Modeling the Bistability of Laminated Composite Toroidal Slit Tubes

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The bistability of a toroidal slit tube is modeled using the Rayleigh-Ritz method. Approximate explicit expressions for the original stable deployed geometry, and the deformed stowed geometry are used to derive forms for the bending and stretching strain energy. The surface of a torus has varying Gaussian curvature, requiring a new approach to the modeling and analysis of the stable configurations. A comparative study with established straight-BRC models was conducted from which the doubly curved-BRC model presented here predicts second stable state coil radii with 96.25% agreement.

I. Introduction

THE invention and investigation of bistable reeled composites¹ (BRCs) has led to many valuable applications in the sectors of security, civil engineering, energy, mining, consumer, defense and aerospace. BRCs are open-section tubular structures that can be rolled up and extended, analogous to tape measures, but are stable in both configurations without the need of support structures. Bistability arises due to the arrangement of fibers in each ply² through an antisymmetric lay-up scheme with respect to the mid-plane which results in a preferential direction for bending to occur³ and theoretically may be exhibited using any material of high Poisson ratio and sufficient stiffness⁴.

Various approaches, modeling and analyses of shells⁵ and BRCs over the years include finite element analysis⁶ to predict and characterize the rolled-up configuration and beam models^{7,8,9} to predict second equilibrium position and stability. Progressing to a shell model¹⁰ enabled the study of edge effects and an inextensional bending model¹¹ utilized a property of developable surfaces - the Gaussian curvature¹² remains zero everywhere - to predict coiled radius by modeling deformation of a plate superimposed upon the surface of a great cylinder.

Figure 1: A torus-BRC rendering. Coiled/stowed (left) to deployed/extended (right)

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In this paper, we begin investigating the doubly curved torus-BRC as illustrated in Figure 1 which until now has only been briefly discussed⁴.

A. Geometric Model

Typically, a complete torus is represented using two radii, R and r to describe major and minor circles respectively as shown in Figure 2. The torus is an example of a surface of revolution whereby the cross-section is swept through an angle about an axis of revolution, resulting in a 3D solid or shell. We consider the point P on the torus surface with longitudinal and transverse principle curvatures $\kappa_{x_0} = \frac{1}{R+r}$ and $\kappa_{y_0} = \frac{1}{r}$. A particular property of the torus is the Gaussian curvature, $K_g = \kappa_x \kappa_y$ which **varies** along the cross-section - negative maxima closest to the axis of revolution (at $R - r$ distance), positive maxima at point P and zero along the top and bottom (at R distance).

We define the local, on-surface longitudinal and transverse directions with curvilinear co-ordinate axes x - y respectively, within the global X-Y-Z Cartesian co-ordinate system in which the torus lies with equatorial width $\in (-(R+r), R+r)$ along the X-,Y-axes and height $\in (-r, r)$ along the Z-axis.

Figure 2: A torus is defined by major (R) and minor (r) circles

The initial, deployed torus cross-section, $\mathbf{T}_0(s)$ is modeled as:

$$\mathbf{T}_0(s) = x_0(s)\hat{\mathbf{e}}_X + z_0(s)\hat{\mathbf{e}}_Z \quad (1)$$

where,

$$x_0(s) = \frac{1}{\kappa_{x_0}} - \frac{1}{\kappa_{y_0}} + \frac{1}{\kappa_{y_0}} \cos(s \kappa_{y_0}) \quad (2)$$

$$z_0(s) = \int_0^{\pm s} \sqrt{1 - \left(\frac{dx_0}{ds}\right)^2} ds \quad (3)$$

and $s \in \left(-\frac{\beta_{y_0}}{2\kappa_{y_0}}, \frac{\beta_{y_0}}{2\kappa_{y_0}}\right)$ are points along the cross-section of circumference or width, $w_0 = \frac{\beta_{y_0}}{\kappa_{y_0}}$, the transverse subtended angle is β_{y_0} , longitudinal and transverse principle curvatures κ_{x_0} and κ_{y_0} and point P is located at $x_0(s = 0)$.

