Large Games with Countable Characteristics*

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Abstract

We consider anonymous games with an atomless probability space of players in which players' characteristics are countable. Our main result shows that the set of equilibrium distributions coincides with the set of distributions induced by equilibrium strategies together with the function assigning characteristics to players. This results implies the existence of Nash equilibria in continuous large games with countable characteristics.

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1 Introduction

Based on the work of Hildenbrand (1974) and Hart, Hildenbrand, and Kohlberg (1974), Mas-Colell (1984) formulated the equilibrium notion of large anonymous games in terms of distributions. In this way, he departed from the usual formalization of an equilibrium as a strategy, i.e., as a measurable function from players into actions. This allowed him to obtain an existence theorem easily and under general conditions.

Since an equilibrium distribution is a probability distribution over characteristics and actions, it does not tell us what actions each player takes in an equilibrium. Rather, and loosely, it tells us only the fraction of players with a certain characteristic that play a certain action. This is obviously in contrast with equilibrium strategies. Motivated by this, and despite the ease and generality that Mas-Colell's approach allows, researchers still investigate properties of equilibrium strategies (see the recent survey by Khan and Sun (2002)).

It is clear that the two approaches are related. In the concrete case of a game described by a measurable function from players into characteristics (i.e., payoff functions), this function together with an equilibrium strategy induces an equilibrium distribution of the game. Our main result shows that for games with countable characteristics the converse also holds. Thus, for such games, a distribution is an equilibrium distribution if and only if it is induced by a Nash equilibrium strategy together with the function assigning payoff function to the players.¹

¹A similar but weaker result holds when the action space is finite (see Rath (1995, Theorem 8) or when it is countable and the distribution over characteristics is atomless

An important consequence of our result is that it yields a general existence theorem for equilibrium strategies in games with an uncountable action space. Thus, in contrast to the view expressed in Khan, Rath, and Sun (1997), it is possible to extend the equilibrium theory for anonymous games beyond the case of countable action space. The extension we present requires, however, that players' characteristics be countable. Hence, our results stress the importance of countability assumptions for the existence of equilibrium strategies in games with a continuum of players, highlighted in Khan, Rath, and Sun (1997), but they show that those assumptions can be imposed either on the action space or on players' characteristics.

2 Definitions and the Main Results

Let A be a non-empty, complete and separable metric space of *actions* endowed with its Borel σ – algebra and $\mathcal{M}(A)$ be the set of Borel probability measures on A endowed with the weak convergence topology. By Parthasarathy (1967, Theorems II.6.2 and II.6.5), it follows that $\mathcal{M}(A)$ is a complete, separable metric space. Let $\mathcal{U}$ be a metric space of utility functions $u : A \times \mathcal{M}(A) \rightarrow \mathbb{R}$ endowed with its Borel σ – algebra. The set $\mathcal{U}$ represents the space of *players' characteristics*.

A *game with a continuum of players* is characterized by an atomless probability *space of players* (T, Σ, λ) and a measurable function $U : T \rightarrow \mathcal{U}$. We represent such game by $G = ((T, \Sigma, \lambda), U, A)$. A game G has *countable characteristics* (see Khan and Sun (1995a, Theorem 1)). The examples of Khan, Rath, and Sun (1997) show that these representation results do not extent beyond the case of either countable actions or countable characteristics.

acteristics if $U(T)$ is a countable subset of $\mathcal{U}$.

Let $\mathcal{U}_c$ denote the space of continuous utility functions $u : A \times \mathcal{M}(A) \rightarrow \mathbb{R}$ endowed with the supremum norm; it is a complete, separable metric space. A game G is *continuous* if A is compact and $\mathcal{U} = \mathcal{U}_c$.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginals of τ on $\mathcal{U}$ and A respectively. The expression $u(a, \tau) \geq u(A, \tau)$ means $u(a, \tau) \geq u(a', \tau)$ for all $a' \in A$.

Given a game G with a continuum of players, a Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution* for G if $\tau_{\mathcal{U}} = \lambda \circ U^{-1}$ and $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$.

We will use the following notation: $B_{\tau} = \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}$. For a general game G with a continuum of players, an implicit requirement for τ to be an equilibrium distribution is that B_{τ} is measurable. Note that if G is continuous, then B_{τ} is closed, and so a Borel set; hence $\tau(B_{\tau})$ is well defined for all distributions $\tau \in \mathcal{M}(\mathcal{U}_c \times A)$.

A *pure strategy* is a measurable function $f : T \rightarrow A$. A pure strategy f is a *Nash equilibrium* of G if $U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})$ for almost all $t \in T$.

For games with countable characteristics we obtain a strong representation theorem: the set of equilibrium distributions coincides with the set of distributions induced by Nash equilibria and the function U describing the game.

Theorem 1 *Let G be a game with countable characteristics. If τ is an equilibrium distribution of G , then there exists a Nash equilibrium $f : T \rightarrow A$ such that $\tau = \lambda \circ (U, f)^{-1}$.*

As noted by Khan and Sun (1999, p. 474), the conclusion of Theorem 1 is valid when the space of players is a hyperfinite Loeb counting space even without the restriction of countable characteristics. In contrast, the strength of Theorem 1 is that it holds for all atomless probability spaces of players.

