Crashing of efficient stochastic bubbles

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Abstract

Efficiency is not commonly related to the crash of bubbles. However in the presence of wary agents, infinite-lived agents that are worried about distant losses, efficient bubbles may occur and, in a stochastic setting, these bubbles can crash. In this paper we characterize the Arrow-Debreu (AD) price and establish the relationship between the agents' concern about distant losses and the existence of pure charges in the AD price. We show that this pure charge induces efficient bubbles in the positive net-supply assets that complete the markets and that, as we enter some sub-tree, that pure charge may no longer present in the AD price for the sub-economy, implying the crash of the bubble. Finally, we give an example in which there is an efficient bubble with infinitely many crashes.

Keywords: crashing, efficient bubbles, complete markets, stochastic economies

1. Introduction

Rational bubbles have been extensively studied since the late 70's pioneering work by Blanchard (1979) (see [1]) and Blanchard and Watson (1982) (see [2]). Santos and Woodford (1997) (see [3]) made a theoretical and systematic study of rational bubbles in a general equilibrium model with finite and infinite-lived agents. However, efficient bubbles in positive net-supply assets seemed to be

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26 ruled out by the same portfolio constraints that avoided Ponzi schemes. Araujo,
 27 Novinski and Pascoa (2011) (see [4]) showed that this is no longer the case when
 28 standard impatience assumptions do not hold. More precisely, if agents are *wary*,
 29 that is, not willing to neglect losses at distant dates, the efficient allocations may
 30 be implemented by trading positive net supply assets with speculative prices.
 31 The drawback of this surprising result is that it was established in a deterministic
 32 setting, where the occurrence of a bubble implies that it will be present at all
 33 dates. Araujo, Gama, Novinski and Pascoa (2019) (see [5]) showed how it can
 34 be implemented sequentially these efficient allocations in economies with money
 35 or other long-lived assets with taxes.

36 In a stochastic economy, bubbles can burst some time later, as we enter some
 37 sub-tree where the reason for the occurrence of the bubble does not prevail.
 38 In this paper we address whether efficient bubbles generated by wariness may
 39 burst in a stochastic economy. Wariness is a lack of impatience that consists in
 40 neglecting distant gains but not distant losses. The lack of impatience can be
 41 interpreted in terms of ambiguous beliefs, in the sense of Gilboa and Schmeidler
 42 (1989) (see [6]) and Schmeidler (1989) (see [7]), making the agents worried about
 43 their losses among the different dates and state of nature. And, as secondary
 44 objective, we provide an analysis of possible increments in asset price variation
 45 due to the crashing of bubbles in some states of nature.

46 We provide an example where the occurrence of a pure charge in the Arrow-
 47 Debreu price may happen and then disappear, depending on whether the in-
 48 fimum of consumption is a cluster point of the consumption sequence or not,
 49 within the relevant sub-tree. This induces the asset price bubble or the burst
 50 of that bubble, respectively, and the bubble may even occur and then crash in
 51 infinitely many different period depending on the path. In all cases, bubbles
 52 cannot reappear in the economy.

53 The article is organized as follows: In section 2, we define the notation used
 54 along the article, and we define the type AD economy that we will work on.
 55 In section 3, we characterize the superdifferential of the type of consumers that
 56 we defined. In section 4, we implement the AD allocation sequentially, and we

analyze the existence of efficient bubbles in this framework and how they affect the variation of the asset prices, and we conclude this section with an example with a positive bubble that the probability of crashing from a very distant date onwards is positive. In section 5, we give some concluding remarks.

2. Model

We consider a model with a enumerable number of periods t , in each period, there exists a finite number of states and finite number of sucessor for each of them. In other words, the set of all the possible states can be seen as tree, and each state as a node of the tree.

Let us denote $N \subset \mathbb{N}$ a finite set that represents all the states that could occur as immediate successors of a node of the tree with $1 \in N$, $s^t = (s_1, s_2, \dots, s_t) \in \{1\} \times N^{t-1}$ as a node of the tree in the date t , $s^{t,-}$ as the predecessor of s^t , $s^{t,-2} = s^{t,-,-}$ as the predecessor of $s^{t,-}$, $\sigma := (s_1, \dots, s_t, \dots)$ as an infinite path of the tree, $\{1\} \times N^\infty$ as the set of all the infinite paths σ , $\mathcal{N}$ as the σ -algebra induced by $\{\sigma : \sigma_t = s^t\}$ for each $s^t \in \{1\} \times N^{t-1}$ and for each $t \in \mathbb{N}$, and $\mathbb{P}$ as a probability measure in $(\{1\} \times N^\infty, \mathcal{N})$. Therefore, $\mathbb{P}[\sigma_t = s^t]$ can be interpreted as the probability of the node s^t to occur.

2.1. Utility function

The space of consumption is $\mathcal{X} := \{X : \cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t \rightarrow \mathbb{R}^+ : \exists K > 0 \text{ such that } X(s^t) \leq K\}$ then the elements of $\mathcal{X}$ can be seen as elements of $L^\infty\left(\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ where $\tilde{\mathcal{N}}$ is the discrete σ -algebra in $\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t$ and P is the σ -additive measure induced by $P(\{s^t\}) = \mathbb{P}[\sigma_t = s^t]^1$.

Since we are in a stochastic economy with infinite number of dates, there are several forms to implement the lack of impatience for losses. We consider a natural extension for the stochastic case of the one used in Araujo et al. (2011) (see

¹The only purpose of the probability P is to define properly the consumption set that we are working on.

[4]), ε -Contamination². It uses ambiguity aversion in the sense of Schmeidler (1989) (see [7])³ which give us a clear relationship between ambiguity in the discount factors and the importance in the worst consumptions in each path, that is,

$$U(X) := \int_{N^\infty \times \mathbb{N}} u \circ X_{\sigma_t} d\mathbb{P} \times \zeta(\sigma, t) + \int_{N^\infty} \left(\beta(\sigma) \inf_t u \circ X_{\sigma_t} \right) d\mathbb{P}(\sigma) \quad (1)$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a differentiable, strictly increasing and strictly concave function, ζ_t is a measure in $\mathbb{N}$ that can be interpreted as the discounted factors for each state, and β is a $\mathcal{N}$ -measurable non-negative function. As can be observed in the form of the utility function, the agents worry about their consumption by considering all the possible paths in the tree.

Agents with this behavior worry about every consumption in each period if the discounted factors ζ are strictly positive, and if the probability of each state s^t , $\mathbb{P}[\sigma_t = s^t]$ is also positive. However, they are also concern about the worst possible outcome of each path. Differently to Araujo et al. (2019) [5], there is ambiguity not just only on the discounted factors, in this case there exists also among all the paths among of the tree.

The left part of the utility function is the standard discounted utility function in stochastic economies. The right part is the $\mathbb{P}$ -average among all possible paths of the worst consumption.