Simulating deformation of the torus slit tube is achieved using the terms c_1 and c_2 to denote change in the major and minor circle radii respectively, shown in Figure 3, enabling use of the Rayleigh-Ritz method. The new geometry, $\mathbf{T}(s)$ can be expressed as:

$$\mathbf{T}(s) = x(s)\hat{\mathbf{e}}_X + z(s)\hat{\mathbf{e}}_Z \quad (4)$$

where,

$$\mathbf{T}(s) = [x_0(s) + \Delta x(s)] \hat{\mathbf{e}}_X + z(s) \hat{\mathbf{e}}_Z \quad (5)$$

$$\Delta x(s) = - \underbrace{c_1}_a - \underbrace{\frac{1}{\kappa_{y0}} \cos(s \kappa_{y0})}_b - \underbrace{\frac{c_2}{\kappa_{y0}}}_c + \underbrace{\frac{1+c_2}{\kappa_{y0}} \cos\left(\frac{s \kappa_{y0}}{1+c_2}\right)}_d \quad (6)$$

$$z(s) = \int_0^{\pm s} \sqrt{1 - \left(\frac{dx}{ds}\right)^2} ds \quad (7)$$

Figure 3: Deformation of the slit tube cross-section due to c_1 and c_2

Eq. 6 is constructed such that:

- a , defines the X-translation effect of coiling/roll up
- b , cancels with the last term in $x_0(s)$ from Eq. 2 describing the initial cross-section's X-component in preparation for term d
- c , negates X-translation of the minor circle's centre-point due to new radius
- d , describes the new, deformed, circular cross-section geometry with radius, $\frac{1}{\kappa_y}$

When $c_1 < 0$, the torus undergoes stowage/coiling, shown in Table 1. c_2 is the most effective term deforming the geometry with respect to principle curvatures; when $0 < c_2$, both the longitudinal and transverse curvature change and describes flattening of the torus cross-section. The effects of c_1 and c_2 are also illustrated in Figure 4 - β_x describes longitudinal stretching and will be made use of later in Eq. 21 for stretching strain energy calculation.

	$c_1 < 0$	$c_1 = 0$
$c_2 = 0$	roll-up	no change
$0 < c_2$	flatten and roll-up	flatten

Table 1: Deformation effects of c_1 and c_2 terms

Given Eqs. 2, 6 and 7, Eq. 5 may be presented as,

$$\mathbf{T}(s) = \begin{bmatrix} x(s) \\ 0 \\ z(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{\kappa_{x0}} - \frac{1}{\kappa_{y0}} - c_1 - \frac{c_2}{\kappa_{y0}} + \frac{1+c_2}{\kappa_{y0}} \cos\left(\frac{s \kappa_{y0}}{1+c_2}\right) \\ 0 \\ \int_0^{\pm s} \sqrt{1 - \left(\frac{dx}{ds}\right)^2} ds \end{bmatrix} \quad (8)$$

Figure 4: Effects of the deformations c_1 , c_2 and β_x

B. Principle Curvatures Calculation

The longitudinal curvature⁵, $\kappa_x(s)$ varies along the cross-section as seen in Figure 5. The longitudinal radius of curvature, $d(s)$ is defined as a straight line through the minor circle center, intersecting a point on the cross-section and axis of revolution. $d(s) \perp ds$ over the cross-section. The transverse curvature, κ_y does **not** vary with arc-length, s and therefore describes the arc of a great circle.

$$\kappa_x(s) = \frac{1}{d(s)}, \quad \kappa_y = \frac{|\mathbf{T}'(s) \times \mathbf{T}''(s)|}{|\mathbf{T}'(s)|^3} \quad (9)$$

where,

$$d(s) = \frac{x(s)}{\cos(\alpha)} \quad (10)$$

We consider the curvature at each point along the cross-section and follow a geometric approach inset in Figure 5 for an arbitrary point Q .

