$

$$\Rightarrow \hat{\mathbf{C}}_{\text{AML}}(i+1) = f_{\text{AML}}(\hat{\mathbf{C}}_{\text{AML}}(i)) \quad (9)$$

such that $\text{Tr}(\hat{\mathbf{C}}_{\text{AML}}) = 1$. It has also been established in [16], that the AML estimator of the normalized covariance matrix is not only unique, but also the algorithm always converges irrespective of the initialization. The convergence can be analyzed using the following criteria [16], [17]:

$$c(i+1) = \frac{\left\| \hat{\mathbf{C}}_{\text{AML}}(i+1) - \hat{\mathbf{C}}_{\text{AML}}(i) \right\|_F}{\left\| \hat{\mathbf{C}}_{\text{AML}}(i) \right\|_F} \quad (10)$$

where $\|\cdot\|_F$ represents the Frobenius norm. Equation (9) is iterated till c becomes smaller than a predefined value.

On the other hand, when the texture is a random variable, the FP estimator of the normalized covariance matrix is not an ML estimator, it is an AML estimator [16]. If the PDF of texture is given by $f_X(x)$, then the ML estimator of the normalized covariance matrix, associated with the PDF generating function $h_p(q)$, is given by [11], [15], [17], [18]:

$$\hat{\mathbf{C}}_{\text{ML}} = f_{\text{ML}}(\hat{\mathbf{C}}_{\text{ML}}) = \frac{1}{N} \sum_{i=1}^N \frac{h_{p+1}(\mathbf{z}_i^{*t} \hat{\mathbf{C}}_{\text{ML}}^{-1} \mathbf{z}_i)}{h_p(\mathbf{z}_i^{*t} \hat{\mathbf{C}}_{\text{ML}}^{-1} \mathbf{z}_i)} \mathbf{z}_i \mathbf{z}_i^{*t} \quad (11)$$

$$\Rightarrow \hat{\mathbf{C}}_{\text{ML}}(i+1) = f_{\text{ML}}(\hat{\mathbf{C}}_{\text{ML}}(i)) \quad (12)$$

where $h_p(q)$ has the expression:

$$h_p(q) = \int_0^{+\infty} \frac{1}{x^p} \exp\left(-\frac{q}{x}\right) f_X(x) dx \quad (13)$$

In [18], it has not only been shown that (12) admits a unique solution, but also that the recursive algorithm converges to a fixed point solution for any initialization. If the PDF of the texture, $f_X(x)$, is known, the PDF generating function, $h_p(q)$, can be analytically computed. If $h_p(q)$ has a closed form, the ML estimator can be recursively computed using (12). However, if $h_p(q)$ does not have a closed form, the ML estimator, $\hat{\mathbf{C}}_{\text{ML}}$, cannot be computed and the AML estimator, $\hat{\mathbf{C}}_{\text{AML}}$, should be used. It must be noted that, in [15], [16], [18], an additional step

to normalize the covariance matrix estimation at each iteration has been introduced, which guarantees improvement in estimation accuracy at each iteration. For both the AML and ML estimators, this normalization step can be generally represented as [15], [16], [18]:

$$\hat{\mathbf{C}}(i) = \frac{1}{\text{Tr}(\hat{\mathbf{C}}(i))} \hat{\mathbf{C}}(i) \quad (14)$$

Practical intricacies in computing ML estimator of the normalized covariance matrix will be discussed further in section V with the proposal of a new iterative estimation algorithm.

C. Limitation of Multi-look Model

The speckle noise in SAR images can be reduced at the expense of decreased spatial resolution by a process called multi-looking [1]. This is achieved by averaging n independent estimates of reflectivity obtained by dividing the synthetic aperture length into n segments each of which is called a look. The averaging of n independent looks reduces the standard deviation of speckle by a factor of $\sqrt{n}$ [1].

If $y_1, y_2, \dots, y_n$ is a sample of n complex valued vectors following the complex Gaussian distribution (5), then the sample Hermitian covariance matrix, $\hat{\Sigma}_y$

$$\hat{\Sigma}_y = \frac{1}{n} \sum_{j=1}^n y_j y_j^{*t} \quad (15)$$

is a maximum likelihood estimator and a sufficient statistic for the Hermitian covariance matrix $\mathbf{C}$ [14]. Let

$$\mathbf{A} = n\hat{\Sigma}_y. \quad (16)$$

The joint distribution of the distinct elements of matrix $\mathbf{A}$ is called a complex Wishart distribution [14] and is given by:

$$f_{\mathbf{A}}(\mathbf{A}) = \frac{|\mathbf{A}|^{n-p} \exp(-\text{Tr}(\mathbf{C}^{-1}\mathbf{A}))}{K(n, p) |\mathbf{C}|^n} \quad (17)$$

$$\text{where } K(n, p) = \pi^{p(p-1)/2} \prod_{j=1}^p \Gamma(n - j + 1) \quad (18)$$

where p is the dimension of the observation vector, n is the number of looks, $\text{Tr}(\cdot)$ represents matrix trace, $\Gamma(\cdot)$ is the Gamma function, and $K(n, p)$ is a scaling function which is similar to the definition of a multivariate Gamma function [19].

The distribution, $f_{\hat{\Sigma}_y}(\hat{\Sigma}_y)$, of the sample Hermitian covariance matrix, $\hat{\Sigma}_y$, can be obtained using (16), and (17) [20]:

$$f_{n\hat{\Sigma}_y}(n\hat{\Sigma}_y) = \frac{n^{np} |\hat{\Sigma}_y|^{n-p} \exp\left(-n \operatorname{Tr}\left(\mathbf{C}^{-1} \hat{\Sigma}_y\right)\right)}{n^{p^2} K(n, p) |\mathbf{C}|^n} \quad (19)$$

$$f_{\hat{\Sigma}_y}(\hat{\Sigma}_y) = n^{p^2} f_{n\hat{\Sigma}_y}(n\hat{\Sigma}_y) \quad (20)$$

$$f_{\hat{\Sigma}_y}(\hat{\Sigma}_y) = \frac{n^{np} |\hat{\Sigma}_y|^{n-p} \exp\left(-n \operatorname{Tr}\left(\mathbf{C}^{-1} \hat{\Sigma}_y\right)\right)}{K(n, p) |\mathbf{C}|^n} \quad (21)$$

In circumstances where retaining high spatial resolution becomes important, multi-look processing may not be desirable, and single-look data can be directly used for SAR image analysis. Therefore, multivariate distributions for single-look PolSAR data are of interest. Unfortunately, the multi-look multivariate distribution of covariance matrix, $f_{\hat{\Sigma}_y}(\hat{\Sigma}_y)$ (21), does not reduce to the single-look ($n = 1$) multivariate case because of an inherent limitation present in the scaling function $K(n, p)$ (18) [19] as:

$$\begin{aligned} K(n, p) &\rightarrow \infty, \quad \text{if } n \leq p - 1 \\ \Rightarrow f_{\hat{\Sigma}_y}(\hat{\Sigma}_y) &\rightarrow 0 \end{aligned} \quad (22)$$

For single-look SAR data ($n = 1$), and a mono-static SAR configuration ($p = 3$), $n < p - 1$, which limits the use of (21) for modeling the speckle noise of single-look multivariate PolSAR data. Therefore in this paper, the speckle noise, $\mathbf{Y}_p$, has been modeled by the zero-mean multivariate complex Gaussian distribution (5). It is interesting to mention here that the relaxed Wishart ($\mathcal{RW}$) distribution [21], whose functional form is identical to (21) except that the number of looks, n , is replaced by a variable parameter $\hat{n} \leq n$, is also not usable for modeling single-look multivariate PolSAR data for the same reasons mentioned above, although for modeling multi-look unfiltered PolSAR data $\mathcal{RW}$ has been shown to compete well with Wishart, $\mathcal{G}^0$ and $\mathcal{K}$ distributions [21] and actually performs better than these distributions for speckle filtered PolSAR data [21].