A non-constant β implies that an agent consistent with Equation 1 has different concerns about distant losses depending on the paths of the tree which is completely intuitive from the decision maker point of view. Therefore it causes that $\beta : N^\infty \rightarrow \mathbb{R}_+$ is a nonnegative, bounded and not necessarily constant function of the infinite paths.

²In the deterministic case, given a probability measure μ and $\varepsilon \in [0, 1)$, let ν_ε be the convex capacity defined by $\nu(A) = (1 - \varepsilon)\mu(A)$ for $A \subsetneq \mathbb{N}$ and $\nu_\varepsilon(\mathbb{N}) = 1$. This is called the ε -contamination capacity with respect to μ . We can rewrite the Choquet integral with respect of ν_ε as $U(x) = (1 - \varepsilon) \int u \circ x d\mu + \varepsilon \inf u \circ x$.

³A preference defined in a set of acts in the Ascombe-Aumann framework is ambiguity averse if $1/2(f + g) \succsim f, g$ for all f, g acts. In the context of Schmeidler (1989) (see [7]), it is equivalent to the capacity being convex.

105 2.2. Arrow-Debreu Equilibrium

106 Consider an economy with I consumers. Each consumer i is characterized by
 107 an endowment allocation $\{W_{s^t}^i\}_{s^t} \in L_+^\infty \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ such that
 108 $\{W_{s^t}^i\}_{s^t} \ggg 0^4$, and a preference that admits a representation consistent with
 109 Equation 1 on $L_+^\infty \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$.

110 **Definition 1.** An *Arrow-Debreu equilibrium* is a couple $\left(\pi, (x^i)_{i=1}^I \right)$ such
 111 that, for all i , x^i maximizes U^i on $B_{AD}(\pi, W^i) := \{x \geq 0 : \pi(x) \leq \pi(W^i)\}$,
 112 $\sum_{i=1}^I (x^i - W^i) = 0$ and $\pi \in L_+^{\infty*} \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ with $\pi \neq 0$.

113 Since the weight of the infimum each path is completely negligible for all
 114 agents, this type of patience is compatible with the hypotheses imposed by
 115 Bewley (see [8]) and Mas-Colell et al. (see [9]) to ensure existence of equilibrium
 116 in economies with $L_+^\infty \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ as consumption set. The price
 117 of this economy is an element of its dual, that is, $ba_+ \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$
 118 the set of the finitely additive measures in the set of nodes.

119 **Theorem 1** (Bewley's Existence Theorems [8]).

120 (T1) Consider an economy with a finite number of agents where each agent has
 121 complete, convex, transitive monotonous, norm continuous and Mackey
 122 usc preferences in l_+^∞ , and $W^i \ggg 0$, then there exists an AD equilibrium
 123 with prices in ba_{++} .

124 (T2) Under the additional assumption of Mackey lsc, $\pi \in l_{++}^1$.

125 Therefore, if $\left(\pi, (x^i)_{i=1}^I \right)$ is an AD equilibrium where U^i is a concave func-
 126 tion that satisfies the condition of (T1), $\pi \in \partial U^i(x^i)^5$ for all i .

127 In the next section, we will prove some important properties of the pure
 128 charges in the supergradient of the optimal solution of an agent consistent with

⁴ $x \geq 0$ means $x_{s^t} \geq 0$ for all s^t , $x > 0$ means $x \geq 0$ and $x \neq 0$, $x \gg 0$ means $x_{s^t} > 0$ for all s^t , and $x \ggg 0$ means that there exists $\varepsilon > 0$ such that $x_{s^t} \geq \varepsilon$ for all s^t .

⁵For any concave function $U : B \rightarrow \mathbb{R}$ where B is a Banach space, $\partial U(x) := \{\pi \in B^* : U(y) - U(x) \leq \pi(y - x) \forall y \in B\}$ is the superdifferential of U at x , and $\pi \in \partial U(x)$ is a supergradient of U at x .

Equation 1. These properties will be extremely relevant to establish the relationship between the pure charges and the existence of efficient bubbles in stochastic economies.

3. Characterization of the Super-differential of the Utility Function

Since the utility function considered has positive weight on the infimum consumption in each path, a wary agent will be concerned about the infimum consumption in each path if the infimum is never attained. Therefore, there is a positive pure charge in the superdifferential in the optimal allocation. The following result is an extension of the deterministic case analyzed in Araujo et al. (2011) [4].

Proposition 1. *Let $X \ggg 0$, $\underline{X} : N^\infty \rightarrow \mathbb{R}_{++}$ such that $\underline{X}_\sigma = \inf_t X_{\sigma_t}$. For U given by Equation 1 with $u \in C^1(0, \infty)$ and $\pi \in L^{\infty*} \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$. We have that π can be written as*

$$\begin{aligned} \pi Y = & \sum_{s^t} \int_{[\sigma : \sigma_t = s^t]} u' \circ X_{\sigma_t} \zeta_t d\mathbb{P}(\sigma) Y_{s^t} + \int_{A_1} \beta(\sigma) u' \circ \underline{X}_\sigma \text{LIM}_\sigma(Y_\sigma) d\mathbb{P}(\sigma) + \\ & \sum_{t_1 \leq \dots \leq t_K} \int_{A_{2,t_1,\dots,t_K}^K} \beta(\sigma) u' \circ \underline{X}_\sigma \left(\sum_{k=1}^K \alpha_2(\sigma, k) Y_{\sigma^{t_k}} \right) d\mathbb{P}(\sigma) + \\ & \sum_{t_1 \leq t_2 \leq \dots} \int_{A_{3,t_1,t_2,\dots}} \beta(\sigma) (u' \circ \underline{X}_\sigma) \left(\sum_{k=1}^\infty \alpha_3(\sigma, k) Y_{\sigma^{t_k}} + \right. \\ & \left. \alpha_3(\sigma, \infty) \text{LIM}_\sigma(Y_\sigma) \beta(\sigma) \right) d\mathbb{P}(\sigma), \end{aligned}$$

where $Y \in L^\infty \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ and

- $A_1 := [\sigma : \underline{X}_\sigma < u(X_{\sigma^t}) \forall t \in \mathbb{N}]$,
- $A_{2,t_1,\dots,t_K}^K := \left[\sigma : \inf_t u \circ X_{\sigma_t} = u(X_{\sigma^{t_k}}) \forall k = 1, \dots, K \wedge u(X_{\sigma_t}) \geq u(X_{\sigma^{t_k}}) + \varepsilon \forall t \neq t_1, \dots, t_K \text{ for some } \varepsilon > 0 \right]$, $A_2 := \bigcup_K \left(\bigcup_{t_1 \leq \dots \leq t_K} A_{2,t_1,\dots,t_K}^K \right)$,
- $A_{3,t_1,t_2,\dots} := \left[\sigma : \inf_t u \circ X_{\sigma_t} = u(X_{\sigma^{t_k}}) \forall k \geq 1 \wedge u(X_{\sigma_t}) > u(X_{\sigma^{t_k}}) \forall t \neq t_1, t_2, \dots \right]$, $A_3 := \bigcup_{t_1 \leq t_2 \leq \dots} A_{3,t_1,t_2,\dots}$,
- $\alpha_2(\sigma, k) \geq 0 \forall k = 1, \dots, K$ and $\sum_{k=1}^K \alpha_2(\sigma, k) = 1, \forall \sigma \in \{1\} \times N^\infty$,
- $\alpha_2(\sigma, k), \alpha_2(\sigma, \infty) \geq 0 \forall k \in \mathbb{N}$ and $\sum_{k=1}^\infty \alpha_3(\sigma, k) + \alpha_3(\sigma, \infty) = 1, \forall \sigma \in \{1\} \times N^\infty$ and

