The virtual element method strength reduction technique for the stability analysis of stony soil slopes

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Abstract: Based on the incremental method of elastic-plastic mechanics and bilinear projection operators, by combining the strength reduction method with the $\varphi-v$ inequality, this paper proposes a virtual element method strength reduction technique for slope stability analysis. The deformations of a homogeneous slope and heterogeneous slope are solved under different strength reduction factors, and the mesh dependency problem of the method is discussed. Numerical examples verify the correctness and effectiveness of the proposed method. The results demonstrate that due to the differences in the physical and mechanical properties of soil and rocks, stress becomes concentrated in the contact zone between soil and rocks, and thus, the plastic zone surrounds the rocks. Therefore, it is difficult to form regular connections in a plastic zone, such as a soil slope. This method can be used to analyse the stability of a stony soil slope and to study the effects of particle size, rock content, rock density, and rock spatial distribution on the mechanical behaviour of stony soil slopes.

Key Words: Slope, Virtual element method, Finite element method, Strength reduction

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1 Introduction

Since Zienkiewicz [1] first proposed the strength reduction finite element method (FEM), this technique has been extensively used in and has gradually become the primary means of performing slope stability analysis, particularly with the advancement of computer technology. The initial application of the strength reduction FEM is limited to the analysis of slope deformation and the engineering construction process. To provide a safety factor similar to other methods, many scholars, such as Naylor [2], Donald & Giam [3], Matsui & San [4], Ugai [5], Dawson [6], and Griffiths & Lane [7], have adopted the strength reduction FEM.

Compared with the classic limit equilibrium method and limit analysis method [8], the strength reduction FEM boasts a few unique advantages. First, the failure mode under the limit state, which is very important in many situations, can be obtained using this technique. For example, if the slope cutting method is used to improve the stability of landslides, the slip surfaces and slip-resistant surfaces of the slopes must be fully considered [9], and an excavation body can play a positive role only if it is located on the slip surface of the slope; otherwise, the excavation body will have the opposite effect. Second, the progressive instability process of a slope due to strength deterioration can be understood using this method; as a result, limited reinforcement measures can be placed at the most critical locations [10], and the effects of different construction procedures on the ultimate safety of the slope can be considered [11]. Finally, some relatively complex factors affecting slope stability, such as groundwater seepage [12], dynamic factors [13], and the adoption of a more realistic constitutive model [14], can be considered.

Matsui et al. [15] applied a shear strength reduction technique to slope stability analysis and analysed the stability of artificial fill slopes and excavation slopes through numerical simulation. Furthermore, the strength reduction method was employed to study the effect of soil dilatancy on slope stability by Manzari et al. [16] and to analyse the stability characteristics of unsaturated soil slopes by Mukhlisin et al. [17]. A two-parameter strength reduction method, which can be used to establish the reduction

principles for the internal friction angle and cohesion, was proposed by Yuan et al. [18]. Additionally, with the strength reduction method and limit equilibrium method, the safety factors and slip surface positions for different slopes were calculated by Cheng et al. [19].

Although the application of the FEM in evaluating slope stability has been refined, the FEM suffers from poor applicability to complex problems, and its calculation accuracy is sensitive to the mesh quality [9]. In the strength reduction method, as the strength parameter continues to decrease, the yield surface can gradually decrease. Under the effect of a deadweight load, the stress concentration region is the first to produce plastic deformation and even suffers from mesh distortion, which can significantly affect the computational accuracy of the FEM [18]. An encrypted mesh or the use of quadratic integral elements could result in significantly increased computational costs [9]. In addition, for problems involving complex geometric shapes, such as stony soil slopes (in which rock blocks of different sizes and different shapes are randomly distributed throughout the slope), the generation of a twisted mesh in the discrete process of the model is unavoidable.

As an alternative, the virtual element method (VEM) [20] proposed in recent years can adapt to any shape and any number of edges. Compared with the FEM, the VEM has obvious advantages when working with complex structural problems. In addition, the VEM stiffness matrix is constructed using the projection operator on the element, and the unit stiffness matrix can be directly solved using the node degree of freedom (DOF). Based on the above characteristics, the VEM has been successfully used to solve the bending of metal plates [21], parabolic problems [22] and linear elasticity problems [23]. Chen et al. [24] proved the equivalence of weak Galerkin methods and the FEM by improving upon the former.

Furthermore, the VEM can be employed to handle complex fracture patterns even for arbitrary nonlinear materials and complex nonlinear material models, see References [25-27]. Actually, elements with any number of edges (not necessarily convex), hanging nodes, flat angles, collapsing nodes, etc., can be easily handled in the VEM framework while retaining the same approximation properties of FEM.