D. Modeling Texture

The scene backscattering texture X is a positive random variable. It represents the fluctuations of radar intensity backscatter, which depend on the heterogeneity of the scene under observation. As a result, different probability distributions can be used to model the texture for various levels of

heterogeneity. For a highly homogeneous scene, the texture has been modeled as a constant, so the observation vector Z_p simply follows the zero mean multivariate complex Gaussian distribution (5) [22].

Some authors have modeled the texture as a Gamma distributed variable for slightly heterogeneous areas, resulting in the well known $\mathcal{K}$ distribution [11], [23]–[26]. Although the $\mathcal{K}$ distribution models slightly heterogeneous and homogeneous areas very well, it fails to model extremely heterogeneous areas e.g. urban areas.

To find a general distribution which models extremely heterogeneous areas as well, in [5] the texture was modeled as a generalized inverse Gaussian (GIG) distribution with Gamma and reciprocal of Gamma as its two special cases. While the Gamma distributed texture resulted in the $\mathcal{K}$ distribution, the reciprocal of Gamma distributed texture resulted in the univariate $\mathcal{G}^0$ distribution [5], which was successfully applied to single channel SAR data. The $\mathcal{G}_0$ distribution has been experimentally shown to be very flexible, capable of modeling from very homogeneous to extremely heterogeneous areas [5]. Another group of researchers have proposed the univariate Fisher distribution for texture modeling [4]. It must be reiterated here that the $\mathcal{G}_I^0$ [5] and Fisher are the same laws, one derived for the return [5] and the other proposed for the backscattering texture [4], respectively. It has also been shown in [27] that Fisher distribution can not only model the backscattering amplitude statistics more accurately than the classical distributions like Nakagami, log-normal, $\mathcal{K}$, Nakagami-Rice, and Weibull, but it can also model homogeneous to extremely heterogeneous backscatter very well.

As the univariate $\mathcal{G}$ distributions have successfully modeled varying degrees of heterogeneity in the data, a logical next step has been the extension to the multivariate (polarimetric) case. In [4], [6], the multivariate, multi-look $\mathcal{G}$ and KummerU distributions have been obtained modeling the texture as GIG and Fisher distributions, respectively, while the speckle noise, in both cases, follows a complex Wishart distribution. Since these multi-look distributions cannot be used to model single-look polarimetric SAR data directly for reasons mentioned in the previous section, it is desirable to formulate closed forms of these distributions for the single-look case. Recently, in [11] the single-look multivariate KummerU distribution has been derived assuming Fisher distributed texture and zero-mean multivariate complex Gaussian distributed speckle noise. Also, the closed form of single-look multivariate $\mathcal{G}^0$ distribution, assuming inverse Gamma distributed texture has been listed in [11]. Further, it has also been shown that the KummerU distribution

asymptotically converges to $\mathcal{K}$ and $\mathcal{G}^0$ distributions.

In this paper the single-look multivariate $\mathcal{G}$ distribution, assuming a GIG distributed texture and zero-mean multivariate complex Gaussian distributed speckle noise, is derived. Also the single-look multivariate closed forms of the two special cases of GIG distribution, Gamma and reciprocal of Gamma distributions, are shown to present the same expressions as the ones given in [11].

III. GENERALIZED INVERSE GAUSSIAN DISTRIBUTED TEXTURE

The GIG distribution is here used to model the intensity texture random variable. The GIG distribution, denoted as $\mathcal{N}^{-1}(\alpha, \gamma, \lambda)$, is defined as [28]:

$$f_X(x) = \frac{(\lambda/\gamma)^{\alpha/2}}{2K_\alpha(2\sqrt{\lambda\gamma})} x^{\alpha-1} \exp\left(-\frac{\gamma}{x} - \lambda x\right), x > 0 \quad (23)$$

where K_ν is the modified Bessel function of the second kind and order ν . The domain of parameters of GIG distribution are given by:

$$\begin{cases} \gamma > 0, \lambda \geq 0 & \text{if } \alpha < 0 \\ \gamma > 0, \lambda > 0 & \text{if } \alpha = 0 \\ \gamma \geq 0, \lambda > 0 & \text{if } \alpha > 0 \end{cases} \quad (24)$$

It must be noted that the definition of GIG distribution in (23) is not the same as the one given in [6]. In an earlier paper of the same author [5], the square root of the GIG distribution modeling amplitude backscatter is given, from which the GIG distribution, modeling intensity, can be derived by using the transformation $f_X(x) = \frac{f_{XA}(\sqrt{x})}{2\sqrt{x}}$. This transformation results in (23) instead of the equation given in [6].

Two special cases of the GIG distribution are the Gamma and reciprocal of Gamma distributions. In order to derive these special cases let us examine (23) and the following relations of modified Bessel functions [6]:

$$K_\nu(\mu) = 2^{\nu-1} \Gamma(\nu) \mu^{-\nu}, \quad \mu \simeq 0, \nu > 0, \quad (25)$$

$$K_\nu(\mu) = K_{-\nu}(\mu). \quad (26)$$

The Gamma distribution $\Gamma(\alpha, \lambda)$ can be derived by assuming $\alpha > 0$ and $\gamma \rightarrow 0$ and using (25)

in (23):

$$\begin{aligned} f_X(x) &= \frac{(\lambda/\gamma)^{\alpha/2} x^{\alpha-1}}{2 \times 2^{\alpha-1} \Gamma(\alpha) (2\sqrt{\lambda\gamma})^{-\alpha}} \exp(-\lambda x) \\ &= \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\lambda x), \quad x > 0 \end{aligned} \quad (27)$$

The reciprocal of Gamma distribution $\Gamma^{-1}(\alpha, \gamma)$ can be derived by assuming $\alpha < 0$ and $\lambda \rightarrow 0$ and using (25), (26) in (23):

$$\begin{aligned} f_X(x) &= \frac{(\lambda/\gamma)^{\alpha/2} x^{\alpha-1}}{2 \times 2^{-\alpha-1} \Gamma(-\alpha) (2\sqrt{\lambda\gamma})^\alpha} \exp\left(-\frac{\gamma}{x}\right) \\ &= \frac{x^{\alpha-1}}{\gamma^\alpha \Gamma(-\alpha)} \exp\left(-\frac{\gamma}{x}\right), \quad x > 0 \end{aligned} \quad (28)$$

IV. SINGLE-LOOK POLARIMETRIC $\mathcal{G}$ DISTRIBUTION

In this section, the single-look polarimetric $\mathcal{G}$ distribution has been derived using the product model (2), assuming GIG distributed texture (23) and zero-mean multivariate complex Gaussian speckle noise (5). Also, the special cases of Gamma and reciprocal of Gamma distributed texture have been considered.