151 • $\{\text{LIM}_\sigma\}_{\sigma \in A_1 \cup A_3}$ is a collection of generalized limits⁶ such that it is a mea-
 152 surable function in $(\{1\} \times N^\infty, \mathcal{N})$,
 153 if and only if $\pi \in \partial U(X)$.

154 The proof can be found in Appendix A.

155 Note that, in our framework, it is not enough to analyze the cases in which
 156 the infimum is never attained, or to analyze the cases in which it is always
 157 attained in finite or infinite number of times only as in Araujo et al. (2011)
 158 [4]. Moreover, any allocation, the set of paths can be separated in three main
 159 sets: the ones in which the infimum is never attained (denoted A_1), the ones in
 160 which the infimum is attained only in a finite number of periods (denoted A_2),
 161 and the ones in which the infimum is attained in an infinite number of periods
 162 (denoted A_3). In each set, the super-differential has different properties. Under
 163 $\beta(\sigma)|_{A_1} \gg 0$ $\mathbb{P}$ -a.c.⁷ in A_1 and $\mathbb{P}[A_1] > 0$, there is a positive pure charge in
 164 the supergradient that is concerned about paths in A_1 only. The pure charge
 165 of the supergradient component in A_2 does not exist since the infimum of X is
 166 attained a finite number of times only. For the latter, there are supergradients
 167 without pure charges component that is concerned about A_3 only (because
 168 $\alpha(\sigma, \infty) = 0$ $\mathbb{P}$ a.c. in A_3 for that supergradient), but, under $\beta(\sigma)|_{A_3} \gg 0$ $\mathbb{P}$ -
 169 a.c. in A_3 and $\mathbb{P}[A_3] > 0$, there are supergradients with a positive pure charge
 170 component that has that property (because $\alpha(\sigma, \infty) > 0$ for a positive measure
 171 of paths in A_3 for that supergradient).

172 From Proposition 1, we know that the existence of pure charges in the in-
 173 the super-gradient is strongly related to the infimum being attained in a finite
 174 number of periods in each path. However, in the case in which the infimum
 175 is attained an infinite number of times, there are supergradients with no pure
 176 charge component, but there are also supergradients with a pure pure charge
 177 taking nonnegative values in the path in which the infimum is attained.

⁶A generalized limit is a continuous, linear functional in $l^{\infty,*}$, LIM, such that $\text{LIM}(x) = \lim_t x_t$ when it exists, and $\text{LIM}(x) \in [\liminf_t x_t, \limsup_t x_t]$ when it does not.

⁷ $x|_A$ means that x is restricted to the paths contained in A .

178 **Remark 1.** If $X \gg 0$ instead of $X \ggg 0$, there might be pure charges generated
 179 by the fact that $\{u'(X_{s^t})\}_{s^t}$ is not bounded. Moreover, if $\{\zeta_t \mathbb{P}[\sigma_t = s^t] u'(X_{s^t})\}_{s^t}$
 180 is not in l^1 , $\partial U(X) \not\subseteq l^1$.

181 4. Bubbles in sequential implementation of an AD equilibrium

182 Let us now define a sequential economy with one good in each state, which is
 183 considered as a *numéraire*, and $|N|$ Long-Lived assets with nonnegative returns
 184 $\{R_{j,s^t}\}_{s^t} \in L_+^\infty \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ for each j . Additionally, each asset
 185 has a *short-sales* constraints. Each agent has an initial nonnegative amount
 186 of each asset $z_{j,0}^i \geq 0$, and an initial endowment $\{\omega_{s^t,i}^i\}_{s^t,i} \geq 0$ where $\omega_{s^t,i}^i =$
 187 $W_{s^t}^i - \sum_j R_{j,s^t} z_{j,0}^i$ for each s^t . Therefore, the consumer's problem is to find an
 188 optimal portfolio $\{z_{s^t}\}_{s^t} \in \mathbb{R}^\infty$ to maximize U^i subject to the sequential budget
 189 constraints

$$\begin{aligned} x_{s^t} - \omega_{s^t} &\leq q_{s^t} (z_{s^t,-} - z_{s^t}) + R_{s^t} z_{s^t,-}, \quad \text{for all } s^t, t \geq 1, \\ z_{s^t} &\geq -K_{s^t}, \end{aligned}$$

190 where x_{s^t} is the consumption of the *numéraire*, ω_{s^t} is the endowment of the
 191 *numéraire*, $q_{1,s^t}, \dots, q_{J,s^t}$ are the prices of the assets, z_{s^t} is the assets' allocation
 192 on the node s^t , and $\{K_{s^t}\}_{s^t}$ is nonnegative and bounded from above.

193 Our goal is to analyze efficient bubble in this economy. By efficient bubbles
 194 we mean the bubbles that occur in the implementation of an AD equilibrium
 195 (π, x^i) of Subsection 2.2 in stochastic sequential economies as the ones described
 196 in the current section with a bounded short-sale constraint adjusted to include
 197 the optimal allocation in the budget constraint of the agent. When agents
 198 are not impatient, *long-run improvement opportunities* are not ruled out by
 199 bounded short-sales constraints, see Araujo et al. (2019) [5], that is, the exis-
 200 tence of a portfolio that allows the agent to increase his/her utility by increasing
 201 the long run consumption over-saving the assets to do so⁸.

⁸Take any non-negative real sequence z and let $x \equiv x(z, i)$. Suppose consumer i replaces z_t^i by $z_t^i + h z_{s^t}$, with $h > 0$, from some date n onward. As we move on the right along this

202 To avoid the *long-run improvement opportunities* mentioned above, we must
 203 impose an additional constraint such as taxes as in Araujo et al. (2019) [5] ar-
 204 gued. Another form to avoid them is to impose some type of *P-constraint*⁹
 205 mentioned in Araujo et al. [4] as a transversality condition in the budget con-
 206 straint to it prohibit the existence of such opportunities. The necessity of this
 207 type of constraints is a direct consequence of the lack of impatience since they
 208 would have incentives to save large quantities of wealth in form of assets to
 209 avoid the worst states in distant dates.

