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Abstract

Uncertainty propagation has been addressed extensively in space missions around the Earth, but much less for missions around small solar system bodies. Small bodies usually have irregular and weak gravity and our knowledge of their gravity, rotation speed and surrounding space environment is largely uncertain. These characteristics make the orbit propagation around these small bodies a challenging task. Focusing on the uncertainty of the small body's gravity, this paper applies the differential algebra (DA) technique to study the orbit propagation problem, and addresses its efficiency for a given the required accuracy. Different from traditional studies that focus on the uncertainty of the initial state, this study assumes an exact initial state and studies the influences that gravity model uncertainties have on the orbit. Taking the asteroid Steins as an example, the accuracy and efficiency of the DA approach are firstly validated by comparison with the traditional Monte Carlo method. Then, the effects of gravity uncertainties on different types of orbits (prograde, retrograde and polar) are studied. The retrograde motion is found to be more robust to the gravity uncertainty than the prograde ones. For near polar orbits, the impact of gravity uncertainty on orbital motion depends significantly on the initial position, and it reaches the maximum if the initial position is near the polar regions. Moreover, short-term effects are found to play an important role in orbit deviation as a result of the gravity uncertainty. These discoveries can help mission designers assess the posed risk and design appropriate mission orbits.

Keywords	small solar system body, uncertain gravity field, state propagation, differential algebra, sensitivity analysis
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Submission Files Included in this PDF

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Cover Letter

Dear Editor,

I'm Jinglang Feng. I got my PhD from Aerospace Engineering of Delft University of Technology, Netherlands, with the efforts on innovatively modelling and solving dynamics and control problems in highly nonlinear system with specification on contact binary asteroid system. I am currently working on my postdoc project about the effect of model uncertainties on orbital motion around asteroid, in Nanjing University, China.

Uncertainty propagation has been addressed extensively in space missions around Earth, but much less for missions around small solar system bodies. Different from traditional studies that focus on the uncertainty of initial state, this paper focuses on the uncertainty of the small body's gravity field. Moreover, different from the case of Earth, the small bodies usually have irregular and weak gravity, together with large uncertainties on their gravity, rotation speed and surrounding space environment. These characteristics bring challenges to the uncertainty propagation. For balancing efficiency and accuracy, the differential algebra (DA) technique is used, which is the first application of DA to orbital propagation in uncertain gravity field of an asteroid. The discoveries help assess the posed risk and design appropriate mission orbits.

Acta Astronautica is an excellent journal for space related topics, e.g. astrodynamics and space technology. Therefore, I believe my paper align well with the scope of this journal.

Thanks for your time and attention.

Yours Sincerely,
Jinglang

Comments from the editors and reviewers:

-Reviewer 1

The manuscript investigates the use of differential algebraic (DA) techniques to study the motion of spacecraft in small-body gravity fields. More specifically, by taking asteroid Stein as testbed, the effect of uncertain gravitational model parameters on the orbital motion is assessed. First of all, the orbital regime in which Taylor expansions fail to describe the effect of uncertainties is identified. Then, DA-based and standard Monte Carlo (MC) simulations are performed to pinpoint the orbits and the spacecraft phasing on them that show the highest sensitivity to the uncertain parameters. The analyses are intended to provide engineers with guidelines to select the most appropriate operational orbit around a small-body.

Evaluation: I think the approach proposed by the Authors has the potential to provide mission analysts with a valuable tool to study the robustness of orbital motion to uncertainties for missions around small-bodies. In addition, the results are supported by a comprehensive set of simulations. However, I think the authors need to address the following major and minor comments before the paper can be accepted for publication.

Major comments

Major comment #1:

Section 2.2: This section should be carefully reviewed: I think the reader can hardly catch how DA is used (which is not acceptable, since this is among the major aims of the paper).

Firstly, thank you very much for this very detailed review, which greatly improves the quality of this paper! We are sorry for the misleading material. Accordingly, we rewrite the major part of this section that are marked in red from line 13 on page 5 to line 2 on page 6, and hopefully addressing all your following concerns and questions.

Some issues (major and minor) are highlighted hereafter:

- Eq. 1: $\mathbf{f}$ should be a vector function, so please use bold font as for the other vectors

It is corrected on page 5, line 13.

- Please be more specific in indicating that $\mathbf{p}$ is the vector embedding all the uncertain parameters of the mathematical model adopted to describe the gravitational field

You can find our correction on page 5, lines 14-15.

- why does $\mathbf{p}$ suddenly have the subscript 0 when it is initialised as DA?

We correct this misleading definition on page 5, line 20.

- Please indicate the final state with a dedicated subscript (e.g. $\mathbf{t}_f$) in the equations

The DA is an iterative process, the final state with a dedicated subscript is described on page 5, lines 31-32.

- Eq. 1 includes the possibility of an explicit dependence on t in the function. That dependence suddenly disappears in the 4th equation on page 5

This error is corrected in all equations of Section 2.2.

- The definition of the map M_p in the sixth equation on page 5 is not consistent with the relation provided earlier in the 3rd equation on the same page

Yes, the definition was indeed misleading in the original version. We modify its definition on page 5, line 30 and correct the inappropriate expression on page 6, lines 24-26.

- M_p is declared to be a k -th order polynomial map. However, the summation is performed to infinity

This error is corrected on page 5, line 28.

- The summation in the sixth equation on page 5 has coefficients f_p^k that are not introduced in the text

We make it up on page 5, line 29.

- The last part of page 5 seems to introduce a way to compute mean and covariance from polynomials that is not used later in the paper. Please clarify the reason why it has been introduced here.

Yes, we don't use these equations to compute in the paper. We remove this misleading part.

Major comment #2:

Section 4.3: This section seems to propose a connection between the convergence of the Taylor polynomial and the chaoticity of the orbital motion. This connection could potentially provide interesting hints on an indicator of chaoticity that is alternative to the ones available in the literature. If this is the case (and if this is Authors' intent), I think the Authors should better elaborate on this connection here to support reader's perception.

Thanks for this suggestion! Yes, we intended to connect the instability of the orbital motion with the convergence of the Taylor expansion. However, we didn't make this clear here. So, we make this up on page 8, lines 7-24 and on page 9, lines 1-11, 18-21, respectively.

Minor comments and typos:

1) Abstract: There is no need to introduce the acronym MC in the abstract (it is used only once)

We remove it on page 2, line 21.

2) Abstract: "short-term effects is found" should be "short-term effects are found"

We correct it on page 2, line 26.

3) Page 2, Introduction: I would replace "During Rosetta's (ESA) distant flyby" with something like "ESA Rosetta mission distant flyby"

We correct it on page 2, line 42.

4) Page 2: “for a spacecraft’s (s/c) rendezvous” should be “for spacecraft’s (s/c) rendezvous”

[We remove ‘a’ on page 2, line 36.](#)

5) Page 2: “JAXA’s Hayabusa’s” should be “JAXA Hayabusa’s”

[We correct it on page 2, line 42.](#)

6) Page 2, last line: “Due to the large mass and fast rotation of Stein, the circular and low-to-moderate altitude science orbits around it were possible” The current form of this sentence and its current context seems to imply that Rosetta’s probe performed close proximity motion around Stein. Maybe, I would use “is possible” instead of “were possible”.