The marginal distribution $f_{Z_p}(\mathbf{z})$ can be calculated by the formula [29]:

$$f_{Z_p}(\mathbf{z}) = \int_0^\infty f_{Z_p}(\mathbf{z}|X) f_X(x) dx \quad (29)$$

where $f_X(x)$ is given in (23) and $f_{Z_p}(\mathbf{z}|X)$, the PDF of observation vector $\mathbf{Z}_p$ given texture X , can be calculated using the following formula [29]:

$$f_{Z_p}(\mathbf{z}|X) = \left. \frac{f_{Y_p}(\mathbf{y}|X)}{\left| \frac{\partial g(X, Y_p)}{\partial Y_p} \right|} \right|_{\mathbf{y}=\frac{\mathbf{z}}{\sqrt{X}}} \quad (30)$$

where $g(X, Y_p) = \sqrt{X} Y_p$. Since X and Y_p are statistically independent, $f_{Y_p}(\mathbf{y}|X) = f_{Y_p}(\mathbf{y})$, and the Jacobian of transformation $\left| \frac{\partial g(X, Y_p)}{\partial Y_p} \right| = x^p$ [30]. Considering this and the expression in (5) $f_{Z_p}(\mathbf{z}|X)$ becomes:

$$f_{Z_p}(\mathbf{z}|X) = \frac{1}{\pi^p |\mathbf{C}| x^p} \exp\left(\frac{-\mathbf{z}^{*t} \mathbf{C}^{-1} \mathbf{z}}{x}\right) \quad (31)$$

Replacing (23) and (31) in (29):

$$\begin{aligned} f_{Z_p}(\mathbf{z}) &= \frac{(\lambda/\gamma)^{\alpha/2}}{\pi^p |\mathbf{C}| 2 K_\alpha(2\sqrt{\lambda\gamma})} \int_0^\infty x^{\alpha-p-1} \\ &\quad \times \exp\left(-\left(\frac{\gamma}{x} + \lambda x\right) - \frac{q}{x}\right) dx \end{aligned} \quad (32)$$

where $q = \mathbf{z}^{*t} \mathbf{C}^{-1} \mathbf{z}$. Using the following integral definition of modified Bessel functions [6],

$$K_\nu(2\sqrt{ab}) = \frac{(a/b)^{\nu/2}}{2} \int_0^\infty x^{\nu-1} \exp(-b/x - ax) dx \quad (33)$$

(32) becomes:

$$f_{\mathbf{Z}_p}(\mathbf{z}) = \frac{\lambda^{p/2}(\gamma + q)^{(\alpha-p)/2}}{\gamma^{\alpha/2} \pi^p |\mathbf{C}| K_\alpha(2\sqrt{\lambda\gamma})} K_{\alpha-p}\left(2\sqrt{\lambda(\gamma + q)}\right) \quad (34)$$

which is the single-look multivariate $\mathcal{G}$ distribution for PolSAR data, denoted by $\mathcal{G}_p(\alpha, \lambda, \gamma, \mathbf{C})$, and is the 1-look counterpart of the multi-look $\mathcal{G}$ distribution given in [6].

Two special cases of the $\mathcal{G}_p(\alpha, \lambda, \gamma, \mathbf{C})$ distribution can be derived. The first case models single-look multivariate polarimetric clutter, varying from homogeneous to slightly heterogeneous, and is obtained by assuming $\alpha > 0$, $\lambda > 0$, $\gamma \rightarrow 0$ and also using (25) in (34):

$$\begin{aligned} f_{\mathbf{Z}_p}(\mathbf{z}) &= \frac{\lambda^{p/2} q^{(\alpha-p)/2}}{\gamma^{\alpha/2} \pi^p |\mathbf{C}| 2^{\alpha-1} \Gamma(\alpha) (2\sqrt{\lambda\gamma})^{-\alpha}} K_{\alpha-p}\left(2\sqrt{\lambda q}\right) \\ &= \frac{2\lambda^{(\alpha+p)/2} q^{(\alpha-p)/2}}{\pi^p |\mathbf{C}| \Gamma(\alpha)} K_{\alpha-p}\left(2\sqrt{\lambda q}\right) \end{aligned} \quad (35)$$

which is the single-look multivariate $\mathcal{K}$ distribution for PolSAR data, denoted by $\mathcal{K}_p(\alpha, \lambda, \mathbf{C})$ and presented for the bivariate case in [31] and for the multivariate case in [11]. This distribution can also be derived using the product model by modeling texture as Gamma distributed variable and speckle noise as zero-mean multivariate complex Gaussian distributed.

The second case models single-look multivariate polarimetric clutter, varying from homogeneous to extremely heterogeneous, and is obtained by assuming $\alpha < 0$, $\gamma > 0$, $\lambda \rightarrow 0$, and also using (25), (26) in (34):

$$\begin{aligned} f_{\mathbf{Z}_p}(\mathbf{z}) &= \frac{\lambda^{p/2}(\gamma + q)^{(\alpha-p)/2}}{\gamma^{\alpha/2} \pi^p |\mathbf{C}| 2^{-\alpha-1} \Gamma(-\alpha) (2\sqrt{\lambda\gamma})^\alpha} \\ &\times 2^{p-\alpha-1} \Gamma(p - \alpha) \left(2\sqrt{\lambda(\gamma + q)}\right)^{\alpha-p} \\ &= \frac{\Gamma(p - \alpha)(\gamma + q)^{(\alpha-p)}}{\gamma^\alpha \pi^p |\mathbf{C}| \Gamma(-\alpha)} \end{aligned} \quad (36)$$

which is the single-look multivariate $\mathcal{G}^0$ distribution for PolSAR data, denoted by $\mathcal{G}_p^0(\alpha, \gamma, \mathbf{C})$, and has been recently introduced in [11]. This distribution can also be derived using the product model by modeling texture as reciprocal of Gamma distributed variable and speckle noise as zero-mean multivariate complex Gaussian distributed. A comparison of (35) and (36) shows that the $\mathcal{G}_p^0$ does not depend on modified Bessel functions.

One of the desirable features of $\mathcal{G}_p^0$ distribution is that it can be used to model from homogeneous to extremely heterogeneous clutter. However, this paper shows the significance of the more general $\mathcal{G}$ distribution as, for sub-meter resolution PolSAR data, the $\mathcal{G}_p^0$ distribution poorly fits some areas whereas the $\mathcal{G}$ distribution fits the data much more accurately.