Figure 5: Definition for the longitudinal radius of curvature, $d(s)$ with the elements at each surface point an arc-distance, s from the X-axis

where,

$$\alpha = \arcsin\left(\frac{dx}{ds}\right) \quad (11)$$

and substituting into Eq. 10 yields,

$$d(s) = \frac{x(s)}{\cos\left(\arcsin\left(\frac{dx(s)}{ds}\right)\right)} = \frac{x(s)}{\sqrt{1 - \left(\frac{dx(s)}{ds}\right)^2}} \quad (12)$$

Given Eq. 12, $\kappa_x(s)$ from Eq. 9 simply becomes:

$$\kappa_x(s) = \frac{\sqrt{1 - \left(\frac{dx(s)}{ds}\right)^2}}{x(s)} \quad (13)$$

Similarly for the initial longitudinal curvature:

$$\kappa_{x_0}(s) = \frac{\sqrt{1 - \left(\frac{dx_0(s)}{ds}\right)^2}}{x_0(s)} \quad (14)$$

κ_y from Eq. 9 is determined using Eq. 8. It follows that κ_y may be expressed as,

$$\kappa_y = \pm \frac{\frac{d^2x(s)}{ds^2}}{\sqrt{1 - \left(\frac{dx(s)}{ds}\right)^2}} \quad (15)$$

It can be seen that the Gaussian curvature is positive along the outermost cross-section edge (when $\frac{dx(s)}{ds} < 0$) and negative along the innermost edge (when $0 < \frac{dx(s)}{ds}$) given the expressions for longitudinal curvature from Eqs. 13 and 14. The transverse curvature may be positive or negative as a result of the mathematical procedure in Eq. 15, but shall remain positive over all deformations modeled given that the cross-section converges to flattening and no further.

C. Torus Strain Energy Model: Bending & Stretching Strain

The strain energy model used here extends past models¹, to take into account variable curvature and stretching along the cross-section of the tube. The bending and stretching energy per unit area¹³ are defined as,

$$u_b(s) = \frac{1}{2} \begin{bmatrix} \Delta\kappa_x(s) & \Delta\kappa_y & \Delta\kappa_{xy}(s) \end{bmatrix} [\mathbf{D}] \begin{bmatrix} \Delta\kappa_x(s) \\ \Delta\kappa_y \\ \Delta\kappa_{xy}(s) \end{bmatrix} \quad (16)$$

$$u_s(s) = \frac{1}{2} \begin{bmatrix} \epsilon_x(s) & \epsilon_y & \epsilon_{xy}(s) \end{bmatrix} [\mathbf{A}] \begin{bmatrix} \epsilon_x(s) \\ \epsilon_y \\ \epsilon_{xy}(s) \end{bmatrix} \quad (17)$$

where change in principle curvature, $\Delta\kappa$ and stretching, ϵ_x are defined respectively as,

$$\begin{aligned} \Delta\kappa &= \kappa_{deformed} - \kappa_{initial} \\ &= \kappa - \kappa_0 \end{aligned} \quad (18)$$

$$\epsilon_x(s) = \frac{l_{deformed} - l_{initial}}{l_{initial}} = \frac{l(s) - l_0(s)}{l_0(s)} \quad (19)$$

where,

$$l_0(s) = \beta_{x_0} x_0(s) \quad (20)$$

$$l(s) = \beta_x x(s) \quad (21)$$

$$w_0 = \beta_{y_0} R_{y_0} \quad (22)$$

are the initial lengths, $l_0(s)$ and deformed lengths, $l(s)$ of imaginary parallel longitudinal “fibers” and initial cross-section circumference, w_0 (which remains constant due to transverse stretching, ϵ_y being neglected). β_{x_0} is the initial longitudinal subtended angle and β_x is the deformed longitudinal subtended angle generally resulting in longitudinal stretching, illustrated in Figure 4.

It follows that changes in longitudinal and transverse curvature are:

$$\Delta\kappa_x(s) = \frac{\sqrt{1 - \left(\frac{dx(s)}{ds}\right)^2}}{x(s)} - \frac{\sqrt{1 - \left(\frac{dx_0(s)}{ds}\right)^2}}{x_0(s)} \quad (23)$$

$$\Delta\kappa_y = \frac{\frac{d^2x(s)}{ds^2}}{\sqrt{1 - \left(\frac{dx(s)}{ds}\right)^2}} - \frac{1}{R_{y_0}} \quad (24)$$