[Thanks for this suggestion! We correct it on page 2, line 44.](#)

7) Page 3: I would replace “the uncertainties of the GNC errors and density of the secondary” with something like “the uncertainties associated to the GNC errors and the density of the secondary”

[We correct it on page 3, line 16.](#)

8) Page 3: “will arrive at asteroid Bennu in 2020” Actually, ORISIS-Rex has already performed its rendezvous with asteroid Bennu

[We correct it on page 3, line 18.](#)

9) Page 3: I would replace “the uncertainty of the gravity field on the state propagation” with “the effect of the uncertain gravity field on the state propagation”

[We correct it on page 3, line 25-26.](#)

10) Page 3: “Due to the increasing requirement on the on-board autonomy” I agree, but I would be more specific (e.g., the necessity of adapting autonomously to the uncertainties of the gravity field)

[We make it more specific on page 3, lines 29-30.](#)

11) Page 3: “The differential algebra technique allows arbitrary order” Please introduce here the acronym DA, since it is used thereafter

[We add it on page 3, line 31.](#)

12) Page 3: I would replace “adjusting the expansion order of the dynamics” with “adjusting the expansion order of the Taylor expansion”

[We correct it on page 3, line 35.](#)

13) Page 3, last line: “Since the SRP is not the dominating perturbation for low-altitude motion around massive small bodies such as Stein [1, 15] only the non-spherical gravity is included in the dynamics for this research”. Previously, the Authors state that “...the main perturbations come from the non-spherical gravity field, the SRP and the solar gravitation”. I understand the decision on the SRP, however no statement is provided on the reason why the solar gravitation is not included

[We add this reason on page 4, lines 1-2.](#)

14) Page 4: I would replace "...to uncertainties of the high-order gravity" with "...to uncertainties on higher-order terms"

We correct it on page 4, line 8.

15) Page 4: "The main contributions of this work include: validating..." I would remove the colon from this sentence

We correct it on page 4, line 9.

16) Section 2, title: I would replace "The DA algorithm" with something like "The DA technique"

We replace it on page 4, line 19.

17) Page 4: I would replace "...with respect to the initial condition, either state or system parameters" with something like "...with respect to either initial state or system parameters"

We correct it on page 4, line 23.

18) Figure 1: I see the picture is very similar to similar ones available in the DA literature. Nevertheless, I would describe (in the text or in the caption) at least the meaning of the operator T

We address your concern on page 4, lines 33-36.

19) Page 4, DACE-2.0: is it possible to provide a link at which the software can be found?

Thanks for this suggestion. You can find the link of the latest version of DACE in the footnote on page 5.

20) Page 6, equation right after Eq. 4: Please introduce the meaning of all $U_{\dots}$ in the text.

We add it on page 6, line 27.

21) Section 4, title: I don't understand what the Authors mean with "feasibility"

We meant that the Section 4 is for evaluating whether DA can be efficiently applied to the current problem. Since it is misleading, we change it to 'Application of DA technique' on page 6, line 29.

22) Page 7: "As it is a large asteroid with a fast rotation rate, orbital motion around it is possible" Please provide a reference for this statement

We intended to express that: the large asteroid has relatively large gravitation that the s/c can orbit around it. Otherwise, the s/c might fly away due to the weak gravitation. Our second point is that the fast rotation can average the effect of the irregular gravity on the s/c orbital motion. Then the s/c can stay close to the asteroid, instead of flying away or impacting on the asteroid, due to the irregularity. Actually, these points can be summarized from the book of Prof. Scheeres, i.e. 'Orbital Motion in Strongly Perturbed Environments, Springer, 2012.' And we add this reference on page 7, line 17.

23) Page 7: Some of the physical parameters of Stein are not introduced in the text (e.g., the rotation period)

The rotation rate of Stein is 6.047 h, which is given as T on page 7, line 18. We make it more clearly by replace T with T_{Period} .

24) Page 7: “Given Stein’s central gravity (ρ) and reference radius (r) as the units of mass and length, respectively” Are the Authors sure that GM stands for mass?

Thanks for pointing out this misleading part. GM stands for the gravitational constant instead of mass. We correct it on page 7, lines 21, 22.

25) Page 7: “covariance matrix is a diagonal” should be “covariance matrix is diagonal”

We remove ‘a’ on page 7, line 25.

26) Page 7: “with a value of 3.969×10^{-5} for the two diagonal components” why is the same value adopted for both components?

As we wrote on page 7, line 26, the $1-\sigma$ uncertainties of C_{20} and C_{22} are the same 0.63%, which is taken from reference [2]. Therefore, their covariance that are the diagonal components are also the same, with the value of 3.969×10^{-5} .

27) Page 7: “by comparing with the standard or pointwise method” Please try to be more specific here in the way the comparison is performed.

We make it clear on page 7, lines 32-33.

28) Page 8: I would replace “i.e. the minimum radius larger than which the Taylor expansion converges in this study” with something like “the limit values of the orbital radius for which the expansion is considered to converge”

We modify it on page 8, lines 12-17.

29) Page 8: “By propagating the dynamics using the DA-MC and MC methods, the final states of the 1000 samples and their nominal values are obtained, respectively” I do not understand the meaning of this sentence. Please consider reformulating it.

We make it clear on page 9, lines 12-14.

30) Page 8: “The absolute difference between their mean values are defined as the error of the DA-MC w.r.t. MC in this study and is illustrated in Fig. 3.” Please be more specific on the way the error is computed

We make it clear on page 9, lines 14-17.

31) Page 8: “For the same altitude and for the same uncertainty, the prograde equatorial orbit (i.e. $i = 0$) is found to have the largest error” Since this is a result that is achieved later in the paper, please point the reader to the dedicated Section

We make it clear on page 8, line 20.

32) Page 8: Please replace “in the simulations in Section 4.3,” with “in the simulations in this section”

We correct it on page 8, line 23.

33) Page 8: Please replace “and increases with the expansion goes to higher order” with something like “and increases with the expansion order”

We correct it on page 9, line 18.

34) Page 9: Please replace “In addition to the comparison of accuracy” with “In addition to the accuracy analysis”

We correct it on page 10, line 6.

35) Page 10: “The study about uncertainty of initial states only requires one n -th order DA integration...However, for the uncertainty of gravity field, the n -th order DA integration is required for different orbital geometry” This concept is not clear to me: I assume that, even when we study uncertainty on initial states, different DA integrations must be performed for different nominal initial conditions. Please try to reformulate these sentences.

Yes, your interpretation is correct. We remove the misleading sentences.

36) Page 10: “Since the error of in-plane position can be even larger than the orbital radius” I think this is a “deviation” or “displacement” rather than an “error”

We correct it on page 11, lines 27-30.

37) Page 11: I would replace “thus eliminating the effect of the uncertain initial condition on the orbital motion.” with “thus eliminating the effect of the different initial condition on the orbital motion.”

We correct it on page 12, line 35.

38) Page 11: “the derivations of the orbits” should be “the deviations of the orbits”. This typo appears more than once in the paper

We correct these on page 12, line 42, and on page 13, line 23.