V. PARAMETER ESTIMATION

Numerical maximization of likelihood is used here for parameter estimation of $\mathcal{G}$, $\mathcal{G}_p^0$, and $\mathcal{K}_p$ distributions. Given the training data $\mathbf{Z} = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \dots, \mathbf{z}_N\}$ for a particular class, the likelihood functions for the $\mathcal{G}$, $\mathcal{G}_p^0$, and $\mathcal{K}_p$ distributions according to the ML theory can be written as:

$$L(\alpha, \lambda, \gamma | \mathbf{Z}, \mathbf{C}) = \frac{\lambda^{p/2}}{\gamma^{\alpha/2} \pi^p |\mathbf{C}| K_\alpha(2\sqrt{\lambda\gamma})} \prod_{i=1}^N (\gamma + q_i)^{(\alpha-p)/2} K_{\alpha-p}\left(2\sqrt{\lambda(\gamma + q_i)}\right), \quad (37)$$

$$L(\alpha, \gamma | \mathbf{Z}, \mathbf{C}) = \frac{\Gamma(p - \alpha)}{\pi^p |\mathbf{C}| \gamma^\alpha \Gamma(-\alpha)} \prod_{i=1}^N (\gamma + q_i)^{\alpha-p}, \text{ and} \quad (38)$$

$$L(\alpha, \lambda | \mathbf{Z}, \mathbf{C}) = \frac{2\lambda^{(\alpha+p)/2}}{\pi^p |\mathbf{C}| \Gamma(\alpha)} \prod_{i=1}^N q_i^{(\alpha-p)/2} K_{\alpha-p}\left(2\sqrt{\lambda q_i}\right) \quad (39)$$

where $q_i = \mathbf{z}_i^{*t} \mathbf{C}^{-1} \mathbf{z}_i$. Note that for $p = 1$ the above equations result in univariate (single-channel) likelihood functions.

Matlab's "mle" method in the Statistics Toolbox has been used for MLE of parameters. This method has been tuned to use the Nelder-Mead Simplex algorithm as described in [32]. The Simplex algorithm is a well known direct search method for multidimensional minimization of an objective function (negative log likelihood function). It attempts to minimize the real valued objective function without utilizing any derivative information (derivative-free).

Three important challenges in parameter estimation of the univariate $\mathcal{G}_A^0$ law and polarimetric (multivariate) $\mathcal{G}$ distributions have been noted in [33]: 1) for a small sample size ($n \in \{9, 25, 49, 81, 121\}$) the conventional algorithms like Simplex (and also Broyden-Fletcher-Goldfarb-Shanno (BFGS)) fail to converge in obtaining parameters of $\mathcal{G}_A^0$ law about 11% of the time, 2) the negative log-likelihood function of the $\mathcal{G}_A^0$ law has "almost" flat likelihood regions around the minimum, which makes finding the minimum a difficult task, and 3) polarimetric distributions are indexed by matrices of complex values, and their computation is prone to severe numerical instabilities. However, in this paper the sample size of the training data is

approximately of the order of 1.5×10^5 , and the convergence criterion is a change in the likelihood value or a change in the step size of the parameter vector (norm of the difference) less than 10^{-9} . Even with such a strict convergence criterion, the Simplex algorithm, which is known to perform worse than BFGS [33] and is only granted to converge to a global minimum in one dimension [32], always converges with sample sizes of this order for both univariate and multivariate $\mathcal{G}$ distributions. Also, it was observed that the negative log likelihood functions of both univariate and multivariate $\mathcal{G}$ distributions showed "almost" flat regions around the minimum as presented in [33], however, the convergence criterion was small enough to avoid "pre-mature" convergence. Some experimental evidence showing the convergence of Simplex algorithm on simulated multivariate PolSAR data will be shown in the next subsection.

It is important to note that this parameter estimation method differs from the one used in [6] for two reasons. Firstly, in [6] the authors used the first and second moments of the multi-look univariate intensity $\mathcal{G}_I^0$ distribution for single channel parameter estimation. Secondly, in [6] the parameters of the multivariate distribution were inferred by averaging the single channel parameter estimates of the three polarimetric channels (averaging method). The averaging method MLEs will also be compared to the multivariate MLEs in the later subsection, however, instead of using a moment based approach, MLE will be used for single channel estimates. This method of MLE will be referred to as the averaging MLE method, in contrast to the multivariate MLE method which utilizes the polarimetric likelihood function in eq. (37).

Prior to estimating the parameters of $\mathcal{G}$, $\mathcal{G}_p^0$, and $\mathcal{K}_p$ distributions, the normalized covariance matrix, $\mathbf{C}$, needs to be estimated following the procedure described in section II-B. The AML estimator of $\mathbf{C}$ can be computed using (8). However, as the texture PDF is known (23) and the PDF generating function, $h_p(q)$ (13), reduces to an analytical form given in (34), it is desirable to find the ML estimator of normalized covariance matrix. In the case of the $\mathcal{G}$ distribution, the ratio $c_p(q) = \frac{h_{p+1}(q)}{h_p(q)}$ is given by:

$$c_p(q) = \frac{1}{\pi} \sqrt{\frac{\lambda}{\gamma + q}} \frac{K_{\alpha-p-1} \left(2\sqrt{\lambda(\lambda + q)} \right)}{K_{\alpha-p} \left(2\sqrt{\lambda(\lambda + q)} \right)} \quad (40)$$

Equation (40) shows that to find the ML estimator of normalized covariance matrix α , λ , and γ must be known, which subsequently requires an initial estimation of the normalized covariance matrix. Algorithm V.1, which uses the AML estimate as an initial guess, has been used to compute

the ML estimator of normalized covariance matrix. It must be noted that in the algorithm shown, f_{ML} and f_{AML} have some additional arguments contrary to their respective equations (11) and (8). This has been done to depict procedure calls with all the required parameters as input arguments.

Algorithm V.1: ESTIMATECOVARIANCEMATRIX($\mathbf{I}_0, \mathbf{Z}$)

procedure ESTIMATEEXACTCOVARIANCEMATRIX($\mathbf{C}_0, \alpha, \lambda, \gamma, \mathbf{Z}$)

repeat

comment: Estimate $\hat{\mathbf{C}}_{\text{ML}}$ using (11) & (40)

$\hat{\mathbf{C}}_{\text{ML}} \leftarrow f_{\text{ML}}(\mathbf{C}_0, \alpha, \lambda, \gamma, \mathbf{Z})$

comment: Calculate MLE of α, λ, γ using $\hat{\mathbf{C}}_{\text{ML}}$ in (37)

$\hat{\alpha}, \hat{\lambda}, \hat{\gamma} \leftarrow mle\left(L(\alpha, \lambda, \gamma | \mathbf{Z}, \hat{\mathbf{C}}_{\text{ML}})\right)$

comment: Calculate convergence criteria

$c \leftarrow \frac{\|\hat{\mathbf{C}}_{\text{ML}} - \mathbf{C}_0\|_F}{\|\mathbf{C}_0\|_F}$

comment: Set inputs for next iteration

$\mathbf{C}_0 \leftarrow \hat{\mathbf{C}}_{\text{ML}} \quad \alpha, \lambda, \gamma \leftarrow \hat{\alpha}, \hat{\lambda}, \hat{\gamma}$

comment: ϵ is an arbitrary constant

until $c < \epsilon$

return $(\hat{\mathbf{C}}_{\text{ML}}, \hat{\alpha}, \hat{\lambda}, \hat{\gamma})$

main

comment: Estimate $\hat{\mathbf{C}}_{\text{AML}}$ using (8)