210 Since in this article we have more interest in analyze efficient bubbles than
 211 to find different ways of implementing efficient allocations, we will assume that
 212 each agent has some type of *P-constraint* that allows us to implement efficient
 213 allocation. However, note that the implementation with money and taxes is
 214 also possible even in stochastic economies with agents concerned about the
 215 average of the worst possible outcomes. To implement this efficient allocations
 216 sequentially with taxes or with a *P-constraint*, it is required that the market
 217 structure is complete (in the sense of $\text{span}(R_{s^t} + q_{s^t}) = N$) in each node. In
 218 the case of taxes in a complete market economy, the taxes would not be defined
 219 by the lim sup as in the deterministic case, it would be defined by the average
 220 of the lim sup in the paths since the taxes depend indirectly on how the agents
 221 concern about the distant losses.

222 Since there is a strong relationship between the AD prices and the supergra-
 223 dients of the equilibrium allocations, we can characterize the AD price based on
 224 the equilibrium allocation. The next section explores the relationship between
 225 the existence of efficient bubbles and the AD prices.

direction, we hoard more at date n and at subsequent dates for which $\omega_{s^t}^i > x_{s^t}$, in order to
 consume more at subsequent dates where $\omega_{s^t}^i < x_{s^t}$. There would be a utility gain by doing
 so if the right derivative were positive. Now, the right derivative is the minimum of the values
 that all supergradients may take at this direction and, therefore, such improvement strategy
 is ruled out if there is a supergradient $T = \mu + \nu$ for which $T(c(z(n))) \leq 0$.

⁹ $\lim_t \sum_{s^t} \mu_{s^t} q_{s^t} z_{s^t} \geq \nu(x(z, i) - \omega^i)$ where ν is the capacity of the AD price.

226 *4.1. Characterization of Efficient Bubbles and their Crashing*

227 Based on Santos and Woodford [3], a rational bubble at a node s^t is defined
 228 as difference between the price of the asset in this state and the *fundamental*
 229 *value* of it

$$q_{s^t} - \frac{1}{a_{s^t}} \sum_{r=t+1}^{\infty} \sum_{s^r, -(r-t)=s^t} a_{s^r} R_{s^r} \geq 0,$$

230 where $\{a_{s^t}\}_{s^t}$ are the state prices that exists due to the lack of arbitrage in the
 231 economy.

232 Due to the Euler equations applied in an optimal efficient allocation $(x^i)_i \gg$
 233 0, the $L^1_{++} \left(\bigcup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right) \equiv \ell^1_{++}$ component of the *AD* price, $\{\mu_{s^t}\}_{s^t}$ ¹⁰,
 234 satisfies that there is a constant $\alpha > 0$ such that $\mu_{s^t} = \alpha a_{s^t}$ for any s^t . There-
 235 fore, without loss of generality it can be assumed that the deflator are $\{\mu_{s^t}\}_{s^t}$.
 236 Moreover, we have that

$$\mu_{s^t} q_{s^t} - \sum_{r=t+1}^{\infty} \sum_{s^r, -(r-t)=s^t} \mu_{s^r} R_{s^r} = \lim_{r \rightarrow \infty} \sum_{s^r, -(r-t)=s^t} \mu_{s^r} q_{s^r}. \quad (2)$$

237 Therefore, the existence of efficient bubbles in a node s^t depends on the asymp-
 238 totic behavior of the subtree that is generated by this node which implies that
 239 if a bubble crashes at some node s^t , a bubble can not reappear in the economy
 240 in any sucessor of s^t .

241 **Proposition 2.** *For any efficient allocation implemented in a sequential econ-*
 242 *omy with a complete set of Long-Lived assets. If there exists one node s^t and*
 243 *one asset j in which there is no bubble for the asset price $q_{s^t}^j$ then there is no*
 244 *bubble for the asset price $q_{s^r}^j$ for any state s^r sucessor of s^t .*

245 Then, after the crashing of a bubble of any asset at the node s^t , there is no
 246 possible reappearance of the bubble in the subtree generated by s^t . However,
 247 it does not mean that there is no bubble in the economy in other subtrees that
 248 are not generated by s^t .

¹⁰Therefore, the MRS for each pair of nodes s^t and $s^{t'}$ in the efficient allocation is equivalent to $\mu_{s^t} / \mu_{s^{t'}}$.

Now, let us analyze conditions in which it is possible the existence of bubbles. To do so, we will establish a relationship between bubbles and the existence of pure charges in the super-gradient of the utility function.

Since the pure charges are related to the agents' concerns of the worst events and the bubble is strongly related to the asymptotic behavior of the asset price, the existence of bubbles in this environment seems to be strongly related with the asymptotic behavior of the agents and the existence of the pure charges in the optimal super-gradient. Moreover, we have that implementing the efficient allocations in a sequential economy we have that

$$\tilde{\nu}^i (X^i - \omega^i) - \nu (X^i - \omega^i) = \sum_j \left(z_{j,0}^i \left(\mu_1 q_{j,1} - \sum_{s^t} \mu_{s^t} R_{j,s^t} - \nu(R_j) \right) \right), \quad (3)$$

where $\tilde{\nu}$ is the capacity that takes the highest value on the direction of the net trade, and ν is the capacity of the AD price. Therefore $\mu_1 q_{j,1} = \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ if $x_t^i - \omega_t^i$ converges for any i , and $\mu_1 q_{j,1} > \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ for at least one asset j and $\mu_1 q_{j,1} \geq \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ for every j if $x_{s^t}^i - \omega_{s^t}^i$ doesn't converge for some i and $R \geq 0$.

Proposition 3. *Given an efficient allocation such that $(x^i)_i \gg 0$. If $\partial U^i(x^i) \subseteq \ell^1$ for all agent i , there is no bubble for any asset in positive net supply.*

Therefore, to find a bubble in assets in positive net supply, we must have that $\partial U^i(x) \not\subseteq \ell^1$. This result is a consequence of Santos and Woodford (1997) (see [3]). Moreover, Equation 3 ensures that the converse is also true under some circumstances.

Proposition 4. *Given an efficient allocation such that $(x^i)_i \gg 0$. If*

1. $x_{s^t}^i - \omega_{s^t}^i$ does not converge in a set of positive measure in $\mathbb{P}$ for some i ,
- or
2. $\nu(R) > 0$,

then there is a bubble for some asset j in positive net supply.

274 However for nodes different than 1 the analysis can not be done for each
 275 asset separately. We will analyze the existence of bubbles for the hole set of
 276 assets, this means that even in presence of a bubble in a node s^t with $t \neq 1$ we
 277 could have no bubble for some assets even in the presence of bubbles for every
 278 asset j at some node s^r predecessor of s^t .