VEM constitutes a powerful and efficient tool to represent the complex geometries of the inclusions in composite materials, such as coarse aggregates in concrete, stony soil. On the other hand, classical zero-thickness interface elements are employed for modeling stress-crack opening processes [28]. While the extension to VEM to three dimensional, the complicated nonlinear materials and models can be realized with the cracking-particle method [29-31], the efficient remeshing techniques [32-34], the screened Poisson equation models [35-36], the numerical manifold method [37-41], the artificial neural network methods [42-44], the phase field modelling [45-47] and the cellular automaton [48-51].

In this article, a VEM procedure for the linear elastic-plastic problem is presented. Then, a VEM strength reduction technique for slope stability analysis with the $\varphi-v$ inequality is proposed. Finally, the classic slope example and a stony soil slope are employed in this paper to study the correctness and effectiveness of the proposed scheme in slope stability analysis.

2. VEM for the linear elastic problem

2.1 Governing equations

As shown in Figure 1, an elastic body is considered to be composed of an open region $\Omega \subset R^2$ and a corresponding boundary Γ . The normal direction on the outer boundary Γ is $\mathbf{n}$ such that $\Gamma = \Gamma_g + \Gamma_h$ and $\Gamma_g \cap \Gamma_h = \emptyset$, where Γ_g is the displacement boundary condition and Γ_h is the continuity boundary condition. In a closed set of regions $\bar{\Omega} \equiv \Omega \cup \Gamma$, when an elastic body is under the action of an external force $\mathbf{h}(\mathbf{x}): \Gamma_h \rightarrow R^2$ and a force $\mathbf{f}(\mathbf{x}): \Omega \rightarrow R^2$, the displacement field of any point $\mathbf{x}$ is $\mathbf{u}(\mathbf{x}): \Omega \rightarrow R^2$. The applied displacement boundary condition is denoted as $\mathbf{g}(\mathbf{x}): \Gamma_g \rightarrow R^2$. The linear elastic-plastic boundary problem can be described as finding a displacement field $\mathbf{u}(\mathbf{x}) \in U$ that satisfies the following conditions:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = 0 & \forall \mathbf{x} \in \Omega \\ \mathbf{u} = \mathbf{g} & \forall \mathbf{x} \in \Gamma_g \\ \boldsymbol{\sigma} \mathbf{n} = \mathbf{h} & \forall \mathbf{x} \in \Gamma_h \end{cases} \quad (1)$$

Under linear elastic conditions,

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{u})] \quad (2)$$

The strain-displacement relationship is as follows:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla^T \mathbf{u}) \quad (3)$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, $\mathbf{f}$ is a force, $\boldsymbol{\varepsilon}$ is the linear strain tensor, and $\mathbf{u}$ is the unknown displacement. For an isotropic elastic material, the characteristic tensor is

$$\mathbf{D}[\boldsymbol{\varepsilon}] = \lambda (\operatorname{tr} \boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon} \quad (4)$$

where λ and μ are Lamé's constants.

The weak form of this problem can be expressed as finding the $\mathbf{u}(\mathbf{x}) \in U$ that satisfies the following condition:

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v}(\mathbf{x}) \in V \quad (5)$$

Where

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega = \int_{\Omega} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{u})] : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega$$

and

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_h} \mathbf{h} \cdot \mathbf{v} \, d\Gamma_h.$$

U and V are the two spaces for the displacement field to be tested:

$$U := \left\{ \mathbf{u}(\mathbf{x}) : u \in [W(\Omega)]^2 \subseteq H^1(\Omega)^2, u = g(\Gamma_g) \right\}$$

$$V := \left\{ \mathbf{v}(\mathbf{x}) : v \in [W(\Omega)]^2 \subseteq H^1(\Omega)^2, v = 0(\Gamma_g) \right\}$$

where the space $W(\Omega)$ includes the linear displacement field.

Fig. 1 A continuum body

2.2 Discrete format of the VEM

The region Ω is divided into non-overlapping polygonal elements. If the lowest order is adopted, nodes are set only at the unit vertex. The discrete space of the test function of region Ω is denoted as $V_h \subset V$. This requires defining the shape function inside V_h , including continuous functions that are restricted to elements and linear functions. Additionally, the shape function of each element is linear at the element boundary.