37) Page 15: “have similar tendencies as these for” should be “have similar tendencies as those for”

We correct it on page 16, line 8.

38) Page 15: I would replace “hardly” with “weakly” in “The deviation in the cross-track direction is hardly affected”

We correct it on page 16, line 9.

39) Page 16: “maximums” should be “maxima”

We correct it on page 17, line 8.

40) Page 16: "...irregular gravity field with uncertain" should be "...irregular gravity field with uncertainty"

We correct it on page 17, line 26.

41) Appendix: There is no reference to the Appendix in the text of the paper

We make it up on page 7, line 19.

-Reviewer 2

The DA technique to study the orbital motion propagation is very promising from the point of view of computer accuracy and efforts. Numerous papers by one of the authors reveal interesting solutions in these fields.

The current work involves the problem of orbital prediction for missions around small solar system bodies. It gives good examples of researches with detailed analysis and various figures. This article deserves attention and publication in the journal.

My personal notes ... Unfortunately, the technique of differential algebra is again used for a very simple force model, only for the gravitational field with the second zone harmonic and the second sectorial harmonic. Perhaps this is indeed due to specifics of the problem of motion around an asteroid (Stern). But I would like to see the estimates of other perturbing factors in comparison with the accuracy of the DA algorithm.

Thank you!

First of all, thank you very much for your positive opinion about our research! The initiative of this work is to assess whether DA is valid and efficient for dealing with the problem of highly nonlinear dynamics with large uncertain parameters. For missions to small solar system bodies, the largest uncertainties of the s/c's dynamical environment come from the inaccurate determination of the gravity and rotation of the small body. Therefore, for this first attempt of introducing DA into characterizing the effect of the gravity uncertainty on the surrounding orbital motion, only the uncertainties of the second order harmonic C20 and C22 are considered, as they dominate the irregular part of the small body's gravity. In addition, for motion close to the small body, the perturbation from the irregular gravity dominates.

Thanks for your concerns of including other perturbing forces. We are currently taking effort on including the uncertainties of the 3rd and 4th order gravity and the perturbation from the solar radiation pressure. This study is still in progress and no systematic conclusions has been obtained yet. But, the DA technique seems to work well even with more perturbing factors.

Highlights

- DA is feasible and more efficient for propagating gravity uncertainty of an asteroid.
- Retrograde motion is more robust to the gravity uncertainty than the prograde one.
- Near polar motion deviates maximally if starting from the polar region of the asteroid.
- Short-term effect plays important role in orbit deviation from gravity uncertainty.

1

2 **Title**

3 Orbit propagation in irregular and uncertain gravity field using differential algebra

4

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Orbit propagation in irregular and uncertain gravity field using differential algebra

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Abstract

Uncertainty propagation has been addressed extensively in space missions around the Earth, but much less for missions around small solar system bodies. Small bodies usually have irregular and weak gravity and our knowledge of their gravity, rotation speed and surrounding space environment is largely uncertain. These characteristics make the orbit propagation around these small bodies a challenging task.

Focusing on the uncertainty of the small body's gravity, this paper applies the differential algebra (DA) technique to study the orbit propagation problem, and addresses its efficiency for a given the required accuracy. Different from traditional studies that focus on the uncertainty of the initial state, this study assumes an exact initial state and studies the influences that gravity model uncertainties have on the orbit. Taking the asteroid Steins as an example, the accuracy and the efficiency of the DA approach are firstly validated by comparison with the traditional Monte Carlo method. Then, the effects of gravity uncertainties on different types of orbits (prograde, retrograde and polar) are studied. The retrograde motion is found to be more robust to the gravity uncertainty than the prograde ones. For near polar orbits, the impact of gravity uncertainty on orbital motion depends significantly on the initial position, and it reaches the maximum if the initial position is near the polar regions. Moreover, **short-term effects are** found to play an important role in orbit deviation as a result of the gravity uncertainty. These discoveries can help mission designers assess the posed risk and design appropriate mission orbits.

Keywords: small solar system body, uncertain gravity field, state propagation, differential algebra, sensitivity analysis

1 Introduction

Missions to small solar system bodies (asteroids and comets) have gained much attention in recent years. For the design of such missions, an important issue is how to describe the dynamics of the spacecraft's (s/c) for different phases (e.g. rendezvous, approaching, orbiting around and even landing) about a body that usually possesses an irregular gravity field. This gives rise to challenges for mission design, not only from the highly nonlinear orbital dynamics, which has been extensively addressed (see for example [1]), but also from the limited capability of determining the physical parameters of the body (e.g., mass, shape, gravity, rotation, density).

However, limited efforts have been put on addressing these uncertainties systematically. During **ESA Rosetta's** distant flyby of Stein and **JAXA Hayabusa's** close proximity to Itokawa, the determination of their gravity fields was with uncertainties of certain level. Due to the large mass and fast rotation of Stein, the circular and low-to-moderate altitude science orbits around it **are**

possible. However, for the small-mass Itokawa, the orbital motion was highly unstable and the mission therefore used a so-called solar terminator orbit (STO) [1], which is perpendicular to the Sun-asteroid line. With the Monte Carlo (MC) simulation, the effects of the uncertain gravity on the evolution of these mission orbits were quantified by Melman [2]. The motion was found to be more sensitive to this uncertainty if asteroid's gravity is weak. This sensitivity is most severe for motions in mean-motion resonances¹ with the asteroid's rotation.

The Asteroid Impact and Deflection Assessment (AIDA) mission, consisting of NASA's Double Asteroid Redirection Test (DART) mission and ESA's Asteroid Impact Mission (AIM), aims to demonstrate the effectiveness of the kinetic impactor deflection technique. AIM did not go into a further study case, but DART is approved and planned to launch in 2021 [3]. It will deliver a kinetic impactor on the smaller body (i.e., the secondary) of the binary asteroid Didymos [4]. For the orbiting phase, the parameters of the asteroid (e.g., mass, shape, rotation and the s/c's mass, surface area and reflectivity) were varied to evaluate their influences on the stability and lifespan of orbital motion in the equatorial plane of the asteroid [5]. With MC simulations retrograde orbits with 90 days free motion arcs were found. Moreover, during the landing phase, the separation error of the deployment was found to dominate the uncertainties associated to the GNC errors and the density of the secondary [6]. Launched in 2016, NASA's first asteroid sample return mission, ORISIS-Rex, rendezvoused with target asteroid Bennu in Dec. 2018, and will collect samples and return to Earth in 2023. On arriving, the s/c will spend several months in various orbits to develop a topographic map of the asteroid, estimate the spin state and the gravity field, and select proper sampling sites [7]. The nominal science orbit is a STO, because Bennu is a small asteroid with weak gravity field. Based on the averaged dynamics and tested against MC simulations, a frozen-STO was found to be more robust against the execution errors than a circular-STO [8].