Given $\kappa_{xy} = \epsilon_{xy} = 0$, $D_{13} = D_{23} = 0$ and the plate has initial curvature $\kappa_{x_0}(s)$ and κ_{y_0} , the bending energy per unit area from Eq. 16 becomes:

$$\begin{aligned} u_b(s) &= \frac{1}{2} \left[D_{11} \Delta\kappa_x^2(s) + 2D_{12} \Delta\kappa_x(s) \Delta\kappa_y + D_{22} \Delta\kappa_y^2 \right] \\ &= \frac{1}{2} \left[D_{11} [\kappa_x(s) - \kappa_{x_0}(s)]^2 \right. \\ &\quad \left. + 2D_{12} [\kappa_x(s) - \kappa_{x_0}(s)] [\kappa_y - \kappa_{y_0}] \right. \\ &\quad \left. + D_{22} [\kappa_y - \kappa_{y_0}]^2 \right] \end{aligned} \quad (25)$$

and given Eqs. 19, 20 and 21, the stretching energy per unit area from Eq. 17 becomes:

$$\begin{aligned} u_s(s) &= \frac{1}{2} A_{11} \epsilon_x^2(s) \\ &= \frac{1}{2} A_{11} \left[\frac{l(s) - l_0(s)}{l_0(s)} \right]^2 \end{aligned} \quad (26)$$

The bending energy per unit length is:

$$\begin{aligned} U_b &= \iint_{A_0} u_b(s) \, dA \\ &= 2 \int_0^{l_0} \int_0^{\frac{w_0}{2}} u_b(s) \, ds \, dl \\ &= 2 \int_0^{\frac{w_0}{2}} u_b(s) \cdot l_0(s) \, ds \end{aligned} \quad (27)$$

Similarly, the stretching energy per unit length is:

$$\begin{aligned} U_s &= \iint_A u_s(s) \, dA \\ &= 2 \int_0^l \int_0^{\frac{w_0}{2}} u_s(s) \, ds \, dl \\ &= 2 \int_0^{\frac{w_0}{2}} u_s(s) \cdot l(s) \, ds \end{aligned} \quad (28)$$

The integral limits for bending (Eq. 27) and stretching energy (Eq. 28) differ because stretching is **not** considered for the bending energy, therefore integration is executed over the initial area, $A_0 = 2 \int_0^{l_0} \int_0^{\frac{w_0}{2}} ds dl$. Locally stable configurations are defined as,

$$\left[\frac{d}{dc_1}, \frac{d}{dc_2}, \frac{d}{d\beta_x} \right] U_t = 0 \quad (29)$$

where $U_t = U_b + U_s$ is the total strain energy.

Figure 6 illustrates the approach presented beginning with a deployed torus, $\mathbf{T}_0(s)$ where induced deformations c_1 and c_2 result in a deformed torus geometry $\mathbf{T}(s)$. The initial surface curvature is then subtracted from the deformed surface curvature to determine the change in curvature at every point along the cross-section, $\Delta\kappa(s)$ and bending energy per unit area, $u_b(s)$. The strain is calculated using the initial and deformed tube lengths including longitudinal stretching due to β_x - leading to the stretching energy per unit area, $u_s(s)$. The bending and stretching energy per unit area are then integrated across the surface area of the deformed torus and summed together to find the total strain energy per unit length, $U_t(c_1, c_2, \beta_x)$ - the total strain energy has units Jm^{-1} due to the surface integral limits for the tube length being zero to unity (1m). The total strain energy is then plotted in order to find local minima at which point corresponds to a specific set of values for c_1 , c_2 and β_x which describe the second stable state geometry.

Figure 6: Flow chart of the method presented

II. Results

The modeled laminated composite toroidal slit tube has unidirectional ply properties in Table 2.

Property	Value	Units
Fiber modulus	240×10^9	Pa
Fiber shear modulus	95×10^7	Pa
Fiber Poisson ratio	0.22	-
Matrix modulus	4×10^9	Pa
Matrix shear modulus	2.7×10^9	Pa
Matrix Poisson ratio	0.35	-
Fiber volume fraction	$\frac{1}{2}$	-
Unidirectional layer thickness	0.213×10^{-3}	m
Number of plies	5	-

Table 2: Unidirectional ply mechanical properties

With **ABD**-matrix for an anti-symmetric 45° layup, $[+45^\circ, -45^\circ, 90^\circ, +45^\circ, -45^\circ]$ calculated^{1, 2, 4, 11} :