Let us define the contribution of an element to bilinear Equation (5) as follows:

$$a^e(\mathbf{u}, \mathbf{v}) = \int_{\Omega_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{u})] : \boldsymbol{\varepsilon}(\mathbf{v}) d\Omega \quad (6)$$

First, the key to constructing a method in which all calculations are based on the boundary is satisfying the following for any linear polynomial $\mathbf{p}$:

$$\begin{aligned} a^e(\mathbf{p}, \mathbf{v}) &= \int_{\Omega_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{p})] : \boldsymbol{\varepsilon}(\mathbf{v}) d\Omega \\ &= \int_{\Gamma_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{p})] \mathbf{n} \cdot \mathbf{v} d\Gamma - \int_{\Omega_e} \text{div}[\mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{p})]] \cdot \mathbf{v} d\Omega \\ &= \int_{\Gamma_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{p})] \mathbf{n} \cdot \mathbf{v} d\Gamma \end{aligned} \quad (7)$$

If Voigt symbols are used, $\boldsymbol{\sigma} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]$ and $\boldsymbol{\varepsilon} = [\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{12}]$; moreover,

the normal vector can be written in matrix form:

$$\mathbf{n} = \begin{bmatrix} n_x & 0 \\ 0 & n_y \\ n_y & n_x \end{bmatrix} \quad (8)$$

For the function $v_h \in V_h$, let us define the projection Πv_h as a linear function, and the following is satisfied for any linear polynomial $\mathbf{p}$:

$$a^e(\mathbf{p}, \Pi v_h) = a^e(\mathbf{p}, v_h) \quad (9)$$

This formula indicates that Πv_h does not depend on rigid body motion, and thus, the conditions must be increased to ensure the uniqueness of the solution. This can be achieved through the following requirements: 1) the mean of the function is equal to the projection of all nodes, and 2) the moment of the function is equal to the projection of all nodes. Therefore, on each element,

$$\frac{1}{N} \sum_{j=1}^N \Pi v_h(\mathbf{x}_j) = \frac{1}{N} \sum_{j=1}^N v_h(\mathbf{x}_j) \quad (10)$$

$$\frac{1}{N} \sum_{j=1}^N \mathbf{x}_j \times \Pi v_h(\mathbf{x}_j) = \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j \times v_h(\mathbf{x}_j) \quad (11)$$

where $\mathbf{x}_j$ is the coordinate of a node j and the sum is performed for all nodes on the element boundaries.

The projection can then be expressed in the following form:

$$\Pi v_h = \mathbf{H} \mathbf{a} \quad (12)$$

For this projection, we can choose a polynomial basis function. Under normal circumstances, this choice is unlimited as long as the applicability of the problem is satisfied [52]. This avoids the singularity problem when calculating the VEM parameters. In the case of linear elasticity, the proposed method must include terms for rigid body translation, rotation, shear strain, axial strain and volumetric strain. Accordingly, we can choose

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & \overset{\text{rotation}}{-y^*} & \overset{\text{shear-}}{x^*} & \overset{\text{shear-}}{y^*} & \overset{\text{shear-}}{x^*} \\ 0 & 1 & x^* & -y^* & x^* & y^* \\ \underset{\text{translation}}{} & & & \underset{\text{axial-}}{} & & \underset{\text{volum-strain}}{} \end{bmatrix} \quad (13)$$

This projection contains some unknown parameters $\mathbf{a} = \{a_1, \dots, a_6\}$ and a group of monomial terms $\mathbf{H}(\mathbf{x})$ [53].

$$x^* = \left(\frac{x - x_D}{h_D} \right) \text{ and } y^* = \left(\frac{y - y_D}{h_D} \right) \quad (14)$$

where $\mathbf{x}_D$ denotes the barycentric coordinates of the element and h_D is the element dimension.

The left-hand side of Equation (9) can be estimated. A test function $\mathbf{p}$ can be used as the approximate projection Πv_h [54]. Because the rigid body model does not contribute to strain energy, the monomial terms $\mathbf{H}(\mathbf{x})$ can be reduced to three terms. The reduced $\mathbf{h}(\mathbf{x})$ and corresponding VEM parameters $\mathbf{a} = \{\alpha_1, \alpha_2, \alpha_3\}$ can be used to express the following test function:

$$\mathbf{p} = \begin{bmatrix} x^* & y^* & x^* \\ -y^* & x^* & y^* \end{bmatrix} \mathbf{a} = \mathbf{h}(\mathbf{x}) \mathbf{a} \quad (15)$$

By substituting the above into the right-hand side of Equation (9), we have the following:

$$\begin{aligned} a^e(\mathbf{p}, \Pi v_h) &= \int_{\Gamma_e} \boldsymbol{\sigma}(\Pi v_h) \mathbf{n} \cdot \mathbf{p} \, d\Gamma \\ &= \mathbf{a}^T \underbrace{\int_{\Gamma_e} \mathbf{h}^T \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{H})] \mathbf{n} \, d\Gamma}_{\hat{\mathbf{G}}_i} \mathbf{a} \end{aligned} \quad (16)$$

The function v_h is a piecewise linear function along the element boundary and can be written as

$$v_h = \mathbf{N} \hat{\mathbf{v}} \quad (17)$$

where $\hat{\mathbf{v}}$ is the DOF for the coordinates of the nodes and $\mathbf{N}$ is a piecewise linear function matrix on the element boundary.