The uncertain gravity field has an obvious influence on determining the s/c state and pose risk for mission operations. Therefore, in the current study, the effect of the uncertain gravity field on the state propagation will be our focus. As described, most of the previous research was performed with the MC method; the precision and time consumption of which increase with the increasing number of samples [9]. Due to the increasing requirement on the on-board autonomy, (e.g., to save control effort of the nominal orbit as a result of the inaccurate estimation of the central body's gravity) more efficient methods should be introduced and investigated. The differential algebra (DA) technique allows one both to compute the arbitrary order Taylor expansion of the flow of the highly nonlinear dynamics w.r.t. the uncertain parameters of the dynamics and to quickly evaluate the flow at several points [10]. The computational time is reduced considerably by replacing thousands of integrations with algebra operations, while the accuracy can be kept arbitrarily high by adjusting the expansion order of the Taylor expansion. DA has been used in asteroid encounter analysis [11], orbit conjunction analysis [12], s/c navigation and guidance algorithms [13], optimal control strategies [14], etc. However, its feasibility and efficiency on studying uncertainty problem in small body systems that possess irregular gravity field and highly nonlinear dynamics have not been explored. Therefore, this research will evaluate the performance of DA in orbit propagation in an uncertain gravity field.

For s/c motion around small bodies, the main perturbations come from the non-spherical gravity field, the SRP and the solar gravitation. The SRP is not the dominating perturbation for low-

¹ The mean motion rate of the s/c is in commensurability with the rotation rate of the small body.

altitude motion around massive small bodies such as Stein [1, 15]. Moreover, the solar gravitation is generally estimated two orders of magnitude smaller than that of the SRP. Therefore, both of them are not included in this preliminary work and only the non-spherical gravity is accounted for. Moreover, the spherical harmonics model is applied, as it is a generalized representation of the gravity field of a small body. Since the C_{20} and C_{22} are the dominant terms of the non-spherical gravity, their uncertainties are believed to contribute most to the gravity uncertainty, and are the only uncertainties considered in this study. Nevertheless, it will not be difficult to generalize this work to **uncertainties on higher-order harmonic terms**.

The main contributions of this work include validating the accuracy and efficiency of the DA algorithm against the traditional MC method in orbital propagation in an irregular gravity field with uncertainties, and systematically investigating the influences of the gravity uncertainty on different types of orbits. This paper is organized as follows. Section 2 introduces the basic idea and knowledge of the DA algorithm. Section 3 describes the dynamic model in this study and applies the DA algorithm to expand the dynamical flow to high orders w.r.t. the gravity uncertainty. Based on large amounts of numerical simulations, the efficiency of the DA algorithm and the preferable orbital geometry for robust motion are analyzed in Section 4. Section 5 concludes this study and gives prospects for future research.

2 The DA technique

2.1 The concept of DA

The DA technique was originally developed to solve analytical problems by an algebraic approach [10], and is explored here to obtain k -th order Taylor expansions of the flow of a set of ODE **with respect to either initial state or system parameters**. Classical numerical algorithms are based on the evaluation of functions at specific points, while DA relies on the observation that more information on a function can be extracted rather than its numeric values. The basic idea of DA is to treat functions and the operations on them in a similar way as the treatment of real numbers in the computer environment, which is further explained by the diagram in Fig.1. The complete DA framework includes composition of functions, inverting them, solving nonlinear systems explicitly, and treating common elementary functions, differentiation and integration operators.

Figure 1 Analogy between the floating point representation of real numbers in a computer environment (left) and the algebra of Taylor polynomials in the DA framework (right). T and P represent the transformations from real numbers to floating points, and from functions to their Taylor expansions, respectively, i.e. the transformations to computer operational environment. $\circ$ and $\odot$ represent the operations on real numbers or functions and their corresponding adjoint operations in computer environment, respectively.

The DA implementation in this study is the second version of Differential Algebra Core Engine (DACE-2.0)² currently maintained by Politecnico di Milano. DACE-2.0 provides a convenient and powerful C++ interface to basic and advanced DA routines.

2.2 High order expansion of the flow

The DA allows to compute the derivatives of any function f to arbitrary order k , together with the function evaluation. Any explicit integration scheme is based on algebraic operations, involving evaluating the right hand side of the **ordinary differential equation (ODE)** at several integration points. Therefore, all the evaluations can be carried out in the DA framework by computing the expansion flow of a general ODE to arbitrary order w.r.t initial condition. For the study of orbital motion around celestial bodies, given the state vector of the s/c as $\mathbf{X} = (x, y, z, v_x, v_y, v_z)$, the equations of motion can be written as

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, \mathbf{p}, t) \\ \mathbf{X}(t_0) = \mathbf{X}_0 \end{cases} \quad (1)$$

in which $\mathbf{X}_0$ is the initial state at t_0 , and $\mathbf{f}$ is the ODE of the dynamical system, and $\mathbf{p} = [p_1, p_2, \dots, p_n]$ is the vector that includes all the uncertain parameters of the dynamical model and will be defined in detail in next section. Since the focus of this study is to investigate the effect of the uncertainty of the gravity field on the state propagation, the DA method is used to obtain the high order expansion of the phase flow with respect to $\mathbf{p}$. The first step is to initialize $\mathbf{p}$ as a DA variable

$$[\mathbf{p}] = \mathbf{p} + \delta \mathbf{p}$$

where $\delta \mathbf{p}$ represents the displacement from the nominal value. For illustrative purpose, we use the forward Euler's scheme:

$$\mathbf{X}(t_i) = \mathbf{X}(t_{i-1}) + \mathbf{f}(\mathbf{X}(t_{i-1}), \mathbf{p}, t_{i-1}) \cdot \Delta t.$$

and replace the nominal value $\mathbf{p}$ with its DA expression $[\mathbf{p}]$. The first time step yields

$$\begin{aligned} \mathbf{X}(t_1) &= \mathbf{X}(t_0) + \mathbf{f}(\mathbf{X}(t_0), [\mathbf{p}], t_0) \cdot \Delta t \\ &= \mathbf{X}(t_0) + \mathbf{f}(\mathbf{X}(t_0), \mathbf{p} + \delta \mathbf{p}, t_0) \cdot \Delta t, \end{aligned}$$

which is the k -th order Taylor expansion of the flow $\varphi(\mathbf{X}_0, \mathbf{p}, t_0)$ in $\mathbf{p}$ for $t = t_1$ as DA allows us to automatically approximate $\mathbf{f}(\mathbf{X}_0, \mathbf{p} + \delta \mathbf{p}, t_0)$ as

$$\mathbf{f}(\mathbf{X}(t_0), \mathbf{p} + \delta \mathbf{p}, t_0) \approx \mathbf{f}(\mathbf{X}(t_0), \mathbf{p}, t_0) + \sum_{q=1}^k \mathbf{f}_p^k(\mathbf{X}(t_0), \mathbf{p}, t_0) \cdot \delta p_1^i \cdot \delta p_2^j \cdots \quad (i + j + \cdots = q)$$

in which $\mathbf{f}_p^k$ is the k -th order Taylor expansion of $\mathbf{f}$ w.r.t $\mathbf{p}$. Therefore, the dependence of the solution $\mathbf{X}(t)$ w.r.t the uncertainties in the gravity model at a time t_i is available in terms of a k -th order polynomial map $M_p(\delta \mathbf{p})$. In a similar way, the result of the final step $\mathbf{X}_{t_f}$ is the k -th order Taylor expansion of the flow $\varphi(\mathbf{X}_0, \delta \mathbf{p}, t_0)$ in $\mathbf{p}$ at the final time t_f . The accuracy of the result depends on the expansion order k and the value of the displacement $\delta \mathbf{p}$. This expansion can be achieved by considering any integration scheme, in addition to the example Euler's scheme, that is characterized by a finite number of algebraic operations. In this study, an 8th order Runge-

² Web link for DACE: <https://github.com/dacelib/dace/releases>

Kutta-Fehlberg (RKF78) scheme with absolute and relative tolerance of 10^{-12} is rewritten as a DA scheme and is used for the integration.