The idea of the proof of Theorem 1 is simple: we define as many probability measures over the action space as the number of possible characteristics. Each such measure describes the distribution of choices made by players with a given characteristic. A modified version of Skorokhod's theorem allows us to represent any such measure by a function defined on the set of players with the corresponding characteristic. Then, it is easy to see that these functions define the equilibrium strategy.

Proof of Theorem 1. Let G be a game with countable characteristics and τ be an equilibrium distribution of G . We claim that there exists a measurable function $f : T \rightarrow A$ such that $\tau = \lambda \circ (U, f)^{-1}$.

The proof of the above claim is as follows: Let $U(T) = \{u_i\}_{i=1}^\infty$. For each $i \in \mathbb{N}$, let $T_i = U^{-1}(\{u_i\})$, $c_i = \lambda(T_i)$ and, if $c_i > 0$, $\lambda_i(E) = \lambda(E)/c_i$ for all measurable subsets E of T_i . Furthermore, again if $c_i > 0$, define $\tau_i(D) = \tau(\{u_i\} \times D)/c_i$ for all measurable subsets D of A . Then, there exists a measurable function $f_i : T_i \rightarrow A$ such that $\tau_i = \lambda_i \circ f_i^{-1}$.²

Define $f : T \rightarrow A$ by $f(t) = f_i(t)$ if $t \in T_i$ and $c_i > 0$, while if $t \in T \setminus (\cup_{i:c_i>0} T_i)$, we define $f(t)$ arbitrarily. Note that $\lambda(T \setminus (\cup_{i:c_i>0} T_i)) = 0$ and so f is measurable.

²The existence of such f_i can be obtained by appropriately modifying the proof of Skorokhod (1956, Theorem 3.1.1, p. 281). We need only to redefine the sets $\Delta_{i_1, \dots, i_k}$, using the fact that T_i is atomless to guarantee that $\lambda_i(\Delta_{i_1, \dots, i_k}) = \tau_i(S_{i_1, \dots, i_k})$, where the sets $S_{i_1, \dots, i_k}$ are as in Skorokhod's proof.

Let B be a measurable subset of $\mathcal{U} \times A$ and define $B_i = \{a \in A : (u_i, a) \in B\}$ for all $i \in \mathbb{N}$. Note that $\{u_i\} \times B_i = (\{u_i\} \times A) \cap B$ and so, in particular, $\{u_i\} \times B_i$ is measurable. Also, note that $\tau(B) = \tau((U(T) \times A) \cap B)$ and $\lambda \circ (U, f)^{-1}(B) = \lambda \circ (U, f)^{-1}((U(T) \times A) \cap B)$. Then,

$$\begin{aligned} \tau(B) &= \tau((U(T) \times A) \cap B) = \sum_i \tau(\{u_i\} \times B_i) \\ &= \sum_i c_i \tau_i(B_i) = \sum_i c_i \lambda_i \circ f_i^{-1}(B_i). \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} \lambda \circ (U, f)^{-1}(B) &= \lambda(\{t \in T : (U(t), f(t)) \in B\}) \\ &= \sum_i \lambda(\{t \in T : U(t) = u_i \text{ and } f(t) \in B_i\}) = \sum_i \lambda(\{t \in T_i : f_i(t) \in B_i\}) \tag{2} \\ &= \sum_i c_i \lambda_i(\{t \in T_i : f_i(t) \in B_i\}) = \sum_i c_i \lambda_i \circ f_i^{-1}(B_i). \end{aligned}$$

Thus, it follows that $\tau = \lambda \circ (U, f)^{-1}$.

To finish the argument, we claim that f is a Nash equilibrium of G . As is easily seen, we have that $(U, f)^{-1}(B_\tau) = \{t \in T : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}$. Since $\tau = \lambda \circ (U, f)^{-1}$ is an equilibrium distribution, then $\lambda((U, f)^{-1}(B_\tau)) = 1$ and so $\lambda(\{t \in T : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}) = 1$. Thus, f is a Nash equilibrium. ■

Theorem 1 has an important consequence when applied to continuous games. If G is a continuous game, then, by Mas-Colell (1984, Theorem 1), G has an equilibrium distribution τ . Moreover, if G is also a game with countable characteristics, then $\tau = \lambda \circ (U, f)^{-1}$ and f is a Nash equilibrium of G . Therefore, G has a Nash equilibrium. Thus, Theorem 1 implies the existence of Nash equilibria for continuous games with a continuum of players and countable characteristics.

Corollary 1 *A Nash equilibrium exists for all continuous games with countable characteristics.*

We note that this result parallels Theorem 10 in Khan and Sun (1995b), which is obtained with a countable action space but without any restriction on characteristics (besides measurability).³ Similarly, by replacing the assumption of countable action by that of countable characteristics, we can obtain results that correspond to Theorem 1, Theorem 2 and Corollary 3 of Khan, Rath, and Sun (1997).

Furthermore, it follows from Khan and Sun (1999, Theorem 1) that the restriction of countable characteristics is not needed for the conclusion of Corollary 1 when the space of players is a Loeb space.

We emphasize that for Corollary 1, as well as for Theorem 1 in Khan and Sun (1999), the action space can be uncountable. The difference is that our result, by imposing the assumption of countable characteristics, dispenses with the use of a particular type of players' spaces. In this way, it is the first result on the existence of an equilibrium strategy for games with an arbitrarily atomless probability space of players and an action space that is just a compact metric space.

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³Theorem 10 in Khan and Sun (1995b) is more general in that it allows for players' externality to depend on the action distribution of a partition of players.

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