The right-hand side of Equation (7) can be expressed as

$$\begin{aligned}
a^e(\mathbf{p}, \mathbf{v}_h) &= \int_{\Gamma_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{p})] \mathbf{n} \cdot \mathbf{v}_h \, d\Gamma \\
&= \int_{\Gamma_e} \mathbf{D}[\boldsymbol{\varepsilon}(\mathbf{h}(\mathbf{x})\boldsymbol{\alpha})] \mathbf{n} \cdot \mathbf{N}\hat{\mathbf{v}} \, d\Gamma \\
&= \mathbf{a}^T \tilde{\mathbf{B}}_1 \hat{\mathbf{v}}
\end{aligned} \tag{18}$$

Equations (12) and (17) can be substituted into Equations (10) and (11):

$$\begin{aligned}
\underbrace{\frac{1}{N} \sum_{j=1}^N \mathbf{H}(\mathbf{x}_j) \mathbf{a}}_{\tilde{\mathbf{H}}_T} &= \frac{1}{N} \sum_{j=1}^N \mathbf{N}(\mathbf{x}_j) \hat{\mathbf{v}} \\
&= \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \end{bmatrix}}_{\tilde{\mathbf{N}}_T} \hat{\mathbf{v}}
\end{aligned} \tag{19}$$

$$\begin{aligned}
\underbrace{\frac{1}{N} \sum_{j=1}^N \mathbf{x}_j \times \mathbf{H}(\mathbf{x}_j) \mathbf{a}}_{\tilde{\mathbf{H}}_R} &= \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j \times \mathbf{N}(\mathbf{x}_j) \hat{\mathbf{v}} \\
&= \frac{1}{N} \underbrace{\begin{bmatrix} -y_i & x_i & \cdots & -y_N & x_N \end{bmatrix}}_{\tilde{\mathbf{N}}_R} \hat{\mathbf{v}}
\end{aligned} \tag{20}$$

Ultimately, the virtual parameter $\mathbf{a}$ is arbitrary, and the complete equilibrium equation is

$$\mathbf{G}\mathbf{a} = \mathbf{B}\hat{\mathbf{v}} \tag{21}$$

where

$$\mathbf{G} = \begin{bmatrix} \tilde{\mathbf{G}}_1 \\ \tilde{\mathbf{H}}_T \\ \tilde{\mathbf{H}}_R \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{N}}_T \\ \tilde{\mathbf{N}}_R \end{bmatrix} \tag{22}$$

Because $\mathbf{G}$ is reversible, Equation (21) allows the parameter $\mathbf{a}$ of projection $\prod v_h$ to be ‘virtual’ for the main unknown variables (the DOF of the node displacement) [55].

Because all integrals are performed along the boundary of the element, the Gauss-Lobatto integral formula can be adopted. Therefore, we can determine the value of $\hat{\mathbf{u}}$ for the nodes by integration [52]. These functions have the following property:

$$\mathbf{u} = \sum_{j=1}^N \mathbf{N}_j \hat{\mathbf{u}} \quad \mathbf{N}_i(\mathbf{V}_j) = \delta_{ij} \tag{23}$$

Where, V_j is the vertices of elements.

Hence, the actual form of this function does not require the boundary integral process.

2.3 Construction of the VEM element stiffness matrix

From the weak form (Equation (5)), we can find that all characteristics should satisfy the equilibrium conditions. As described in section 2.2, the displacement is approximated by a polynomial decomposition of the projection $\Pi \mathbf{v}$, which includes a rigid body motion mode $\Pi_R \mathbf{v}$ and a constant strain mode $\Pi_C \mathbf{v}$. The remainder must consider the decomposed mode, which leads to the complete decomposition of the virtual displacement [52]:

$$\mathbf{v} = \Pi_R \mathbf{v} + \Pi_C \mathbf{v} + (\mathbf{v} - \Pi \mathbf{v}) \quad (24)$$

Substituting the above decomposition into a weak form leads to the following:

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= a(\Pi_R \mathbf{v} + \Pi_C \mathbf{v} + (\mathbf{v} - \Pi \mathbf{v}), \Pi_R \mathbf{v} + \Pi_C \mathbf{v} + (\mathbf{v} - \Pi \mathbf{v})) \\ &= a(\Pi_C \mathbf{v}, \Pi_C \mathbf{v}) + 2a(\Pi_C \mathbf{v}, \mathbf{v} - \Pi \mathbf{v}) + a(\mathbf{v} - \Pi \mathbf{v}, \mathbf{v} - \Pi \mathbf{v}) \end{aligned} \quad (25)$$

As a result of orthogonality, the coupling term is equal to zero. Finally, the weak form of each element can be obtained, including the constant strain term and the high-order or stable term, whose approximation is as follows:

$$a^E(\mathbf{u}, \mathbf{v}) = a^E(\Pi_C \mathbf{u}, \Pi_C \mathbf{v}) + S^E((\mathbf{u} - \Pi \mathbf{u}), (\mathbf{v} - \Pi \mathbf{v})) \quad (26)$$

According to the chosen projection, the details of the first term and the stable terms of stiffness matrix can be seen in [55].