$\hat{\mathbf{C}}_{\text{AML}} \leftarrow f_{\text{AML}}(\mathbf{I}_0, \mathbf{Z})$

comment: Calculate MLE of α, λ, γ using $\hat{\mathbf{C}}_{\text{AML}}$ in (37)

$\alpha, \lambda, \gamma \leftarrow mle\left(L(\alpha, \lambda, \gamma | \mathbf{Z}, \hat{\mathbf{C}}_{\text{AML}})\right)$

$\hat{\mathbf{C}}_{\text{ML}}, \hat{\alpha}, \hat{\lambda}, \hat{\gamma} \leftarrow \text{ESTIMATEEXACTCOVARIANCEMATRIX}(\hat{\mathbf{C}}_{\text{AML}}, \alpha, \lambda, \gamma, \mathbf{Z})$

return $(\hat{\mathbf{C}}_{\text{ML}}, \hat{\alpha}, \hat{\lambda}, \hat{\gamma})$

TABLE I
SPECKLE COVARIANCE MATRIX FOR ALL TEXTURES.

$[C]_{11}$	$[C]_{22}$	$[C]_{33}$	$[C]_{12}$	$[C]_{13}$	$[C]_{23}$
0.317	0.100	0.323	$-0.012 - 0.028i$	$0.179 + 0.018i$	$-0.021 - 0.0036i$

Fig. 1. GIG texture PDFs used for simulation.

Convergence of $\mathcal{G}$ Distribution Parameters using Simulated PolSAR Data

In this section the convergence of polarimetric $\mathcal{G}$ distribution MLEs is analyzed using simulated PolSAR data. The data is generated using a simulation procedure similar to the one detailed in [34], [35] for known PDFs. A summary of steps for simulated PolSAR data generation are listed here:

- 1) Compute $C^{1/2}$ for a given covariance matrix C , where $C^{1/2}(C^{1/2})^{*t} = C$, by using a unitary transformation U to diagonalize C ,

$$U^{*t}CU = \Lambda \Rightarrow C^{1/2} = U\Lambda^{1/2}. \quad (41)$$

- 2) Generate zero-mean complex Gaussian vectors, V , with identity covariance matrix.
- 3) The single-look speckle vector Y_p is obtained by

$$Y_p = \left(C^{1/2}V\right)^*. \quad (42)$$

- 4) Generate the backscattering texture, X , using a GIG random number generator presented in [36], with the limitation that the parameter $\alpha > 0$.

Fig. 2. MLEs of (a) α , (b) λ , and (c) γ using the multivariate MLE method 'x' and single channel MLE averaging method 'o', compared to the original parameter values '-' for all 60 scenarios.

5) Obtain the simulated PolSAR observation vector Z_p as:

$$Z_p = \sqrt{X}Y_p \quad (43)$$

The PolSAR data is generated with six different realizations of GIG textures, while the zero-mean complex Gaussian speckle is generated with the same covariance matrix (Table I) for each type of texture. The GIG parameters for the six textures are listed in Fig. 1, which shows the PDFs corresponding to these textures.

In order to analyze the convergence properties of the Simplex algorithm, MLE of parameters for the six textures using the Simplex algorithm is compared to the MLEs obtained using a global

Fig. 3. Mean MLEs and standard deviations of (a) α , (b) λ , and (c) γ using multivariate MLE method 'o', single channel MLE averaging method '□' and SIMPSA estimates 'x' for both methods. The original texture parameters are also shown '+'. The MLEs are considerably close to the original texture estimates for lower values of α , λ , and γ , however they deviate significantly for higher values as shown. Also, the MLEs obtained using the averaging MLE method show higher deviations than the multivariate MLE method.

optimization algorithm, the Simplex Simulated Annealing (SIMPSA) approach presented in [37]. This is done for both multivariate MLE method and for averaging MLE method mentioned earlier. Each texture realization is repeated 10 times making a total of 60 scenarios and each time 0.1 million pixels of PolSAR data are generated. Figure 2(a)-(c) shows the parameter estimates using the multivariate MLE method 'x' and averaging MLE method 'o' compared to the original texture estimates '-'. The MLEs are considerably close to the original texture estimates for lower values of α , λ , and γ , however they deviate significantly for higher values as shown. Also, the MLEs obtained using the averaging MLE method show higher deviations than the multivariate MLE method.

Fig. 4. Fittings of texture PDFs using MLEs from multivariate MLE method ' G_p ' and single channel MLE averaging method ' G_{avg} ' to texture realization (a) 1, (b) 2, (c) 3, (d) 4, (e) 5, and (f) 6.

The MLEs of parameters are also averaged over the 10 repetitions of each texture realization of PolSAR data for both Simplex and SIMPSA algorithms. Figure 3(a)-(c) shows the mean and standard deviation of α , λ , and γ parameter MLEs obtained by the Simplex ('o' and '□') minimization algorithm using the multivariate MLE and the averaging MLE methods, respectively. Also, the MLEs obtained using the SIMPSA algorithm 'x' are shown for both the methods. Further, the original texture parameter '+' is also shown. It is evident from the figures that the MLEs obtained using Simplex algorithm for both estimation methods are exactly aligned with the ones obtained using the global minimization algorithm, SIMPSA. In fact, it was observed that the Simplex algorithm always converged to global minimum MLEs for each of the 60 scenarios. It is therefore reasonable to assume that the Simplex algorithm not only converges for PolSAR data of the order of 0.1 million pixels, but it also obtains MLEs corresponding to global minimum values of the negative log likelihood function. It is interesting to note in Fig. 3(a)-(c) that for both α and λ , the averaging MLE method mean estimates are mostly closer to the original values than the multivariate MLE method, specially for higher values, although the

standard deviation is higher as well. For γ both methods provide similar estimates except for texture realization 5, where multivariate MLE method gives a closer estimate. In order to verify if the averaging MLE method provides better parameter estimates than the multivariate MLE method, the original texture PDFs were compared to those obtained using the mean MLEs of the two methods and the fittings corresponding to the six textures are shown in Fig. 4(a)-(f). In each of the six cases the PDF from the MLEs obtained using the averaging MLE method shows a much better fitting to the original texture PDF, corroborated by a Mean Squared Error (MSE) approximately 2 to 3 orders of magnitude less (χ^2 test is not used to avoid using averaged texture histograms as the MLEs are averages over 10 repetitions). Even for the 5th texture realization, where the MLEs from the averaging MLE method were not clearly closer to the original estimates, the PDF from averaging MLE method fits significantly better. In fact this was observed to hold true for each of the 60 scenarios. It is, therefore, recommended that the averaging MLE method should be used for the MLE of the multivariate $\mathcal{G}$ distribution parameters.