279 The following propositions study the condition for the existence or nonexis-
 280 tence of a bubble in the assets at a node s^t . This conditions are strongly related
 281 to the sets A_1 , A_2 and A_3 defined in Proposition 1.

282 **Proposition 5.** *For any efficient allocation with $(x^i)_i \gg 0$ and any state s^t , if*
 283 *there is one agent i such that*

- 284 1. $x^i|_{[\sigma:\sigma_t=s^t]} \gg 0$, and
- 285 2. *all paths that contain s^t attain the infimum consumption in a finite number*
 286 *of dates $\mathbb{P}$ almost certainly in the set $\{\sigma : \sigma_t = s^t \text{ and } \beta(\sigma) > 0\}$,*

287 *then there is no bubble for any asset at the node s^t .*

288 Note that this proposition is strongly related with the set A_1 in Proposition
 289 1 for $U = U^i$ and the comments that follow this result.

290 This implies that, to have a crashing of a bubble at a node s^t , it is enough to
 291 ensure that the infimum is attained in a finite number of dates for all the paths
 292 that contain s^t , and that there is a set of nodes with a positive probability in
 293 which the pure charge is positive.

294 Let us expose some conditions in which we can ensure a positive bubble at
 295 a node s^t . These conditions are related to the infimum not being attained in
 296 finite time.

297 **Proposition 6.** *For any efficient allocation with $(x^i)_i \gg 0$ and any state s^t , if*
 298 *there exists one agent i and a subset of paths A^i with the following properties*

- 299 1. $x^i|_{[\sigma:\sigma_t=s^t]} \gg 0$,
- 300 2. A^i has positive $\mathbb{P}$ -probability,
- 301 3. any σ in A^i satisfies that $\sigma_t = s^t$, and

302 4. for each σ in A^i , the infimum of $\{X_{\sigma^r}^i\}_{r \in \mathbb{N}}$, the sequence of consumption
 303 of the agent i in the path σ , is a cluster point never attained.

304 Then, there is a bubble at the node s^t if (1) there exists at least one asset in
 305 positive net supply such that $\nu(R^{j,s^t+}) > 0$ or (2) the net trade $x^i - \omega^i$ are not
 306 convergent in A^i .

307 From Proposition 5 and 6 and their proof, the existence of efficient bubbles
 308 in this economy is related to the existence of positive pure charges in the super-
 309 gradient of the agents as it was mentioned in Araujo et al (2011) (see [4]) in
 310 deterministic economies. In this case, Proposition 1 allows us to know the
 311 conditions that must be satisfied to ensure the existence of pure charges in the
 312 superdifferential of each agent.

313 Note that, in Proposition 6, the set A^i is a subset of A_1 defined in Proposition
 314 1 for $U = U^i$.

315 Similar to the deterministic case, the desire of the WARY agent of increasing
 316 their consumption in the worst events in the subtree that contains the node s^t ,
 317 produces a pure charge in the AD price that imply the existence of bubbles for
 318 the set of assets at the node.

319 4.2. Variation of prices in presence of Efficient Bubbles and Crashing

320 If, at some period of time, there is a bubble in some nodes, and there is not
 321 in some other nodes, it means that, before that, the bubble crashes in at least
 322 one node. Therefore, if we analyze the asset prices in these subtrees, we can
 323 analyze the changes in the asset the variation of the assets prices in short and
 324 long run.

325 Under the conditions described in Proposition 5 and 6, it is possible to know
 326 when and where, in the tree, there is a bubble and also when it will crash. Since
 327 these bubbles increase the price above the fundamental value of the assets, the
 328 crashing of them will, naturally, increase the dispersion of the asset prices when
 329 the price's variations are analyzed for all successors s^t of a node s^r with a bubble
 330 in at least one asset.

331 After the bubble crashes at the node s^t , the fundamental value of the asset
 332 and the market price are equal, this means that

$$\max_{\{s^r: s^r, -(r-t)=s^t\}} \mu_{s^r} q_{j,s^r} = \max_{\{s^r: s^r, -(r-t)=s^t\}} \left\{ \sum_{k>r} \sum_{s^k, -(k-r)=s^r} \mu_{s^k} R_{s^k}^j \right\} \rightarrow 0 \text{ when } r \rightarrow \infty.$$

333 However if there is a path σ such that there is always a bubble for the asset j
 334 we have that

$$\mu_{\sigma_t} q_{j,\sigma_t} = \sum_{r>t} \sum_{s^r, -(r-t)=\sigma_t} \mu_{s^r} R_{s^r}^j + \lim_r \sum_{s^r, -(r-t)=\sigma_t} \sum_j \mu_{s^r} q_{j,s^r}, \quad (4)$$

335 and, as we have already mentioned, there exists a relationship between the
 336 existence of bubbles and the pure charges in the super-gradient of the agents.
 337 Therefore, by proposition 1, we have:

338 **Proposition 7.** *Under the conditions of Proposition 1 and for any path σ , the*
 339 *component in Equation 4 that defines the bubble of any asset j at the node σ_t*
 340 *tends to zero when $t \rightarrow \infty$.*

341 This means that the bubble is being distributed among all nodes where the
 342 pure charge is positive, reducing its weight in the price of the asset in which the
 343 bubble is maintained.

344 Nevertheless it does not mean that the price would be bounded, in fact we
 345 have:

346 **Proposition 8.** *Under the conditions of Proposition 6 satisfying (1') $R^j|_{A_i} \gg$*
 347 *0 instead of (1). For any path $\sigma \in A^i$, we have that*

$$\limsup_{t \rightarrow \infty} \left\{ \frac{\lim_{r \rightarrow \infty} \sum_{s^r, -(r-t)=\sigma_t} \sum_j \mu_{s^r} q_{j,s^r}}{\mathbb{P}([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t])} \right\} = \beta < \infty, \quad (5)$$

and there exists a path $\sigma \in A^i$ such that

$$\liminf_{t \rightarrow \infty} \left\{ \frac{\lim_{r \rightarrow \infty} \sum_{s^r, -(r-t)=\sigma_t} \sum_j \mu_{s^r} q_{j,s^r}}{\mathbb{P}([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t])} \right\} = \alpha > 0. \quad (6)$$

348 And since there is a strong relationship between $\mathbb{P}([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t])$ and μ_{s^t} due
 349 to the EE, we will have:

350 **Corollary 1.** *Under the conditions exposed in Proposition 8, the bubble will*
351 *tend to infinity when $t \rightarrow \infty$.*

352 This result is also a consequence of Santos and Woodford (1997) (see [3]),
353 and it implies in our framework that, when t is large, there are large variations
354 of prices between states due to the existence of bubbles in some nodes s^t , which
355 make the asset prices goes to infinity in several paths, but not in everyone.
356 Moreover, the set of paths in which the bubble crashes could have positive
357 value.