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 3.49 \times 10^7 & 2.48 \times 10^7 & 0 & 0 & 0 & -2.60 \times 10^3 \\ 2.48 \times 10^7 & 5.94 \times 10^7 & 0 & 0 & 0 & -2.60 \times 10^3 \\ 0 & 0 & 2.80 \times 10^7 & -2.60 \times 10^3 & -2.60 \times 10^3 & 0 \\ 0 & 0 & -2.60 \times 10^3 & 3.90 & 2.85 & 0 \\ 0 & 0 & -2.60 \times 10^3 & 2.85 & 3.99 & 0 \\ -2.60 \times 10^3 & -2.60 \times 10^3 & 0 & 0 & 0 & 3.15 \end{bmatrix} \quad (30)$$

where **A**, **B** and **D** have units Pa m, Pa m² and Pa m³ respectively.

Figure 7: Strain energy contour plot using Iqbal's model.¹ The curvature is in terms of R_0

A. Straight-BRC Model

The straight-BRC modeled has initial geometry of $R_{y0} = 3.5\text{cm}$ and $\beta_{y0} = \pi$. Figure 7 predicts two stable states at $\kappa_{x0} = 0, \kappa_{y0} = 1$ and $\kappa_x \approx 0.73, \kappa_y = 0$. This equates to a predicted coil radius of approximately 2.56cm and for a BRC of length 1m, the coil subtended angle is approximately 39 radians or 6.2 turns. This value, $\beta_x \approx 39^\circ$ narrows the search for where the second stable point is expected to be in the curved-BRC model results.

B. Doubly Curved-BRC Model

To validate the doubly curved model, toruses with very small longitudinal curvature were compared to straight-BRCs. Predictions for the coil radii of straight- and shallowly doubly curved-BRCs are presented in Table 3. For the curved-BRC modeling results, initial longitudinal curvature was gradually decreased in order to investigate the hypothesized convergence to the straight-BRC coil radius result.

The most precise result in Table 3, 2.66cm for Curved- (5) compares very well to the Straight-BRC prediction of 2.56cm and lies within approximately 3.75%.

BRC Model	κ_{x0} (m^{-1})	R_x (cm)	l (m)	β_x (radians)	κ_y (m^{-1})
Straight-	0	2.56 ± 0.32	1	39.74 ± 4.9	0
Curved- (1)	0.3142	2.38 ± 0.13	1.0061	≈ 28.57	less than 0.468
Curved- (2)	0.1571	2.38 ± 0.13	1.0004	≈ 28.57	less than 0.468
Curved- (3)	0.0393	2.62 ± 0.08	1.0069	≈ 38.42	less than 0.468
Curved- (4)	0.0196	2.62 ± 0.08	1.0069	≈ 38.42	less than 0.468
Curved- (5)	0.0196	2.66 ± 0.02	1.0062	37.90 ± 1.78	less than 0.468

Table 3: BRC model prediction comparison for the second stable state coil radius of a 1m long tube with cross-section radius 3.5cm and subtended angle π

The total strain energy, $U_t = U_b(c_1, c_2) + U_s(c_1, c_2, \beta_x)$ is presented in Figure 8 and plotted with respect to independent variables c_1 , c_2 and β_x .

Figure 8: Total strain energy surface plot over all deformations through c_1 - c_2 - β_x space. ($0 \leq c_1 \leq (R_{x_0} - 0.001\text{m})$, $0 \leq c_2 \leq 60$ and $\beta_{x_0} \leq \beta_x \leq 100^\circ$ where $R_{x_0} \approx 25.4\text{m}$ and $\beta_{x_0} = \pi$)

Figure 9: A more detailed analysis of the strain energy surface plot from Figure 8 in the region; $(R_{x_0} - 0.05\text{m}) \leq c_1 \leq (R_{x_0} - 0.001\text{m})$, $0 \leq c_2 \leq 60$ and $\beta_{x_0} \leq \beta_x \leq 100^\circ$ where $R_{x_0} \approx 25.4\text{m}$ and $\beta_{x_0} = \pi$. Strain energy surface plot showing a low energy channel appearing to form a qualitative $c_1 = \beta_x^2$ -path and from a different perspective (right), the plot showing the low energy channel continues through c_2