VI. APPLICATION TO REAL POLSAR DATA

The $\mathcal{G}$, $\mathcal{G}_p^0$, and $\mathcal{K}_p$ distributions have been applied to two different categories of datasets acquired by two different SAR sensors with different spatial resolution (decameter & sub-meter) and frequency. The coarser decameter resolution X-band PolSAR data has been acquired in April, 2009 using the space-borne sensor TerraSAR-X (TSX) over Wallerfing, Germany. TSX is an X-band SAR operating at a frequency of 9.65 GHz. This acquisition has been done at an incidence angle of approximately 32.68° at the centre coordinates of the scene. The spatial resolution of this image is approximately 1.17 m x 6.59 m (slant range x azimuth). On the other hand, the fine sub-meter resolution X- & S-band (9.65 & 3.2 GHz, respectively) PolSAR data has been acquired in the summer of 2010 using Astrium UK airborne SAR demonstrator (Astrium demonstrator) over Baginton, Southern England. Also, both X- and S-bands have used the same bandwidth (200 MHz) resulting in the same range resolution, which is approximately 0.835 m, while the azimuth resolution is 0.35 m. Astrium demonstrator's capability to acquire SAR imagery simultaneously in X- & S-band has been utilized for this dataset. This is very significant as it suggests that all the imaging geometry parameters are the same and the difference in the radar backscatter can be attributed only to frequency change.

The goodness-of-fit of the data to the proposed distribution is examined by using univariate

Fig. 5. (a) Two portions of TSX pauli color coded image over Wallerfing © DLR (2010) and (b) three portions of Astrium demonstrator image over Baginton © EADS Astrium Ltd. showing the training areas: urban (red), trees (orange), and fields (green).

amplitude distributions for each polarimetric channel as the PDF is multivariate and it is hard to examine it using multidimensional histograms. χ^2 test, MSE and coefficient of correlation (ρ) have been used to examine the goodness-of-fit of univariate amplitude PDFs to amplitude data histograms (the higher the p value of χ^2 test and the closer the values of MSE to zero and those of ρ to unity, the better the PDF fitting to amplitude histograms). Finally, the proposed $\mathcal{G}_p$ distribution is used as the underlying statistical model for three training classes (urban, trees and fields) to classify an independent S-band test data from Astrium demonstrator using a naïve Maximum A posteriori Probability (MAP) classifier. It must be pointed out here that although a careful statistical modeling of the data is desirable, other sources of statistical information, e.g. multifrequency acquisitions and contextual knowledge, lead to improved classification products [38]. Also, if a choice is to be made between improving classification by either enhancing the signal modeling or by incorporating multifrequency acquisitions and contextual information, the latter should be preferred [38]. However, this paper concentrates on an improved statistical model for single-look PolSAR data and therefore incorporating other sources of statistical evidence for improving classification products is beyond its objective.

Fig. 6. Fitting of $\mathcal{G}$, $\mathcal{G}^0$ and $\mathcal{K}$ to (a) homogeneous (fields), (b) moderately heterogeneous (trees), and (c) extremely heterogeneous (urban) areas for decameter resolution TerraSAR-X data.

Fig. 7. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ to homogeneous areas (fields) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution X-Band Astrium demonstrator data.

The training data for each class have been selected from different sub-regions of the TSX and Astrium demonstrator images. The selection of training data has been performed solely on a visual criteria (including comparison with Google Earth images) as ground truth data were not available. Figure 5(a) shows the training areas for the classes: urban (red), trees (orange), and fields (green) in Pauli decomposed [39] PolSAR image of TSX, while fig. 5(b) shows the same for the Astrium demonstrator PolSAR image.

Fig. 8. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions to moderately heterogeneous areas (trees) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution X-Band Astrium demonstrator data.

A. Goodness-of-fit to Amplitude Histograms

The marginal intensity distributions of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}_p$ can be obtained by putting $p = 1$ in (34), (35), and (36), respectively, which reduce to the following intensity distributions:

$$f_{Z_I}(z_j) = \frac{\sqrt{\lambda} \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)^{(\alpha-1)/2}}{\gamma^{\alpha/2} \pi \sigma_j^2 K_\alpha(2\sqrt{\lambda\gamma})} K_{\alpha-1} \left(2\sqrt{\lambda \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)} \right) \quad (44)$$

$$f_{Z_I}(z_j) = \frac{2\lambda^{(\alpha+1)/2}}{\pi \sigma_j^2 \Gamma(\alpha)} \left(\frac{|z_j|^2}{\sigma_j^2} \right)^{(\alpha-1)/2} K_{\alpha-1} \left(2\sqrt{\lambda \frac{|z_j|^2}{\sigma_j^2}} \right) \quad (45)$$

$$f_{Z_I}(z_j) = \frac{\Gamma(1-\alpha) \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)^{\alpha-1}}{\gamma^\alpha \pi \sigma_j^2 \Gamma(-\alpha)} \quad (46)$$

where z_j is the j^{th} complex polarimetric channel, $j \in \{hh, hv, vv\}$, and σ_j^2 is the j^{th} diagonal element of the ML estimator of normalized covariance matrix, $\hat{\mathbf{C}}_{\text{ML}}$, corresponding to the j^{th} polarimetric channel. The marginal univariate amplitude distributions can be derived by using the transformation $f_{Z_A}(\sqrt{z_j}) = 2f_{Z_I}(z_j)\sqrt{z_j}$, resulting in the following closed forms:

$$f_{Z_A}(z_j) = \frac{2\sqrt{\lambda} \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)^{(\alpha-1)/2}}{\gamma^{\alpha/2} \pi \sigma_j^2 K_\alpha(2\sqrt{\lambda\gamma})} K_{\alpha-1} \left(2\sqrt{\lambda \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)} \right) |z_j| \quad (47)$$

$$f_{Z_A}(z_j) = \frac{4\lambda^{(\alpha+1)/2}}{\pi \sigma_j^2 \Gamma(\alpha)} \left(\frac{|z_j|^2}{\sigma_j^2} \right)^{(\alpha-1)/2} K_{\alpha-1} \left(2\sqrt{\lambda \frac{|z_j|^2}{\sigma_j^2}} \right) |z_j| \quad (48)$$

$$f_{Z_A}(z_j) = \frac{2\Gamma(1-\alpha) \left(\gamma + \frac{|z_j|^2}{\sigma_j^2} \right)^{\alpha-1}}{\gamma^\alpha \pi \sigma_j^2 \Gamma(-\alpha)} |z_j| \quad (49)$$

Fig. 9. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions to extremely heterogeneous areas (urban) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution X-Band Astrium demonstrator data.

Fig. 10. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ to homogeneous areas (fields) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution S-Band Astrium demonstrator data.

It must be noted that (48) and (49) are the single-look counterparts of the multi-look marginal amplitude distributions ($\mathcal{K}_A$ and $\mathcal{G}_A^0$) presented in [5], and are thus called the single-look $\mathcal{K}_A$ and $\mathcal{G}_A^0$ distributions, respectively, while (47) is the single-look $\mathcal{G}_A$ distribution.