358 Therefore, when there is a crashing of a bubble in the node s^t , the variations
359 of the asset prices of a subtree that contains the node s^t and the predecessor
360 of s^t tends to infinity in the long-run. Therefore, the existence of bubble is
361 associated with high variations of the assets' price if there is a chance of the
362 bubble to crash.

363 4.3. Example

364 As can be noticed through this article, there is a variety of possibilities for
365 bubbles that can occur in the economy. The following example shows the large
366 variety of possibilites of bubbles that can be made in this framework. It shows
367 that bubbles can occur with positive probability even in presence of a infinite
368 number of paths in which the bubbles crash.

369 **Example 1.** Consider an economy with two agents and two states, $N = \{1, 2\}$,
370 with utility index given by $u^i(x) := \ln(x)$ ¹¹, $\xi_t^i := 1/2^t$, $\beta^i = \beta > 0$ for each
371 $i = 1, 2$, $W : \cup_{t \in \mathbb{N}} \{1\} \times N^t \rightarrow \mathbb{R}_+$ given by

¹¹In this case, each U^i is defined in all $x \gg 0$ uniformly bounded, but the preference induced by this utility function is well-defined for $x \geq 0$ uniformly bounded. Therefore, the utility function can be extended to the case in which $U^i(x) = -\infty$ for all $x = 0$ in at least one node.

$$W_{s^t} = \begin{cases} 8 + 1/2^{t-4} & \text{if } s^{t,-(t-2)} = (1, 2) \text{ and } \exists k \in \mathbb{N} \text{ such that } 2k + 1 \leq t \text{ and} \\ & s_{2k+1}^{t,-(t-2k+1)} = 2, \\ 9 & \text{if } s^t = (1, 2), (1, 2, 1, 1) \text{ or } s^{t,-} = (1, 2, 2), \\ 10 & \text{if } t = 1 \text{ or } s^{t,-} = (1, 2), \\ 11 & \text{otherwise if } t \text{ is even,} \\ 12 & \text{otherwise if } t \text{ is odd,} \end{cases}$$

372 $W_{s^t}^1 = W_{s^t} + A_t$ and $W_{s^t}^2 = W_{s^t} - A_t$ where A_t is 1 if t is even and $-1/2$ if t is
 373 odd. The probability of each node s^t to occur is symmetric with all the nodes
 374 at the date t , that is, that $\mathbb{P}[\sigma : \sigma_t = s^t] = 1/2^{t-1}$ for all $s^t \in \{1\} \times N^{t-1}$. Let
 375 us find an AD equilibrium that exists in this cases because $W^i \gg 0$. Given
 376 $a : \{1, 2\} \rightarrow \mathbb{R}_{++}$, the consumption plan $x^i = a(i) (W^1 + W^2) = 2a(i)W$ is
 377 optimal under the budget constraint $\pi x \leq \pi W^i$ when the prices are given by

$$\pi x = \int \frac{x_{\sigma_t}}{2^{t-1}W_{\sigma_t}} d\mathbb{P}(\sigma) + \beta \left(x_{(1)} / 20 + x_{(1,2)} / 72 + \sum_{s \geq 2} \frac{x_{\tilde{\sigma}_{2s+1}}}{2^{2s+1}(8+1/2^{2t-3})} + \int_{\bigcup_{t \in \mathbb{N}} [\sigma : \sigma_{2t+1} = (1, 2, 1, 1, \dots, 1, 2)]} \text{LIM}_{\sigma} (x_{\sigma}) d\mathbb{P}(\sigma) \right) \quad (7)$$

378 where $\tilde{\sigma}_{2t+1} = (1, 2, 1, 1, \dots, 1)$ and $a(i) = \pi(W^i) / \pi(W)$. One possible generalized
 379 limit is the Banach limit, B , which satisfies that $B((x_t)_{t \in \mathbb{N}}) = \lim_{n \rightarrow \infty} 1/n \sum_{t=1}^n x_t$
 380 when the limit of the sequence $\{1/n \sum_{t=1}^n x_t\}_n$ exists. Under this condition
 381 (π, x^1, x^2) is an AD equilibrium.

382 For the sequential implementation with two long-lived assets, one that pays
 383 1 in each state 1 and 0 in each state 2, and the other one that pays 0 in each state
 384 1 and 1 in each state 2, we can fix the prices $q_1 \in \mathbb{R}_{++}^2$ and z_0^i such that (x^i, z^i)
 385 is implemented sequentially which implies that Equation 3 must be satisfied,
 386 that is, $\tilde{\nu}^i(x^i - \omega^i) - \nu(x^i - \omega^i) = \sum_{j=1}^2 (z_{j,0}^i (\mu_1 q_{j,1} - \sum_{s^t} \mu_{s^t} R_{j,s^t} - \nu(R_j)))$
 387 where ν , the capacity of the AD price, and ν^i , the pure charge in the subgradient
 388 of U^i in x^i that takes the largest value in the net trade $x^i - \omega^i$, have the form
 389 of Equation 7, $\{\mu_{s^t}\}_{s^t}$ the l^1 component of the AD price and the no-arbitrage
 390 state prices that are defined by

$$\mu_{s^t} = \begin{cases} \int_{[\sigma: \sigma_t = s_t]} \frac{x_{\sigma_t}}{2^{t-1} W_{\sigma_t}} d\mathbb{P}(\sigma) & \text{if } s^{t, -(t-2)} = (1, 2) \text{ and } \exists k \in \mathbb{N} \text{ such that} \\ & 2k+1 \leq t \text{ and } s_{2k+1}^{t, -(t-2k+1)}, \\ \int_{[\sigma: \sigma_t = s_t]} \frac{x_{\sigma_t}}{2^{t-1} W_{\sigma_t}} d\mathbb{P}(\sigma) + & \\ \sum_{s \geq 2} \frac{x_{\tilde{\sigma}_{2s+1}}}{2^{2s+1} (8 + 1/2^{2t-3})} & s^{2k+1} = (1, 2, 1, 1, \dots, 1) = s^t \text{ for some } k \geq 1, \\ 1/20 + \beta/20 & t = 1, \\ 1/72 + \beta/72 & \text{if } s^t = (1, 2), \end{cases}$$

391 $\{q_{s^t}\}_{s^t, t \geq 2}$ is defined based on q_1 and the Euler equation $\mu_{s^t} q_{s^t} = \mu_{s^{t,1}}(q_{s^{t,1}} +$
 392 $R_{s^{t,1}}) + \mu_{s^{t,2}}(q_{s^{t,2}} + R_{s^{t,2}})$ for all s^t with $t \geq 1$. To ensure optimality of the
 393 efficient allocation, it must be imposed a *P-constraint*, $\lim_t \sum_{s^t} \mu_{s^t} q_{s^t} z_{s^t} \geq$
 394 $\nu(x(z, i) - \omega^i)$ where ν is the capacity of the AD price, (see Araujo et al. (2019)
 395 [4] to avoid the *long-lived improvement opportunities*.