3. Elastic-plastic solution scheme of the VEM

3.1 The principle of elastic-plastic analysis

In classic plastic mechanics, the stress-strain relationship mainly includes incremental theory and total strain theory. The former can reflect the correlation with the stress history and more comprehensively reflect the plastic deformation properties of a material than the latter.

3.2 Yield criterion

Let $\mathbf{D}_e$ be an elastic matrix, $\boldsymbol{\sigma}$ be a stress vector, $\boldsymbol{\kappa}$ be the hardening parameter, $I_1 = \sigma_{kk}$, and $J_2 = \frac{1}{2} s_{ij} s_{ij}$. The yield function can be expressed as

$$F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = f(\boldsymbol{\sigma}) - k(\boldsymbol{\kappa}) = 0 \quad (27)$$

Alternatively, the yield function can be expressed by the invariant of the stress tensor:

$$F = F(\mathbf{I}_1, (\mathbf{J}_2)^{1/2}, \theta) \quad (28)$$

The flow vector is as follows:

$$\mathbf{a}_d = \frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}} = \frac{\partial \mathbf{F}}{\partial \mathbf{I}_1} \frac{\partial \mathbf{I}_1}{\partial \boldsymbol{\sigma}} + \frac{\partial \mathbf{F}}{\partial (\mathbf{J}_2)^{1/2}} \frac{\partial (\mathbf{J}_2)^{1/2}}{\partial \boldsymbol{\sigma}} + \frac{\partial \mathbf{F}}{\partial \theta} \frac{\partial \theta}{\partial \boldsymbol{\sigma}} \quad (29)$$

The elastic-plastic matrix is written as follows:

$$\mathbf{D}_{ep} = \mathbf{D}_e - \frac{\mathbf{D}_e \mathbf{a}_d \mathbf{a}_d^T \mathbf{D}_e}{\mathbf{A} + \mathbf{a}_d^T \mathbf{D}_e \mathbf{a}_d} \quad (30)$$

When using the associated flow rule, $\mathbf{A} = -\frac{\partial \mathbf{F}}{\partial \boldsymbol{\kappa}} \boldsymbol{\sigma}^T \frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}}$.

3.3 Elastic-plastic stress adjustment method

In this paper, each element is treated as an element of constant stress, and a stress adjustment is carried out on each element. When each incremental load is applied, the stress of some elements in the system may exceed the yield stress, following which plastic strain could occur. The stress and strain on these elements must be adjusted to meet the yield criterion and constitutive laws.

The stress, strain and plastic strain of each element after iteration $r-1$ are $\boldsymbol{\sigma}^{r-1}$, $\boldsymbol{\epsilon}^{r-1}$, and $\bar{\boldsymbol{\epsilon}}^{r-1}$, respectively. $d\boldsymbol{\epsilon}^r$ is the strain increment of iteration r , and the elastic test stress at iteration r is

$$\boldsymbol{\sigma}_e^r = \boldsymbol{\sigma}^{r-1} + \mathbf{D}_e d\boldsymbol{\epsilon}^r \quad (31)$$

From step $r-1$ to step r , the plastic strain increment is

$$d\boldsymbol{\epsilon}_p^r = \lambda \frac{\partial \mathbf{g}}{\partial \boldsymbol{\sigma}} \quad (32)$$

where g is the plastic potential function. The stress at step r should then satisfy the following equilibrium equation:

$$\boldsymbol{\sigma}^r = \mathbf{D}_e(\boldsymbol{\varepsilon}^{r-1} + d\boldsymbol{\varepsilon}_e^r) = \mathbf{D}_e(\boldsymbol{\varepsilon}^{r-1} + d\boldsymbol{\varepsilon}^r - d\boldsymbol{\varepsilon}_p^r)$$

i.e.,

$$\boldsymbol{\sigma}^r = \boldsymbol{\sigma}_e^r - \lambda \mathbf{D}_e \frac{\partial g}{\partial \boldsymbol{\sigma}} \quad (33)$$

Additionally, the strength condition should be satisfied.

$$f(\boldsymbol{\sigma}^r) = 0 \quad (34)$$

By solving Equations (33) and (34) in parallel, we can obtain the stress and plastic strain increments at step r .