Analysis of Results: Figures 6-12 show the fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions to the histograms of the training classes (urban areas, trees and fields) for each polarimetric channel of the images considered along with corresponding parameter MLEs listed in the plot legend.

Figure 6 shows this fitting to decameter resolution TSX amplitude histograms for each channel and each training class. It is evident from the figure that both the $\mathcal{G}$ and $\mathcal{G}^0$ distributions fit the data equally good for all classes, while the $\mathcal{K}$ distribution fails to fit the data for urban areas.

Figures 7-9 and Figs. 10-12 show the $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distribution fittings to the sub-meter

TABLE II
MSE AND ρ OF FITTED UNIVARIATE $\mathcal{G}$, $\mathcal{G}^0$, AND $\mathcal{K}$ DISTRIBUTIONS FOR SUB-METER RESOLUTION ASTRIUM
DEMONSTRATOR DATA.

Class	Pol.	p_{χ^2}			MSE			ρ		
		$\mathcal{G}$	$\mathcal{G}^0$	$\mathcal{K}$	$\mathcal{G}$	$\mathcal{G}^0$	$\mathcal{K}$	$\mathcal{G}$	$\mathcal{G}^0$	$\mathcal{K}$
S-band										
Urban	hh	0.53	5.9e-5	0.00	7.99e-9	1.48e-8	4.50e-7	0.9991	0.9984	0.9501
	hv	0.66	0.00	0.00	9.72e-9	8.36e-8	6.71e-7	0.9992	0.9940	0.9467
	vv	0.60	2.0e-3	0.00	8.49e-9	1.44e-8	3.36e-7	0.9991	0.9986	0.9651
Trees	hh	0.39	0.00	0.00	9.41e-9	1.29e-7	1.16e-7	0.9992	0.9906	0.9909
	hv	0.30	0.00	0.00	3.95e-8	2.81e-7	3.33e-7	0.9978	0.9861	0.9821
	vv	0.11	0.00	0.00	7.66e-9	1.17e-7	1.14e-7	0.9994	0.9921	0.9917
Fields	hh	0.81	0.83	2.4e-3	1.38e-8	1.38e-8	4.55e-8	0.9996	0.9996	0.9988
	hv	0.89	0.90	0.00	2.96e-8	2.96e-8	1.83e-7	0.9996	0.9996	0.9976
	vv	0.87	0.87	0.00	7.43e-9	7.44e-9	4.77e-8	0.9998	0.9998	0.9987
X-band										
Urban	hh	0.61	0.64	0.00	1.08e-8	1.07e-8	9.18e-7	0.9994	0.9994	0.9482
	hv	0.69	4.3e-8	0.00	2.41e-8	2.36e-8	8.05e-7	0.9985	0.9984	0.9485
	vv	0.69	0.72	0.00	4.19e-9	4.15e-9	7.72e-7	0.9998	0.9998	0.9632
Trees	hh	0.29	0.00	0.00	6.46e-9	2.05e-7	4.53e-8	0.9997	0.9927	0.9983
	hv	0.13	0.00	0.00	5.84e-9	1.05e-7	1.51e-8	0.9998	0.9963	0.9994
	vv	0.18	0.00	0.00	7.58e-9	2.80e-7	6.77e-8	0.9998	0.9918	0.9979
Fields	hh	0.97	0.11	2.6e-5	7.23e-9	8.63e-9	2.01e-8	0.9999	0.9998	0.9997
	hv	0.89	0.05	0.06	5.84e-9	6.80e-9	9.96e-9	0.9999	0.9999	0.9998
	vv	0.89	9.5e-5	0.07	8.97e-9	1.46e-8	1.57e-8	0.9999	0.9998	0.9998

resolution X- and S- band Astrium demonstrator data, respectively, for each channel and each training class. Table II lists the goodness-of-fit measures (χ^2 , MSE and ρ) for these fittings. The p_{χ^2} values have been used to assess the goodness-of-fit but in some cases $p_{\chi^2} = 0$ was obtained for visually reasonable fittings due to the test's high dependency on the histogram binning. When this is the case MSE and ρ can be used to further examine the goodness-of-fit. It can be noticed from the figures and the goodness-of-fit measures in Table II that the $\mathcal{G}$ distribution fits the X- and S-band data very accurately for all the polarimetric channels in all the training classes (high p_{χ^2} values). The $\mathcal{G}^0$ distribution does not fit trees areas as well as the $\mathcal{G}$ distribution ($p_{\chi^2} = 0$, $\text{MSE} \approx 1\text{e-}7$ and $\rho \approx 0.990$) in both S- and X-band. The same can also be observed for the $\mathcal{K}$

Fig. 11. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions to moderately heterogeneous areas (trees) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution S-Band Astrium demonstrator data.

Fig. 12. Fitting of $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions to extremely heterogeneous areas (urban) in (a) hh, (b) hv and (c) vv channels for sub-meter resolution S-Band Astrium demonstrator data.

distribution. However, the $\mathcal{G}^0$ distribution fits the fields areas much better (better in S-band than in X-band) and also fits urban areas reasonably well (better in X-band than in S-band) but still not better than the $\mathcal{G}$ distribution (lower p_{χ^2} values). Note that there are some cases where $\mathcal{G}^0$ shows slightly higher p_{χ^2} values than the $\mathcal{G}$ distribution e.g. S-band fields hh, hv and X-band urban hh, vv but the Δp_{χ^2} is very small and can be ignored, so these cases can be treated as equally good fittings. In addition to this, the $\mathcal{K}$ distribution, as expected, fails to model urban areas and performs reasonably well for fields (better in X-band than in S-band), although the p_{χ^2} values are very close to zero.

A direct comparison of the heterogeneity of the observed scene can also be made between the X- and S-bands data to see the effect of frequency change when all other radar parameters are the same. For this purpose, the PDFs fitting to amplitude histograms of homogeneous areas

TABLE III
MULTIVARIATE & AVERAGED UNIVARIATE MLEs OF $\mathcal{G}$ DISTRIBUTION PARAMETERS FOR SUB-METER RESOLUTION
ASTRIUM DEMONSTRATOR DATA.

Class	$\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$	
	Multivariate MLE	Averaging MLE
Astrium demonstrator S-band		
Urban	-0.5955, 0.0040, 0.3221	-0.6414, 0.0048, 0.3692
Trees	0.0312, 0.1035, 0.2696	8.95e-4, 0.1020, 0.2951
Fields	-2.4995, 3.92e-14, 1.1226	-2.7777, 9.58e-13, 1.2809
Astrium demonstrator X-band		
Urban	-0.8725, 8.59e-5, 0.2148	-0.8127, 0.0058, 0.1948
Trees	0.7506, 1.0967, 0.0924	0.7599, 1.0555, 0.0711
Fields	-2.6967, 3.6935, 1.1520	-1.2858, 4.7501, 0.8326
TSX X-band		
Urban	-0.8120, 3.14e-4, 0.2870	-0.8125, 3.27e-4, 0.2817
Trees	-6.2319, 4.02e-12, 3.3217	-2.5794, 2.8164, 2.2900
Fields	-7.1559, 2.6592, 2.1587	-4.8883, 14.8534, 2.6238

between X- and S-bands can be compared between Fig. 7 and Fig. 10, respectively. It can be readily distinguished that the amplitude histogram of S-band data shown in Fig. 10 shows a deviation from the K-distribution fitting (representing the Rayleigh distributed amplitude case), while, in contrast to this, the amplitude histogram of X-band data in Fig. 7 shows nearly full agreement with the K-distribution fitting. This shows that the observed scene is slightly more heterogeneous in S-band than in X-band at the above mentioned radar parameters.