Figure 10: Energy surface plots against β_x - c_1 at $c_2 = 15$ from Figure 9. The maximum energy is limited to 10^9 Jm^{-1} (left) and the energy valley is easily observed now the maximum energy is limited to 10^7 Jm^{-1} (right)

Modeling of highly curved toroidal-BRCs has been conducted and predicts bistability with the second stable state coil radii presented in Table 4 and Figure 11 - coil radius predictions for toroidal-BRCs with ascending initial longitudinal curvature. All of the toroidal-BRCs modeled have initial longitudinal centerline “fiber” length, $l_0(s = 0) = 1\text{m}$ transverse radius of curvature, $R_{y_0} = 3.5\text{cm}$ and transverse subtended angle, $\beta_{y_0} = \pi$.

β_{x_0} (radians)	κ_{x_0} (m^{-1})	$R_x \pm 0.07$ (cm)	$\beta_x \pm 0.63$ (radians)	$\kappa_y \pm 0.33$ (m^{-1})
$\frac{\pi}{4}$	0.79	5.84	17.16	1.36
$\frac{\pi}{3}$	1.05	5.43	18.42	1.36
$\frac{\pi}{2}$	1.57	4.78	20.79	1.36
$\frac{3\pi}{4}$	2.36	2.84	34.84	1.36
π	3.14	2.64	37.47	1.36
$\frac{5\pi}{4}$	3.93	2.29	42.90	1.36
$\frac{3\pi}{2}$	4.71	2.10	45.90	1.36
$\frac{7\pi}{4}$	5.50	1.60	60.26	1.36

Table 4: The predicted second stable state coil radii for toroidal-BRCs with ascending initial longitudinal curvature

Figure 11: The predicted second stable state coil radius versus initial longitudinal curvature for toroidal-BRCs

Figure 12: Initial longitudinal curvature, $\kappa_{x_0}(s)$ vs. cross-section, s (left) and initial transverse curvature, κ_{y_0} vs. cross-section, s (right)

C. The Geometric Model

Modeling doubly curved-BRCs with initially non-zero transverse and longitudinal curvature and subtended angle has been performed with a verification presented in Figure 12. Figure 12 (left) presents the initial longitudinal curvature, κ_{x_0} along the cross-section of a fully deployed i.e. stable toroidal-BRC. As expected, κ_{x_0} varies between extrema of zero along the edges and $\left(\frac{1}{R_{x_0} + R_{y_0}}\right)$ along the centerline of the BRC. Figure 12 (right) presents the uniform transverse curvature across the circular cross-section.

Figure 13 illustrates the effects of simulating deformation, c_1 and c_2 on the longitudinal and transverse curvatures. Top-left in the figure simulates the effect of rolling up only on the longitudinal curvature, when $c_1 < 0$, $c_2 = 0$. Top-right in the figure shows the independence between only rolling up and the transverse curvature. Bottom-left in the figure simulates the effect of flattening only on the longitudinal curvature, where $c_1 = 0$ and $0 < c_2$. Bottom-right in the figure simulates flattening only and its effect on transverse curvature.

Figure 13: Longitudinal and transverse curvature plotted against cross-section after deformation due to c_1 and c_2 ; top-left) $\kappa_x(s)$ for c_1 top-right) κ_y for c_1 bottom-left) $\kappa_x(s)$ for c_2 and bottom-right) κ_y for c_2

D. Modeling Bistability with Non-Uniform Transverse Curvature

Removing the assumption of uniform transverse curvature is achieved by adding higher frequency cosine terms in Eq. 6 - higher order polynomial terms are appropriate also. As a result, the second stable state cross-section of several toroidal-BRCs with various initial longitudinal radii of curvature, R **ABD**-matrix in Eq. 30, minor radius $R_{y_0} = 1.6\text{cm}$, transverse subtended angle, $\beta_{y_0} = 160^\circ$ and thirteen parameters (including c_1 , c_2 and β_x) are modeled and presented in Figure 14.