Because the yield function is based on varying stiffness among the iterations, before the start of every iteration step, we first must determine the contribution of the element integration point (the constant stress element is used in this paper) to the element stiffness matrix and then assemble the system's total stiffness matrix. Except for the first iteration (the first load increment) using the system's elastic total stiffness matrix, the calculation of the stiffness matrix of a single element in the subsequent iterative steps is as follows:

The effective stress, effective plastic strain, uniaxial yield stress and linear strain hardening parameter of each element are $\bar{\boldsymbol{\sigma}}^{r-1}$, $\bar{\boldsymbol{\varepsilon}}_p^{r-1}$, σ_Y and H' respectively existing at the end of $(r-1)^{th}$ iteration. During the application of an increment of load an element, or part of an element, may yield. All stress and strain quantities are monitored at each Gaussian integration point and therefore we can determine whether or not plastic deformation has occurred at such points. For example, the $(r-1)^{th}$ iteration has been completed. Consequently the adopted procedure of r^{th} iteration is follows.

Step a. Compute the tentative incremental stress changes, $d\boldsymbol{\sigma}_e^r$ as $d\boldsymbol{\sigma}_e^r = \mathbf{D}d\boldsymbol{\varepsilon}^r$

where the subscript e denotes that we are assuming elastic behaviour.

288 Accumulate the total stress for each element Gauss point as $\sigma_e^r = \sigma^{r-1} + \sigma_e^r$

289 where σ^{r-1} are the converged stresses for iteration $(r-1)^{th}$.

290 **Step b.** Check if $\bar{\sigma}^{r-1} > \sigma_Y = \sigma_Y^0 + H'\bar{\epsilon}_p^{r-1}$. σ_Y^0 is initial uniaxial yield stress. If

291 the answer is YES, go to *Step c*. If not, go to *Step f*.

292 **Step c.** Now check to see if $\bar{\sigma}_e^r > \bar{\sigma}^{r-1}$ where $\bar{\sigma}_e^r$ is the effective stress based on

293 stresses σ_e^r . If the answer is YES, go to *Step e*. If not, go to *Step d*.

294 **Step d.** The Gauss point is unloading elastically and therefore the following

295 procedure is omitted altogether. $\sigma^r = \sigma_e^r$.

296 **Step e.** The Gauss point had yielded previously and the stress is still increasing.

297 Therefore all the excess stress $\sigma_e^r - \sigma^{r-1}$ must be reduced to the yield

298 surface. Therefore the factor R , which defines the portion of stress which

299 must be modified to satisfy the yield criterion is equal to 1.

300 **Step f.** Now check to see if $\bar{\sigma}_e^r > \sigma_Y^0$. If the answer is YES, go to *Step g*. If not,

301 go to *Step h*.

302 **Step g.** The reduction factor $R = \frac{\bar{\sigma}_e^r - \sigma_Y}{\bar{\sigma}_e^r - \bar{\sigma}^{r-1}}$, then go to *Step i*.

303 **Step h.** The Gauss point is still elastic and therefore the following procedure is

304 omitted altogether. $\sigma^r = \sigma_e^r$.

305 **Step i.** The excess stress $Rd\sigma_e^r$ is divided into m parts where m is given by the

306 nearest integer which is less than $m = (\frac{\bar{\sigma}_e^r - \sigma_Y}{\sigma_Y^0})8 + 1$.

307 **Step j.** Calculate $\sigma^r = \sigma^{r-1} + (1-R)d\sigma_e^r + Rd\sigma_e^r / m - d\lambda d_D / m$.

308 **Step k.** Equivalent plastic strain increment $\bar{\epsilon}_p^r = \bar{\epsilon}_p^r + d\lambda a^T \sigma / \bar{\sigma}$.

309 **Step l.** Go to *Step i*, and repeat m times.

310 **Step m.** Radial return, and $\sigma^r = \sigma^r (\frac{\sigma_Y^0 - H'\bar{\epsilon}_p^r}{\bar{\sigma}^r})$.

4 Strength reduction technique

The fundamental principle of the strength reduction method [56] is to gradually reduce the material shear strength parameters (c and φ) until damage occurs to the slope.

$$c' = \frac{1}{F^i} c \quad (35)$$

$$\varphi' = \arctan\left(\frac{1}{F^i} \tan\varphi\right) \quad (36)$$

where φ' and c' are the internal friction angle and cohesion, respectively, after the strength is reduced and F^i is the strength reduction rate. As shown in Figure 2, c and φ are the cohesion and internal friction angle, respectively, of the slope.