B. Naïve MAP Classification of S-band PolSAR Image

The MLEs of the more accurate $\mathcal{G}$ distribution have been numerically computed over fields, trees, and urban areas by using the multivariate MLE method and the averaging MLE method described in section V. These parameter estimates have been listed in Table III. A comparison of the corresponding MLEs from multivariate MLE method and averaging MLE method in Table III shows that both provide very close parameter estimates for lower values of the parameters as observed earlier for simulated PolSAR data, but deviate considerably from each other for higher parameter values. The MLEs obtained from the averaging MLE method have been used

Fig. 13. (a) Portion of the pauli color coded Astrium demonstrator Baginton S-band image (750×1000 pixels) used as an independent test data for MAP classification © EADS Astrium Ltd., and (b) the corresponding optical image © 2011 Infoterra Ltd.

for classification as it has been noticed in section V that they are more accurate than the ones obtained using multivariate MLE method.

A simple MAP classifier can be used to classify an independent test data (750×1000 pixels) extracted from Astrium demonstrator Baginton S-band image shown in Fig. 13 alongside its optical counterpart. The MAP classifier can be conveniently represented by the following expression:

$$\begin{aligned}
 \mathbf{z} &\rightarrow \omega_i \quad \text{if} \\
 P(\omega_i|\mathbf{z}) &= \max_{j=1}^m P(\mathbf{z}|\omega_j)P(\omega_j)
 \end{aligned} \tag{50}$$

where ω_j represents the j^{th} class, and the input vector $\mathbf{z}$ is assigned the class ω_i with the maximum a posteriori probability.

Analysis of Results: Figure 14 shows the image after applying the MAP classifier using the $\mathcal{G}$ distribution with MLEs from averaging MLE method listed in Table III for each class. Also, the color codes of different classes have been listed in the figure.

A visual comparison of Fig. 14 and its optical counterpart in Fig. 13 shows that the classification identifies urban areas, trees, and fields considerably well. Some of the errors could be attributed to inaccurate training data, as manual procedures have been used for selection of training data. Further, the temporal difference between the acquisition of radar and optical images

Fig. 14. MAP classification of Astrium demonstrator Baginton S-band test data with $\mathcal{G}$ distribution using multivariate MLEs. Urban = red, trees = blue, and fields = green.

would also result in some errors. It must be reiterated here that this is a naïve classification, intended to show the applicability of $\mathcal{G}$ distribution. Other sources of statistical information e.g. multiple frequency sources and contextual information should be incorporated for improvement [38] as mentioned earlier.

VII. CONCLUSIONS AND FUTURE WORK

In this paper the specialized case of single-look polarimetric $\mathcal{G}$ distribution has been derived using the product model, considering a GIG distributed texture and a zero-mean multivariate complex Gaussian distributed speckle. This is the single-look counterpart of the multi-look $\mathcal{G}$ distribution presented in [6]. The single-look $\mathcal{K}_p$ and $\mathcal{G}_p^0$ distributions, which are special forms of this distribution corresponding to Gamma and reciprocal of Gamma distributed textures, respectively, have also been derived. The utility of single-look $\mathcal{G}$ distribution becomes evident as multi-look $\mathcal{G}$ distribution does not reduce to its single-look form when fully polarimetric data are available. The single-look $\mathcal{G}$ distribution can be useful in applications like urban structures detection, road mapping, marine structures, and ship detection etc., where retaining high spatial resolution becomes vital, and multi-looking is not an option. Further, the importance of $\mathcal{G}$ distribution manifests as it fits very high resolution PolSAR data considerably better than the $\mathcal{G}_p^0$ and $\mathcal{K}_p$ distributions.

The proposed $\mathcal{G}$ distribution has been found to fit sub-meter resolution X- & S-band Atrium demonstrator PolSAR data better than, and decameter resolution X-band TerraSAR-X PolSAR data as good as the $\mathcal{G}_p^0$ distribution. Under the given radar parameters it has also been observed that the scene shows more heterogeneity in S-band compared to X-band. Although the proposed distribution is computationally expensive as it has three parameters, it out performs the fitting accuracy of $\mathcal{G}_p^0$ & $\mathcal{K}_p$ distributions. The fitting of these distributions to PolSAR data has been presented by using univariate fitting to amplitude histograms. The $\mathcal{G}$ distribution fits the amplitude histograms of homogeneous, moderately heterogeneous, and extremely heterogeneous areas accurately even where the $\mathcal{G}_p^0$ and $\mathcal{K}_p$ perform relatively poorly. The application of $\mathcal{G}$ distribution to statistically model PolSAR data has also been shown by using a naïve MAP classifier on an S-band PolSAR image. A qualitative visual evaluation of the classification shows that $\mathcal{G}$ distribution can be used as an effective underlying statistical model as all the three classes of urban, trees, and fields areas are reasonably identified.

Maximum Likelihood Estimation (MLE), using Matlab's Simplex algorithm, has been used for parameter estimation. The convergence of the Simplex algorithm to globally maximum likelihood function values has been shown by using simulated PolSAR data and comparing the results with a global maximization algorithm based on Simulated Annealing and Simplex algorithms (SIMPSA). It has been found that the Simplex algorithm always converges to globally maximum values of likelihood function for data of the order of 0.1 million points. It has also been shown that the parameter estimates of the multivariate $\mathcal{G}$ distribution can be computed more accurately by using the average of single-channel estimates instead of computing estimates from multivariate PolSAR data. This has been shown on simulated PolSAR data for a variety of backscattering textures. Also, a new algorithm for accurate estimation of speckle covariance matrix has been proposed.

One of the drawbacks of the current analysis is that the texture has been modeled as the square root of a positive random variable instead of more accurately modeling it as the square root of a diagonal matrix variate with the texture of each polarimetric channel separated along the diagonal. Such a technique has been very recently adopted in [40], and is one of the areas of future work. Also, the GIG texture in simulated PolSAR data was generated with $\alpha > 0$ only, due to limitation of the used GIG random number generator. A GIG random number generator with $\alpha \in \mathbb{R}$ would be desirable. Another area of improvement is the selection of accurate training

data for fitting analysis instead of a visual selection approach. Nevertheless, the superior fitting accuracy of the proposed single-look $\mathcal{G}$ distribution, especially in the case of sub-meter resolution PolSAR data, will be significant in improving classification, segmentation, and feature extraction algorithms for VHR single-look PolSAR data in various applications including but not limited to urban structures detection, road mapping, marine structures and ship detection.

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