3 Dynamical model

In the inertial frame, the equation of motion for an object located at $\vec{r} = (x, y, z)$ in the vicinity of a small body is expressed as

$$\ddot{\vec{r}} = \frac{\partial U}{\partial \vec{r}} \quad (2)$$

where U is the gravitational potential of the small body and is expressed in spherical harmonics as follows [16]

$$U = \frac{GM}{r} \left\{ 1 + \sum_{n \geq 2} \sum_{m=0}^n \left(\frac{R_e}{r} \right)^n P_{nm}(\sin \theta) [C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda)] \right\} \quad (3)$$

where GM is the gravitational constant of the small body; r , θ and λ are spherical coordinates (the radial distance $|\vec{r}|$ from the center of mass to a given point P , latitude and longitude, respectively) in the body-fixed frame; R_e is a characteristic physical dimension and is usually defined as half of the largest dimension of the whole body, i.e. the reference radius; P_{nm} is the associated Legendre polynomial. C_{nm} and S_{nm} are the coefficients of the spherical harmonics expansion which are determined by the mass distribution within the body. This potential is actually defined in the body-fixed frame of the small body, and it is transformed to the inertial frame for the following numerical simulations. It is pointed out here that this study uses the 4th degree and order gravity field, as it captures the main characteristics of the whole gravity and meanwhile reduces the computational burden of the model. As discussed in Section 1, only the uncertainty of the second order gravity field is considered. Therefore, the gravity parameter vector is given as $\mathbf{p}_0 = (C_{20}, C_{22})$ and $\delta \mathbf{p}_0 = (\delta C_{20}, \delta C_{22})$. Since the equation of motion Eq.(2) can be written as

$$\begin{cases} \ddot{x} = \frac{\partial U}{\partial x} = U_x \\ \ddot{y} = \frac{\partial U}{\partial y} = U_y \\ \ddot{z} = \frac{\partial U}{\partial z} = U_z \end{cases} \quad (4)$$

the corresponding first-order expansion of the dynamics, i.e. $f(\mathbf{X}_0, \mathbf{p} + \delta \mathbf{p}, t_0)$ defined in Section 2.2, is given as

$$\begin{cases} U_x^1 = U_{xC_{20}} \cdot \delta C_{20} + U_{xC_{22}} \cdot \delta C_{22} \\ U_y^1 = U_{yC_{20}} \cdot \delta C_{20} + U_{yC_{22}} \cdot \delta C_{22} \\ U_z^1 = U_{zC_{20}} \cdot \delta C_{20} + U_{zC_{22}} \cdot \delta C_{22} \end{cases}$$

in which $U_{sC_{20}}$ and $U_{sC_{22}}$ ($s = x, y, z$) are the derivatives of U_s w.r.t C_{20} and C_{22} , respectively.

4 Application of the DA technique

4.1 The DA based Monte-Carlo method

Usually, the accurate computation of statistics in nonlinear dynamics is addressed with MC

simulations, in which thousands of pointwise integrations are performed for an assigned distribution of the uncertain variables. Moreover, a large number of sample trajectories is required for the convergence of the statistics. As a pure numerical method, MC does not provide analytical information for additional analysis. By introducing DA, the pointwise integrations are replaced with an equal number of map evaluations, i.e. fast polynomials evaluations. This is the DA based MC method (denoted as DA-MC) that was first introduced in [11]. The steps of DA-MC can be summarized as:

- 1) Perform a single DA integration selecting the expansion order according to the demanded accuracy;
- 2) Generate random samples based on the statistical distribution of the uncertainty to be investigated;
- 3) Evaluate the flow expansion map for all the samples and perform statistical analysis of the results.

4.2 The uncertainty of the gravity field

Asteroid Stein is used here as an example. As it is a large asteroid with a fast rotation rate, orbital motion around it is possible [1]. Some of its physical parameters are given as [2, 17]

$$GM = 7.7 \times 10^{-6} \text{ km}^3 / \text{s}^2, R_e = 3.35 \text{ km}, T_{\text{period}} = 6.047 \text{ h}$$

$$C_{20} = -9.78 \times 10^{-2}, C_{22} = 1.32 \times 10^{-2}$$

In addition, its 4th degree and order gravity coefficients are provided in Appendix. Before the application of DA, an analysis on the accuracy of the flow expansion is mandatory. Given Stein's gravitational constant (GM) and reference radius (R_e) as the units of the gravitational constant and the length, respectively, other variables are scaled during the numerical simulations. Assuming a Gaussian distribution of the uncertainties of both C_{20} and C_{22} , their $1-\sigma$ uncertainties are given as [0.63%, 0.63%] [2]. In addition, assuming that there is no correlation between C_{20} and C_{22} , the corresponding covariance matrix is diagonal, with a value of 3.969×10^{-5} for the two diagonal components. For Stein, the $1-\sigma$ uncertainty of GM is 0.00025%, which is three orders of magnitude smaller than that of the second order gravity field. Therefore, it is not considered for the low and moderate altitude motion in this study. Given the $1-\sigma$ uncertainties, one thousand samples are generated for (C_{20}, C_{22}) and then propagated through the DA-MC for three orbital periods, which is long enough to investigate the short-term effects from a practical navigation standpoint. Before proceeding with systematic simulations, the accuracy of the DA expansion of the flow as a function of the expansion order is addressed by computing the difference between numerically propagated states and those obtained by DA evaluations. Based on this analysis the expansion order to obtain reliable results is selected for the upcoming simulations. Fig.2 illustrates the gravity samples (blue points) generated with the given nominal values (red point) and uncertainties.

Figure 2 The gravity samples (blue points) with the nominal values (red point) and $1-\sigma$ uncertainty.

4.3 Convergence radius, validity radius and efficiency

Firstly, the convergence radius (CR), defined as the size of the uncertainty set that can be handled by a Taylor expansion with a demanded accuracy, is investigated. The CR can be used as an indicator of the stability of orbital motion. Specifically, CR decreases quickly when the motion moves closer to the asteroid and the propagation becomes longer, i.e. when the semi-major axis becomes smaller and the integration time increases. This phenomenon is easy to understand due to the facts that a stronger gravity perturbation increases the nonlinearity of the dynamics and the expansion error accumulates with time, making the Taylor polynomial less accurate for a given expansion order. In DACE-2.0, the CR can be estimated with the 'convRadius' function that is based on measuring the overall error of the expansion [18]. Specifically, the two inputs of this function are one component of the final state from the propagation and the tolerance vector corresponding to each component, respectively. If the output CR is smaller than one, it indicates that the expansion accuracy is acceptable in a domain smaller than the one of interest. On the other hand, if CR is larger than one the expansion is accurate for an uncertainty domain larger than the one of interest.