Fig. 2 Schematic diagram for the strength reduction technique in the Mohr stress space.

The strength reduction rate F^i is the safety factor when the slope is in a critical equilibrium state. Currently, many standards are employed to evaluate the critical state of a slope. However, while these criteria all have their respective scope of application, under normal circumstances, the use of different standards has little

impact on the ultimate safety factor. In this paper, the criterion based on the distribution of the plastic zone is adopted [56] to determine the critical equilibrium state of the slope.

Mohr-Coulomb failure and yield criteria are widely applied in geotechnical engineering. Experimental studies have shown that Mohr-Coulomb conditions are consistent with the yield and failure characteristics of geomaterials. Currently, many geotechnical engineering calculation and design schemes adopt the Mohr-Coulomb strength criterion. Therefore, this paper uses an ideal elastic-plastic model and the Mohr-Coulomb yield criterion:

$$\tau_n = c + \sigma_n \tan \varphi \quad (37)$$

or

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(\sigma_1 + \sigma_3) \sin \varphi + c \cos \varphi \quad (38)$$

The Mohr-Coulomb yield condition of an ideal elastic-plastic model can be expressed as

$$f = \frac{1}{3} \mathbf{I}_1(\sigma_{ij}) \sin \varphi - \left(\cos \theta_\sigma + \frac{1}{\sqrt{3}} \sin \theta_\sigma \sin \varphi \right) \sqrt{\mathbf{I}_2(S_{ij})} + c \sin \varphi = 0 \quad (39)$$

where θ_σ is the Lode angle.

According to Zheng et al. [56], when the FEM is used to calculate a safety factor, the strength parameters c and φ are reduced, and the soil body must satisfy the inequality

$$\sin \varphi \geq 1 - 2\nu \quad (40)$$

Simultaneously adjusting the stiffness parameters ν and E ensures that the obtained safety factors are similar to those acquired with the classic limit equilibrium method, and the distribution of the plastic zone is reasonable. The flowchart of the VEM strength reduction technique is shown in Figure 3.

Fig. 3 Flow chart for the VEM strength reduction technique.

5 Examples

5.1 Example 1: homogeneous soil slope

Similar to the FEM, the VEM is a Galerkin method that divides the study area

into a mesh. However, in contrast to the FEM, the VEM has a low-quality requirement for the mesh and can contain non-convex elements. This significantly reduces the mesh generation difficulty; moreover, the mesh of a complex geometric model can be divided more flexibly and conveniently with the VEM than with the FEM. Next, a model of quadrilateral or polygonal elements is used to compute the safety factor and plastic zone distribution of a homogeneous soil slope, and the results are compared with the ABAQUS calculation results to illustrate the reliability of the VEM.

Figure 4 shows the geometric model of the slope. The boundary condition is the constraint at the bottom of the model, and the normal constraints on the two sides are subjected only to gravity. The rock body parameters are listed in Table 1.

Fig. 4 Geometric parameters for Example 1: homogeneous soil slope (unit: m).

Figure 5 shows the results calculated by the ABAQUS quadrilateral mesh and by the VEM quadrilateral mesh or polygon mesh. Using different reduction coefficients, the obtained plastic zone distributions during the slope failure processes are generally consistent. We use the connection of the plastic zone as the standard for determining the ultimate slope safety, and the safety factors of all three models are 1.38. As the strength reduction coefficient increases, the area of the plastic zone becomes larger, and when the reduction coefficient reaches 1.38, the plastic zone is connected.

Fig. 5 Equivalent plastic strain contours of the homogeneous soil slope assessed with the VEM and FEM (different computational steps).

Example 2: Stony soil slopes

Figure 6 shows the dimensions of the geometric model of the soil-rock mixed slope. The discrete models of different rock contents (A, B and C) are shown in the first row of Figure 7, and the values of the material parameters are given in Table 2. Only the rock contents in models A, B and C are different. Model A has a total rock area of approximately 16.178 m², model B has a total rock area of approximately 27.624 m², and model C has a total rock area of approximately 48.0625 m². The boundary condition is the constraint at the bottom of the model, and the normal constraints on the two sides are subjected only to gravity.

Fig. 6 Geometric parameters for Example 2: stony soil slope (unit: m).

This example uses an irregular geometry (convex and concave polygons) to randomly generate the two-dimensional structure of a stony soil. The rock gradation of each stony soil model is shown in Table 3. For different strength reduction coefficients, the equivalent plastic strain contours corresponding to the homogeneous slope model are shown in Figure 7; the equivalent plastic strain contours corresponding to stony soil models A, B and C with different rock contents are also shown in Figure 7.

Fig. 7 Equivalent plastic strain contours of the stony soil slope assessed with the VEM (different area ratios of rock).