396 As a consequence, the sequential prices $(q_{s^t}^1, q_{s^t}^2)_{s^t}$ satisfy the conditions
 397 exposed before, which implies the relationship between the optimal paths and
 398 the existence of bubbles in the economy.

399 Therefore, using the relationship between W and the existence of bubble,
 400 we have that, in $s^2 = (1, 1)$, $s^4 = (1, 2, 1, 2)$, $s^6 = (1, 2, 1, 1, 1, 2)$, $\dots$, $s^{2t} =$
 401 $(1, 2, 1, \dots, 1, 2)$, $\dots$, the bubble crashes. And, in spite of the infinite number of
 402 paths for which the bubbles crash, the proportion of paths with positive bubble
 403 at any state is positive, in fact,

$$\begin{aligned} \mathbb{P}[\sigma : \inf_s X_{\sigma^s} < X_{\sigma_t} \forall t \in \mathbb{N}] &= \mathbb{P}[\sigma : \inf_s W_{\sigma^s} < W_{\sigma_t} \forall t \in \mathbb{N}] \\ &= \mathbb{P}[\sigma : \text{the bubble in the path } \sigma \text{ does not crash}] \\ &= 1/3. \end{aligned}$$

404 This example helps us to notice that in stochastic economies, the existence of
 405 bubbles in considerably large set of paths is consistent with the crashing of bub-
 406 bles in a large amount of period, for different set of paths at each time. We can
 407 say also that the existence of rational efficient bubbles, as in the deterministic
 408 case, is related to the lack of impatience of the agents.

Figure 1: Distribution of W , existence of bubbles (with red circles) and crashing (with red $\times$'s)

5. Concluding Remarks

Given that the crash of bubbles is impossible in deterministic intertemporal economies, it is natural to consider economies that allow for a large variety of bubble patterns. One of these possible settings is to consider stochastic intertemporal economies with lack of impatience. We observed that the form that agents are worried about distance losses affects the pure charges in the price of the AD economy and the existence of bubbles in the sequential economy. These results (Section 4.1) are independent on the specific form of considering distant losses.

When the agents are worried about distant losses in each path, it is possible

419 to establish a relationship between the existence of positive pure charges in the
420 AD price and the existence of bubbles, as in the deterministic case.

421 To do so, we considered a utility function that is compatible with the model
422 defined by Schmeidler (see [7]) and Gilboa and Schmeidler (1989) (see [6]) that
423 is a generalization of the ε -contamination utility function.

424 The way we modelled impatience in a stochastic framework is also compatible
425 with the existence of AD equilibrium, as in the result by Bewley (see [8]). We
426 also noticed that for this type of utility function, the existence of a positive
427 charge in the AD price is strongly related to the asymptotic consumption of the
428 agents, and then, the existence of efficient bubbles in the intertemporal economy
429 at the first date. Moreover, we proved that the relationship between bubbles
430 and the existence of pure charges is also maintained for every node of the tree.
431 Therefore, to analyze the existence of efficient bubbles at any node, it is enough
432 to know the efficient consumption patterns of the agents. We also analyzed the
433 variation of the assets' price and its relationship with bubbles and their crashes,
434 proving that the crashing of bubbles will increment the variation of the assets'
435 price in the long-run.

436 Finally we gave an example that suggests that the stochastic economies
437 admit bubble patterns that are considerably more diverse than the one observed
438 in deterministic economies. The existence of an infinite number of periods in
439 which crashes can occur is not only possible, it is completely compatible with
440 the existence of bubbles in an infinite number of subtrees.

441 **Appendix A. Other proofs**

442 *Proof of Proposition 1.* The idea of the proof is to separate each set A_1 , A_2
443 and A_3 to analyze them separately, and then, apply the dominated convergence
444 theorem to have the result. For each set A_1 , $A_{2,t_1,\dots,t_K}^K$, $A_{3,t_1,t_2,\dots}$ and σ belonging
445 to any of the previous sets, we have that the analysis that can be done in the
446 path σ is analogous to the deterministic case with a, ε -contamination utility
447 function (see Araujo et al. [4]). Therefore the results will be true for each path

448 σ that belongs to any of the sets described before.

449 Since the collection of all sets that have been described before is 2–2 disjoint
 450 and non enumerable, there is an enumerable sub-collection with positive mea-
 451 sure. Therefore we can rewrite the utility function in terms of these enumerable
 452 sub-collection only, and apply the deterministic case in each path that belongs
 453 to any of this subsets of the collection. Finally, if we apply the dominated con-
 454 vergence theorem (for a collection of generalized limits that are measurables in
 455 $(\{1\} \times N^\infty, \mathcal{N})$), we conclude one part of the proof.

456 To prove the other part, notice that what we have done is to prove that the
 457 integral in σ of elements of the super-differential of the utility function in each
 458 path σ , are in the super-differential of the utility function. And it can be easily
 459 observed that to prove second part is enough to prove that the super-differential
 460 is contained in the composition between the integral and the super-differential
 461 for each path.

462 Using some results of non differential analysis in Banach spaces, we have
 463 that, under the condition that we exposed before, the definition of super-
 464 gradient and Clarke super-gradient are equivalents (see Clarke [10], page 36
 465 Proposition 2.2.7). We also have that, under our hypothesis, the Clarke super-
 466 differential of the utility function is contained in the integral in σ of elements
 467 in the Clarke super-gradients of $\int_{\mathbb{N}} u \circ X_{\sigma_t} d\zeta_i(t) + \beta(\sigma) \inf_t u \circ X_{\sigma_t}$ for each σ
 468 (see Clarke [10], page 76 Proposition 2.7.2). Which concludes the proof of the
 469 proposition. $\square$

470 *Proof of Proposition 5 and Proposition 6.* Note that, for each node s^t , the op-
 471 timal allocation is also efficient in the subtree generated by s^t . This subtree will
 472 be denoted by s^t+ . Therefore there is $\{W^{i,s^t}\}_i$ new “endowment allocation”
 473 such that:

- 474 1. $\sum_i W_{s^k}^{i,s^t} = \sum_i W_{s^k}^i \quad \forall s^k, \pi W^{i,s^t} = \pi W^i \quad \forall i$ where π is the AD price,
- 475 2. $W_{\sigma^r}^{i,s^t} = X_{\sigma^r}^i$ for all σ and r such that $r \leq t$ or σ^r is not in the subtree

476 generated by s^{t12} , and
 477 3. $W_{s^{t+k}}^{i,s^t} = W_{s^{t+k}}^i \forall k \geq 1$ for the rest of the endowment distributions in the
 478 subtree generated in s^t .