In this study, for the same altitude and for the same uncertainty, the prograde equatorial orbit (i.e. $i = 0$) is found to have the largest error, as indicated in Section 5.2.1. Therefore, the simulation is performed for different semi-major axes at $i = 0$. The tolerance for all the elements of the state are set to 10^{-12} . The orbital elements Ω, ω, f are all fixed zero in the simulations in this section, and their influences on the output distribution of the final states will be discussed in Section 5. The values of CR of the orbital motion at different semi-major axes are given in Fig.3.

Figure 3 The convergence radius CR of the 3rd, 5th and 7th order Taylor expansion of orbital motion at different semi-major axes.

It can be seen that the values of the semi-major axis where the CR approaches one are 3.35, 2.25, 2.12, for 3rd, 5th and 7th order expansions, respectively. As expected, higher orders are required to accurately handling lower semi-major axes. However, there seems to be an asymptotic value of the semi-major axis under which the expansion is not accurate even by employing very high expansion orders. This value of the semi-major axis is approximately 2.1, which indicates a strong instability of orbital motion for semi-major axis less than 2.1. This phenomenon is also proved by the following simulation.

The final state that is obtained after the propagation of the dynamics for a nominal value of the gravity field is referred to as the nominal state. One thousand sample states are instead computed by propagating the initial state with the samples of the uncertain gravity provided in Fig.2. By using both the DA-MC and MC methods, the sample states and their mean are obtained for both methods. The error of the DA-MC w.r.t. MC is defined as the absolute difference between the two mean states and is illustrated in Fig. 4. For $a < 2.25$, the error of DA-MC w.r.t. MC is generally more than one (i.e. above the error line $\epsilon = 10^0 = 1$) and increases with the expansion order. This means that the Taylor expansion diverges in this region, which is in consistence with the result in Fig.3. For this highly unstable region, the automatic domain splitting [18] will be needed to accurately map the uncertainties. However, for $a > 2.25$, the error decreases quickly with the increase of both the expansion order and the semi-major axis. The accuracy improves significantly as the expansion order increases from the first to the fifth, as indicated by the color lines in Fig. 3. When the order is higher than 7, the accuracy improvement becomes negligible. For instance, for $a = 2.3$, the error of the 1st, 3rd, 5th and 7th orders expansions are at about 10^{-2} , 10^{-3} , 10^{-4} and 10^{-4} . Therefore, we define $a = 2.25$ as the validity radius for the motion around Stein. Moreover, it can be seen that the errors of the position and velocity are comparable for the same semi-major axis and expansion order.

Figure 4 The errors of the position (left) and velocity (right) of the final mean state of the DA-MC (for the 1st, 3rd, 5th and 7th order expansions) w.r.t. the MC at different semi-major axes for integration time of three orbital periods.

In addition to the accuracy analysis, the study of the efficiency of the DA-MC is very relevant in this study since a simulation needs to be performed for multiple initial conditions spanning all the possible orbital elements of interest. The main results are summarized in Fig. 5. The abscissa is the expansion order, and the ordinate is the ratio of computational time of DA-MC to that of MC, which is computed by

$$\frac{t_{DA} + N \cdot t_{eva}}{N \cdot t_{MC}} \approx \frac{t_{DA}}{N \cdot t_{MC}}$$

in which t_{DA} , t_{eva} and t_{MC} are the time of one k -th order DA integration, one DA flow map evaluation and one pointwise integration, respectively; and N is the number of the samples. Since the flow map evaluation is only a simple polynomial calculation, the t_{eva} is negligible and the computational time is approximated by $t_{DA}/N \cdot t_{MC}$, which is highly dependent on the expansion order. For the expansion order of 5, the ratio is about 0.1 for $N = 1000$, i.e. the time consumption of DA is only one tenth of that of MC. For the expansion order of 7, the computational time of DA almost doubles, compared with that of the expansion order 5.

Figure 5 The ratio of time consumption of DA to MC at different expansion orders for 1000 samples evaluated for three orbital periods.

Therefore, to balance the accuracy and efficiency, a 5th expansion order is selected for the simulations in the following research. However, for other studies the expansion order can be determined depending on a specific requirement of the accuracy. Moreover, the following simulations will be performed for investigating the sensitivities of the semi-major axis and inclination of the orbit on the same uncertainty of the irregular gravity field.

5 The systematic orbit propagation in uncertain gravity field

5.1 The circle finder

Before performing the uncertainty propagation, a circle finder [2] is applied to find the most circular orbit in the irregular gravity field, which is a preferable orbit type for mission. The strategy is briefly introduced here. The initial position $\mathbf{r}_0$ ($|\mathbf{r}_0| = r_{target}$) is fixed, the velocity vector $\mathbf{v}_0$ is perpendicular to $\mathbf{r}_0$ and these two vectors determine the orbital plane. If the orbital inclination is fixed, the magnitude of $\mathbf{v}_0$ is the only variable to adjust. A least-square method is then applied to find the velocity that makes the sum of the squared deviations from the target radius r_{target} the minimum in the nominal gravity field. Further details about this circle finder strategy can be found in the work of Melman et al. [2]. In that paper, the circle finder is applied for finding the initial state for every sample gravity field, to constrain the divergence of the motion due to the uncertain gravity. Different from their work, the circle finder is only performed once here for obtaining the initial state in the nominal gravity field. Thus, the same initial state is used in both the nominal and uncertain gravity fields. Therefore, the disturb from the variation of the initial state is completely eliminated and only the uncertainty of the gravity is included in this study. This different choice may explain the differences between our results and theirs.

5.2 The statistics of the state propagations

The final states (propagated in the uncertain gravity field) computed by the DA-MC and MC methods are compared with the nominal states (i.e. the states propagated in the nominal gravity field) and the corresponding errors are projected in the radial, along-track and cross-track directions for different initial nominal states. For the sake of clarity, it is worth stressing that in the remainder of the paper we will indicate with error the difference between the DA-MC and MC results, while deviation and dispersion will indicate the effect of gravity uncertainty on the final state. Since the deviation of in-plane position can be even larger than the orbital radius, definitions of the radial and along-track errors are defined in Fig. 6. The radial difference is the difference between the orbit radius, and the along-track difference is actually the orbit arc between the two states, as denoted in Fig. 6.

Figure 6 The sketch of the radial and along-track directions, and the cross-track direction is

perpendicular to this plane and points to the reader. The XY represents the inertial frame.

5.2.1 Propagation for different orbital inclination

As an example, the errors of orbits at $a = 1.98$ and $a = 2.3$ in the three directions are given in Fig. 7. For $a = 1.98$, the DA-MC method shows huge difference from the results of the accurate MC method, especially for the prograde orbit that lies in divergence region of DA for $a < 2.25$ (referring to Fig. 4). When the orbital inclination i is increased, the difference between the DA-MC and the MC are small. When i is larger than about 75° (marked by the black triangles in the plot), the deviations computed by DA-MC become comparable with that of MC. For the retrograde motion, the output of DA-MC is reliable and can be used for analysis. The different performance of the DA-MC for prograde and retrograde motions is related to the stability property of the orbits. Low-altitude prograde orbits around irregular bodies are generally **unstable** [19]. Analytic solutions fail in these **unstable regions**, so does the DA that is based on the literal expansion of the solution. On the other hand, retrograde orbits are generally stable, as the perturbations from irregular gravity averages out due to the fast relative motion between the mean motion of the s/c and the rotation of the small body [1, 20]. Therefore, the DA is more accurate for the retrograde motion, even for $a=1.98$, a value smaller than the estimated validity radius 2.25 (the minimum value within which prograde orbits are chaotic).