Figure 14: The modeled doubly curved-BRC cross-section for several values of initial longitudinal radius of curvature, $R(\text{m})$ illustrating the physical result of modeling non-uniform transverse curvature

III. Discussion

The method used in this paper is presented in Section I and outlines a mathematical approach treating doubly curved-BRCs as a torus. This approach allows the modeling of non-developable surfaces (where $K_g \neq 0$), surfaces with variable Gaussian curvature and specifically BRCs with initially non-zero longitudinal and transverse curvature. Given that this approach considers surfaces with variable Gaussian curvature, computation is akin to finite element modeling and requires long computation times to produce results with good accuracy and precision.

In order to verify the doubly curved-BRC model developed, a comparison with other existing models for straight-BRC predictions for the second stable state is performed and presented in Table 3. This is achieved with the curved-BRC model by setting the initial longitudinal curvature, κ_{x_0} to be very small i.e. very large radius of curvature, R_{x_0} and very small longitudinal subtended angle, β_{x_0} . Using this approach, it is hypothesized that the curved-BRC model will converge to approximately predict the second stable state for a straight-BRC.

The effects of c_1 and c_2 on the geometry of the BRC have been established as rolling up and flattening respectively (Table 1). The effect of changing β_x is to alter the longitudinal subtended angle and therefore the length of the BRC consequently inducing longitudinal stretching, ϵ_x .

In Figure 8, there are regions for which certain values of c_1 , c_2 and β_x equate to relatively low strain energy - enclosed by blue surface contours with yellow and red indicating ascending strain energy surfaces. A “slab” of relatively low energy extends across c_2 and β_x at very high c_1 - this region corresponds to geometry resembling a coil.

A more precise investigate of the aforementioned region is shown in Figure 9 exhibiting a low strain energy channel with a general shape resembling $c_1 = \beta_x^2$ across c_2 . The open gaps in the plot are trivial and caused by the data intervals used to produce the data and do not reflect the strain energy which is continuous throughout the c_1 - c_2 - β_x volume. A local minima is highlighted in Figure 9.

Given that the relatively low strain energy region in Figure 9 continues uniformly through values of c_2 ,

this shows that flattening is least influential on the strain energy in this example and so we can analyze the energy landscape using a slice in β_x - c_1 space, presented in Figure 10. Upon further analysis, by decreasing the energy limit from 10^9 Jm^{-1} in Figure 10 (left) to 10^7 Jm^{-1} in Figure 10 (right), the low strain energy profile becomes visible. A valley, the same as in Figure 9 is observed implying that particular pairings of c_1 and β_x values are energetically favorable during deformation of the surface. This observation is intuitive given that coiling up a BRC i.e. decreasing the longitudinal radius of curvature which the term c_1 simulates, will result in increasing the number of coil turns i.e. increasing the longitudinal subtended angle which the term β_x represents.

For validation and direct comparison with established straight-BRC models, the initial geometric conditions of the doubly curved-BRC model may be adjusted to converge to straight-BRCs by using infinite longitudinal radius of curvature, $R_{x_0} \approx \infty$. This was conducted and presented in Section II with coil radius predictions that agree to approximately 3.75% with the straight-BRC modeling.

Figure 14 presents the physical result of modeling non-uniform transverse curvature via an alternative computation method using minimization. Adding (ten) higher order cosine terms in Eq. 6 to describe the cross-section shape was found to result in lower energy second stable states with cross-sections that resemble those of Galletly's shell model¹⁰ where a short "boundary layer" is predicted to form at the edges. For the BRC parameters modeled, the predicted coil radii are in good agreement with the straight-BRC model which predicts 2.22cm. Increasing the initial longitudinal radius of curvature (labeled R in Figure 14) results in convergence of the doubly curved-BRC model to the straight-BRC result.

IV. Conclusion

The past methods used to investigate bistability were categorized and chronologically ordered as the simple beam, extended beam, shell and inextensional models. These methods have progressively improved our understanding and analysis of bistability by considering more parameters and effects, however, they do not account for non-zero initial longitudinal curvature and to do so requires an alternative approach. The concept of Gaussian curvature helps define two groups of surfaces; developable and non-developable surfaces or rather, surfaces which can be flattened onto a plane and those that cannot unless stretched or torn. This property of surfaces defines the major departure between preceding BRC research that focussed on developable surfaces and the work presented in this report investigating non-developable surfaces e.g. toroidal-BRCs.

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