479 Note that for some nodes s^t the new “endowments allocation” might be neg-
 480 ative in some states of the economy for some agents. However, it does not imply
 481 that we cannot analyze the economy that it defines¹³ and its relationship with
 482 the original AD economy.

483 To establish this relationship, let us restrict the AD economy such that the
 484 agents maximize their consumption in the subtree generated by s^t only, that is,
 485 the agent consumption set will be:

- 486 1. $\{X_{\sigma^r}^i\}$ for all path σ and r such that $r \leq t$ or σ^r is not in the subtree
 487 generated by s^t , and
- 488 2. $\mathbb{R}_+$ for all s^r in the subtree generated by s^t .

489 Since the wealth of every agent and the efficient allocation are the same as their
 490 counterparts in the initial AD equilibrium, we have that the initial equilibrium
 491 price is in fact an equilibrium price for this restricted economy.

492 Let us analyze the stochastic sequential economy defined by the initial and
 493 the “new” endowment allocation for each s^t . To do so, let us define the new
 494 endowment distribution $\{\omega^{i,s^t}\}_i$ as $\omega^{i,s^t} = W^{i,s^t} - \sum_j R_j z_{j,s^t}^i$. Since the assets’
 495 prices are given by the Euler equations, and the FOC are the same in both cases,
 496 the prices are also the same in both economies. Using the same optimality
 497 conditions that were exposed in Araujo et al. (2019) (see [5]), we know that
 498 using the pure charge, $\nu_i^{s^t}$, the one that takes the highest value on the net trade
 499 $\{X^i - \omega^{i,s^t}\}$, we can implement the efficient allocation if we have

¹²This includes $\{s^{t,-(j)}\}_{j=1}^{t-1}$ all the predecessors of s^t .

¹³The economy defined by $\{W^{i,s^t}\}_i$ only differs with a standard economy on the fact that
 it may have a negative endowment in at most one date or event. In the case that you want to
 avoid the negative endowment in the node s^t , you can choose an endowment allocation $W_{s^{t+k}}^{i,s^t}$
 such that $W_{s^{t+k}}^{i,s^t} > 0$ for $k = 0, \dots, K^{s^t}$ and $W_{s^{t+K+r}}^{i,s^t} = W_{s^{t+K+r}}^i$ for $r \in \mathbb{N}$ satisfying the
 other conditions mentioned above.

$$\nu_i^{s^t} \left(X^i(z) - \omega^{i,s^t} \right) \leq \lim_r \left(\sum_j \sum_{s^r, -(r-t)=s^t} \mu_{s^r} q_{j,s^r} z_{j,s^r} \right)$$

500 for every portfolio z , and with equality for X^i . Then, in order to implement
 501 sequentially this allocation, we have that

$$\lim_r \sum_{s^r, -(r-t)=s^t} \sum_j \mu_{s^r} q_{j,s^r} = \sum_j z_{j,s^t}^i \nu(R_{j,s^t+}) + \nu_i^{s^t} \left(X^i - \omega^{i,s^t} \right) - \nu \left(X^i - \omega^{i,s^t} \right), \quad (\text{A.1})$$

502 and since we have that

$$\mu_{s^t} q_{j,s^t} = \sum_{r>t} \sum_{s^r, -(r-t)=s^t} \mu_{s^r} R_{s^r}^j + \lim_r \sum_{s^r, -(r-t)=s^t} \sum_j \mu_{s^r} q_{j,s^r}$$

503 for each node s^t , the existence of a bubble in the economy at the state s^t is
 504 given by the right part of Equation A.1. Finally, since the subtree generated
 505 by s^t satisfies that $x^i|_{[\sigma:\sigma_t=s^t]} \gg 0$, Proposition 1 can be used restricted to
 506 this set proving that, under the conditions of Proposition 5, the pure charges
 507 in the super-gradient are null in this subtree, and that under the conditions of
 508 Proposition 6, the pure charges in the subtree are such that the right part of
 509 Equation A.1 is positive.

510 □

511 **Remark 2.** In Equation A.1, seems that the right part of the equality could
 512 be negative, something that does not make any sense since it would imply a
 513 negative bubble in the economy. However, this term is at least nonnegative due
 514 to the relationship between $\nu_i^{s^t}$ ¹⁴ and the rest of the pure charges that belong
 515 to the super-differential of U^i when they are evaluated in the direction of the
 516 new “net trade”, $X^i - \omega^{i,s^t}$. Moreover, if we sum Equation A.1 over the agents
 517 we have that

¹⁴The pure charge was chosen to be the pure charge with the highest value on the net trade $\{X^i - \omega^{i,s^t}\}$.

$$\begin{aligned}
I \lim_r \sum_{s^r, -(r-t)=s^t} \sum_j \mu_{s^r} q_{j,s^r} &= \sum_i \sum_j z_{j,s^t}^i \nu(R_{j,s^t+}) + \sum_i \nu_i^{s^t} (X^i - \omega^{i,s^t}) \\
&= \sum_i \sum_j z_{j,0}^i \nu(R_{j,s^t+}) + \sum_i \nu_i^{s^t} (X^i - \omega^{i,s^t}) \\
&\geq 0,
\end{aligned}$$

518 since $\nu_i^{s^t} (X^i - \omega^{i,s^t}) - \nu (X^i - \omega^{i,s^t}) \geq 0$ for each i . Therefore, we have the
519 following results.

520 *Proof of Proposition 7.* Since:

- 521 • the bubble in the economy is characterized by the pure charges that exists
522 in the super-gradient of the agent, more precisely given by A.1,
- 523 • the pure charges that exist in the super-gradient of the agents are integral
524 in σ of a collection of generalized limits, and
- 525 • the probability of each path, $\mathbb{P}(\{\sigma\})$, is zero;

526 we have that $\lim_r \sum_{s^r, -(r-t)=\sigma_t} \sum_j \mu_{s^r} q_{j,s^r} \rightarrow 0$ when $t \rightarrow \infty$. □

527 *Proof of Proposition 8.* From the proof of Proposition 5 and 6, we know that the
528 bubbles depends completely on the behavior of the pure charges in the super-
529 differential of the optimal allocation in the subtree generated by the analyzed
530 node σ_t . Moreover, Proposition 1 ensures that Equation 5 is satisfied.

531 Under condition (1') or (2), we have that the right part of Equation A.1 is
532 bounded from below by a positive constant multiplied by the capacity evaluated
533 in the set $[\tilde{\sigma} \in A^i : \tilde{\sigma}_r = \sigma^r]$. Since $\mathbb{P}[A^i] > 0$, for each $r \geq t$, there is s^r such
534 that $\mathbb{P}[\sigma \in A^i : \sigma_r = s_r] \geq \frac{\mathbb{P}[A^i]}{2|N|\mathbb{P}[\sigma: \sigma_r = s_r]}$ where $|N|$ is the number of elements
535 in N , obtaining Equation 6. □

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