For the homogeneous slope model, Chen [57] obtained a safety factor of 1.0 by using the limit analysis method, while Zheng [56] obtained safety factors of 1.06 and 0.99 via the FEM and VEM, respectively, as shown in Figure 8. When strength reduction is performed on the soil body only, for model A, the plastic zone is connected when the reduction coefficient reaches 1.07, so the safety factor of stony soil slope A is considered to be 1.0. On the other hand, the contrastive analysis models of stony soil slope A from ABAQUS (element number is 1373) and VEM employed (element number is 1358) are built as shown in Figure 9. The strength reduction factor and the correspondent contour map of equivalent plastic strain are as also shown in Figure 9, which reveals that the results from ABAQUS and VEM employed are coincident. For model B, the plastic zone is connected when the reduction coefficient reaches 1.17, so the safety factor of stony soil slope B is considered to be 1.17. For model C, the plastic zone is connected when the reduction coefficient reaches 1.15, so the slope safety factor of stony soil slope C is considered to be 1.15.

Fig. 8 Equivalent plastic strain contours of the stony soil slope assessed with the VEM (the area ratio equals zero).

Fig. 9 The contrastive analysis of stony soil slope A (ABAQUS and VEM employed).

6 Discussion

A comparison of Figures 7-8 shows that due to the existence of rock blocks, the rock failure mode of the stony soil slope is significantly different from that of the

homogeneous slope. For a homogeneous slope (Fig. 8), the plastic zone is first formed at the foot of the slope and then extends from the foot of the slope to the top of the slope. In addition, the slip surface corresponding to the homogeneous slope is smooth and almost circular. For the soil-rock mixed slope (Fig. 8), the plastic zone is first formed at the foot of the slope. However, the plastic zone could also form in other areas of the stony soil slope, such as the area near the rock block. More importantly, it is very difficult to form a smooth slip surface from the foot of the stony soil slope to the top of the slope. Therefore, the slip surface area of the homogeneous slope is larger than that of the stony soil slope.

Rock blocks have certain positive effects on slope stability. The safety factor of the homogeneous slope (Fig. 8) is only 0.99; in contrast, as shown in Figure 8, the safety factor of model A is 1.07, that of model B is 1.17, and that of model C is 1.15. All of the safety factors of the stony soil slopes are better than that of the homogeneous slope. In addition, the safety factors of models B and C are even higher than that of model A.

The rock gradation and distribution influence the stability of a slope [58-59]. The rock gradation of model B is good, and the rock is distributed around the slip surface; therefore, the safety factor of model B is 1.17, which is higher than that of model C (rock area of 48.062 m²). Although the rock content of model C is higher than that of model B, the distribution of rock is relatively scattered, and the content of small stones is relatively high; these small stones do not play a positive role in slope sliding.

Furthermore, the displacement monitoring points are set at the crest of slope A, B and C. The horizontal displacements of these points from different results are shown in Figure 10. The displacements increase with the strength reduction factor, as we have seen, in the three models. However, the deformation is subjected to the distribution of rocks. The specific regularity of displacement affected by the rock distribution is statistical and complicated. Then there should be some further research.

Fig. 10 The contrastive analysis stony soil slope A from ABAQUS and VEM employed.

7 Conclusions

The VEM stiffness matrix is constructed using the projection operator on the element. The DOF of the node is directly used to solve for the element stiffness matrix. The mathematical foundation of the VEM is solid, its realization is simple, and the VEM has no dependence on the element shape. The strength reduction method and $\varphi - \nu$ inequality are used in the VEM to simulate the asymptotic instability of homogeneous and heterogeneous slopes; the results demonstrate that the VEM can overcome the poor adaptability of the FEM for complex problems.

In particular, the results of different examples show that the VEM can solve for the safety factors of homogeneous and heterogeneous slopes, and the obtained safety factors are in good agreement with those obtained by the FEM. This study verifies that the established technique can effectively handle any arbitrary polygon and can be used to investigate the effects of rock on slope stability.

This method can be used to determine the stability of a two-dimensional stony soil slope. In this example, a single rock represents an element (including a non-convex element), which can realistically simulate the diversity of morphology. The findings confirm that due to the differences in the physical and mechanical

properties of soil and rocks, under the action of gravity, stress becomes concentrated in the contact zone between soil and rocks, and thus, the plastic zone surrounds the rocks. Therefore, it is difficult to form regular connections in a plastic zone, such as a soil slope. In future work, the proposed method will be further improved to study the effects of particle size, rock content, rock density, and rock spatial distribution on the mechanical behaviour of stony soil slopes, which will provide a basis for the strengthening design of stony soil slopes (landslides).

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