For $a = 2.3$, the DA-MC has very good performance even at the zero inclination and is therefore reliable for the following systematic studies. In the radial direction, the error reduce from its maximum at zero inclination to its low limit value at about 120° , and stay at this value till 180° . Also in the along-track direction the deviation has a similar tendency: it is maximum at zero inclination and then reduces gradually to reach a minimum at inclination around 100° , after which it increases slowly till $i=180^\circ$. The minimum deviation at $i=100^\circ$ means that the uncertain 2nd order gravity field brings about the weakest in-plane perturbation, i.e. the motion is more robust to this uncertainty. The detailed influence of including the additional order gravity field is out of the scope of this study and is recommended for a future research. One remark on the along-track deviation is made here. In Melman's study [2], the minimum deviation is for retrograde orbit but not for orbit at $i=100^\circ$ (near-polar orbits), in contrast with our results. This difference is believed to relate to the different initial condition adopted. Denote the initial condition of the nominal gravity (given by the red circle in Fig. 2) in our study as X_0 . In their study, for each sample gravity (given by the blue stars in Fig. 2) a new initial condition different from X_0 is computed by the circle-finding algorithm, which we denote as X'_0 . So the deviation after integration includes the contributions from differences between both the initial state (e.g. $X'_0 - X_0$) and the gravity (e.g. the sample gravity w.r.t. the nominal gravity). For the simulations in Fig. 6, we fix the initial condition as X_0 for both the nominal and sample gravities, thus eliminating the effect of the **different** initial condition on the orbital motion.

In the cross-track direction, the minima is located at 0° , 90° , 180° , while the maxima is at around 50° and 130° . This can be explained by the fact that the gravity field is the most symmetric w.r.t. the equatorial and vertical planes. Thus, the orbital motion on these planes suffers the minimum perturbation in the out of plane direction and is more robust to 2nd order gravity uncertainty. Similarly, the orbital motion is most strongly perturbed on the planes with inclinations at about 50° and 130° , as the gravity field is the least symmetric w.r.t. these planes.

Moreover, in the radial direction, the **deviations** of the orbits at inclinations of 0° , 100° , 180° are about 72 m, 4 m and 4 m, respectively. In the along-track direction, for the same inclinations

the deviations are approximately 1858 m, 142 m and 747 m. In the cross-track direction, the maximum and minimum deviations are about 123 m and 0.5 m, respectively. The largest deviation appears in the along-track direction, indicating the motion in this direction is most sensitive to the gravity. While the motion in other two directions are robust to the gravity uncertainty. These quantitative deviations can help mission designers in the design phase of an operative orbit.

Figure 7 The radial, along track and cross track deviations of position of the final state w.r.t. the nominal final state for both DA and MC at different inclinations.

5.2.2 Propagation for different orbital semi-major axis

Figure 8 The radial, along-track and cross-track deviations of position of the final state w.r.t. the nominal final state for both DA (red line) and MC (blue line) at different semi-major axis for $i = 0^\circ$ and $i = 100^\circ$.

For the semi-major axis a varying between the range $[2.3, 5]$, the deviations of the orbit at $i = 0^\circ$ and $i = 100^\circ$ (obtained by both the DA-MC and MC methods), the inclinations where these are largest and smallest respectively, are given in Fig. 8. It is obvious that the DA method performs well for nearly all the simulations. For $i = 0^\circ$, the deviations in the along-track and cross-track directions are the largest and smallest, respectively. While for $i = 100^\circ$, the deviations in the along-track and radial directions are the largest and smallest, respectively. These results are consistent with those shown in Fig. 7 for $a = 2.3$. Generally, the error monotonously reduces with the increase of a in all three directions. This can be explained by the obvious fact that the influence of gravity uncertainty on the orbital motion becomes weaker when the motion moves further away from the small body. Specifically, for $i = 0^\circ$, in the radial, along-track and cross-track directions, the deviations of the orbit reduces from 131 m to 0.24 m, 1818 m to 266 m, 2 m to 1

mm, respectively. For $i = 100^\circ$, the corresponding errors of the orbit in the three directions reduces from 12 m to 0.15 m, 150 m to 66 m, 52 m to 22 mm, respectively.

5.2.3 Sample Distribution and 3σ error ellipse

For a more intuitive analysis, the distributions of the final states are displayed on the *radial-along track* plane in Fig. 9, as the dominant dispersion is in the along-track direction. For comparison, the distributions of the DA-MC and the MC method are given in blue and red points, respectively. They coincide well with each other for most of the plots, except for a few sample points of both Fig. 9B at $i = 0^\circ$. For $a = 2.3$, the 1000 samples are integrated over 1, 2, 2.5 and 3 orbital periods for comparison, respectively, and the resulting distributions are plotted in Figs. 9A, 9B, 9C and 9D for inclinations at $0^\circ, 60^\circ, 120^\circ, 180^\circ$. As the integration time grows, from Fig. 9A to 9B, to 9C and to 9D, the dispersion increases significantly for the same inclination, e.g. the along-track dispersions expand from range $[-1, 0.6]$ to $[-2, 1.5]$ for $i = 0^\circ$, and from range $[-0.3, 0.3]$ to $[-0.7, 0.7]$ for $i = 180^\circ$. Moreover, as a considerable part of the samples are outside the 3σ error ellipse, the highly non-linear feature is more obvious in Fig. 9B. This phenomenon is gradually weakened in Fig. 9C and 9D. For the same integration time of three orbital periods, the semi-major axis is increased from 2.3 (Fig. 9D) to 3 (Fig. 9E) and to 4 (Fig. 9F), respectively. For $i = 0^\circ$, the along-track dispersion shrinks remarkably to $[-0.4, 0.4]$. For other inclinations, the dispersions also reduce significantly in the along-track direction, which is explained by the fact that the further the motion is from the small body, the less it is influenced by the gravity field and also the related uncertainty. For all the simulations, it can be seen that the along-track dispersions reduce from their maximums at $i = 0^\circ$ through $i = 60^\circ$ to their minimum around $i = 120^\circ$, and goes up again till $i = 180^\circ$.

Moreover, the black circle in each plot is the 3σ error ellipse (99% confidence) of the final states projected on the *radial-along track* plane, which is statistically obtained from the linearization of the dynamics of Eq.(2). It can be seen that a small portion of the ellipses cannot cover 99% of the samples due to the non-Gaussian distribution of the sample states resulting from the high non-linearity. For $a = 2.3$, the ellipses cover more samples for inclinations of $60^\circ, 120^\circ, 180^\circ$ than of 0° , showing the stronger non-linear property of the prograde equatorial motion in the close vicinity of the asteroid. Fig. 9 confirms the phenomenon and our analysis: the retrograde motion and motion with large semi-major axis are more robust to gravity uncertainty; the minimum error locates at the inclination ranging from 100° to 120° for $\Omega = \omega = f = 0^\circ$. Since the DA-MC performs well for all these simulations, it will be used for studying the non-zero Ω, ω, f on the outputs in next section.

Figure 9 Sample distributions on the radial-along track plane for inclinations at $0^\circ, 60^\circ, 120^\circ, 180^\circ$. The blue and red points are from the DA-MC and MC methods, respectively, and the black circles are the 3σ error ellipses. Subplots A, B, C, D are for $a = 2.3$, but with simulation time of 1, 2, 2.5 and 3 orbital periods, respectively. Subplots E and F are for $a = 3$ and $a = 4$, respectively, with the integration time of three orbital periods.

5.4 The effects of Ω, ω, f on uncertainty analysis

Since all the orbits studied here are circular, or a given orbit inclination, the longitude of the ascending node Ω and $u = \omega + f$ fully describe orientation and the position of the initial point on this circular orbit, respectively. The semi-major axis is fixed at 2.3. Similar with simulations in the previous section, the inclination is sampled at a grid of 5° , and the effects of Ω, u are investigated separately with a grid of 45° for each element.

For studying the effect of Ω , Fig.10 shows the dispersion in the radial, along-track and cross-track directions, where the deviations at different inclinations have similar tendencies as those for $\Omega = 0$. The deviation in the cross-track direction is weakly affected by Ω . In the radial direction, the deviations for $\Omega = \pi/2$ and $\Omega = \pi$ rank the maximum from $i = 0^\circ$ to about 45° and from $i = 45^\circ$ to about 120° , respectively, while the deviations for other Ω are relatively smaller. For all the inclinations in the along-track direction, the black and green lines, corresponding to $\Omega = \pi$ and $\Omega = 3\pi/4$, represent the largest and smallest deviations, respectively, for both retrograde and prograde motions. Moreover, the second largest and smallest deviations appear at $\Omega = \pi/2$ and $\Omega = \pi/4$, respectively. This phenomenon can be explained by different mass distributions (related to spherical harmonics) in different quadrants of Ω . As can be seen from Eq.(3), the potential U is related to the longitude λ (or Ω) through the spherical harmonic terms when $m \neq 0$. The uncertainty of the C_{22} term is considered in this study and its coefficient is

$$P_{2,2}(\sin\theta) \cdot \cos(2\lambda) = 3\cos^2\theta \cdot \cos(2\lambda).$$

Therefore, the absolute value of this coefficient reaches a maximum at $\lambda = 0, \pi/2, \pi$ and minimum at $\lambda = \pi/4, 3\pi/4$. Specifically, for $\Omega = 0, \pi/2, \pi$, the influence of the second order gravity is strong and the motion is more sensitive to its variation. And in the first and second quadrants of Ω , the impacts of the second order gravity are weak and the motion is resultantly more robust to the deviations of the gravity. Moreover, for all Ω , the minimum deviation is also located at an inclination between 100° and 120° , and the retrograde motion is still more robust than the prograde one.

Figure 10 The errors in the radial, along-track, cross-track directions for different Ω with $u = 0^\circ$

In Fig. 11, it can be seen that varying u has a significant impact on the deviations in the cross-track direction, which are almost symmetrically distributed w.r.t. $i = 90^\circ$. However, for $u = \pi/2$ the cross-track error near the polar region significantly reduces. In the radial direction, except for the low-inclination region from 0° to about 40° , the error is the largest for $u = \pi$. Errors in the along-track direction are still dominant. In general, the largest and smallest errors appear for $u = \pi$ and $u = 3\pi/4$, respectively. Near the polar region, the errors approach the local maxima for $u = \pi/2, \pi/4$ and $3\pi/4$. Therefore, the conclusions can be made as: when the simulation starts near the polar area of the small body, the near polar motion is the most sensitive to the uncertain 2nd order gravity field. And the short-term effect can also be proved, as the deviation of orbital motion obviously depends on the initial value of u . Moreover, except the polar case, for the same inclination the motion that starts closer to the equatorial region, i.e. u close to 0 or π (indicated as the black line in the along-track direction), is more sensitive to the 2nd order gravity field uncertainty. Nevertheless, whatever the value of u , the retrograde motion is still more robust than the prograde one.

Figure 11 The errors in the radial, along-track, cross-track directions for different u with $\Omega = 0^\circ$.

In summary, the motion is very sensitive to (i.e. the local maximum of dispersion) the uncertainty of the 2nd order gravity field in the following situations: near the equator for both the prograde and retrograde cases; near the pole of the small body (i.e. $u = \pi/2$) for near polar orbits (i.e. $i = 90^\circ$).

6 Conclusions

This study proves the feasibility of applying the DA technique in propagating the orbital motion in an irregular gravity field with uncertainty. For the study case of asteroid Stein, the validity radius of DA is found to be around 2.25. To balance the computational accuracy and efficiency, a 5th order expansion is selected for this study. The retrograde motion is found to be more robust against the

uncertainty of the second order gravity field with respect to the prograde motion. Moreover, the longer integration time is and the closer the motion is to the small body, the larger dispersions of the final states w.r.t. the nominal states are, due to the highly nonlinear and strongly unstable dynamics in the vicinity of the small body. For the case of zero Ω, ω, f , the orbit with minimal dispersion locates at the inclination ranging from 100° to 120° . For Ω in the first and second quadrants, the effects of the uncertain second order gravity are weak and the motion is resultantly more robust to the deviations of the gravity. For the near polar orbit, the orbital motion starting in the polar region of the small body are more sensitive to the uncertainty of the second order harmonics of the gravity field. In addition, the orbit deviation is found to depend on the initial state, as a result of the short-term effects.

The outcome of this research can assist mission analysts in designing proper orbital geometry and planning mission operations. For future efforts, the SRP should be included in the dynamical model especially for asteroid with a weak gravity field or for middle to high altitude motions. In addition to the uncertain gravity field, the effect of uncertainty of the initial state on the distribution of final states can also be studied, for a more complete analysis considering both navigation and modeling errors. Last but not the least, the uncertainty of gravity field higher than the second order should also be included in the analysis for a further investigation of the gravity uncertainty on orbital motion around small solar system bodies.

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Appendix

The normalized and non-zero coefficients of the 4th order gravity field of Stein [2]

Stein					
C_{20}	-9.78×10^{-2}	C_{33}	-3.55×10^{-4}	C_{42}	-8.55×10^{-4}
C_{22}	1.32×10^{-2}	S_{31}	1.23×10^{-3}	C_{43}	-1.79×10^{-5}
C_{30}	1.37×10^{-2}	S_{32}	-1.08×10^{-4}	S_{41}	-2.03×10^{-4}
C_{31}	1.99×10^{-3}	S_{33}	-1.04×10^{-3}	S_{42}	-1.27×10^{-4}
C_{32}	7.18×10^{-4}	C_{40}	2.52×10^{-2}	S_{43}	-7.64×10^{-6}
-	-	C_{41}	-2.96×10^{-4}	S_{44}	-1.36×10^{-5}

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Vitae

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