Performance Analysis of Ultra-Dense Networks with Regularly Deployed Base Stations

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Abstract—The concept of Ultra Dense Networks (UDNs) is often seen as a key enabler of the next generation mobile networks. The massive number of BSs in UDNs represents a challenge in deployment, and there is a need to understand the performance behaviour and benefit of a network when BS locations are carefully selected. This can be of particular importance to the network operators who deploy their networks in large indoor open spaces such as exhibition halls, airports or train stations where locations of BSs often follow a regular pattern. In this paper we study performance of UDNs in downlink for regular network produced by careful BS site selection and compare to the irregular network with random BS placement. We first develop an analytical model to describe the performance of regular networks showing many similar performance behaviour to that of the irregular network widely studied in the literature. We also show the potential performance gain resulting from proper site selection. Our analysis further shows an interesting finding that even for over-densified regular networks, a non-negligible system performance could be achieved.

Index Terms—Ultra dense networks, stochastic geometry, regular networks, irregular networks, SINR.

I. INTRODUCTION

THE concept of Ultra Dense Networks (UDN) is often seen as a key enabler of the next generation mobile networks [1], [2]. UDNs are built on the concept of Heterogeneous cellular Networks (HetNet) which provide an effective way for increasing network capacity and achieving higher data rates. In contrast to HetNets, UDNs are expected to provide full network coverage and thus they usually consist of significantly higher number of cells densely deployed within a region. Besides, further densification of cells also offers higher area spectral efficiency which can lead to further increase in network capacity. This potential network capacity gain has triggered growing interest in UDNs from research community, industry and standardization bodies.

In general, densifying a sparse network usually leads to linear increase in the overall network capacity. However, many recent works indicate that as the network reaches a certain high density, interference begins to dominate the network performance leading to declining of network capacity [3], [4]. The precise network density where the network capacity reaches its peak performance is highly influenced by the pathloss model. To study this performance behaviour, existing works seek Stochastic Geometry (SG) to investigate UDN performance under various pathloss models. It is shown that pathloss models play a significant role in the network performances, as some simple pathloss models such as standard power-law unbounded pathloss model may show continuing network capacity improvement as network densifies [5], other more

practical pathloss models indicate a peak network capacity at a certain network density followed by a monotone drop as network continues to densify.

To ensure that a UDN can always operate at its peak capacity, it is necessary to find the optimal density. However, this is difficult to achieve in practice since different environments exhibit different pathloss models. Some recent works focus on other deployment factors that can improve network capacity. The work done by [3] and by us [6] show that antenna height can influence the network capacity. Precisely, lowering antenna height can improve network capacity even when the network density is high. This feature offers network operators a means to maintain peak network capacity even the network is over-densified.

In this paper, we study the behaviour of networks in which the placement of base stations (BSs) is properly selected as well as the impact that the proper BS placement can have on the UDN performance. It is a common belief that proper BS placement can lead to higher network capacity compared with random placement. When BS location are selected randomly producing irregular network, a BS can experience high level of interference caused by neighbouring BS, and hence reducing the overall network performance. On the other hand, proper BS placement that produces regular network maintains a certain distance among neighbouring BSs, leading to more even interference and higher overall network performance. However, proper site selection may incur high cost especially in UDNs since the number of BSs involved in the process is high. We seek SG to investigate the performance impact of regular networks versus irregular networks. Although regular deployments for dense networks may be difficult to achieve, such deployments are often considered by the industry (see e.g. [7], [8], [9]), in particular for indoor deployments in large open spaces such as exhibition halls, stadiums, airports or train stations [9]. Motivated by its tractability, and backed by a number of works which showed that the actual networks deviate from the idealized regular network topology, irregular network topology became widely accepted for modeling cellular networks. Given the above, and considering the complexity of analysis, regular networks have not been well investigated despite the interest from the industry. Our objective is to study performance similarities and differences between both deployments, and to indicate the potential benefits of careful site selection and how such benefits change with network density. The insights provided in our work could help network operators to better understand behavior of UDNs and thus optimize their investments in the infrastructure and network planning.

A. Related work

One of the first works investigating the relationship between network density and system performance is due to Andrews et al. [5]. The work shows surprising results where coverage probability as well as mean achievable data rate per cell, do not depend on BS density (also known as *SINR invariance property* [10]). The constant mean achievable data rate per cell implies that continually increasing the number of BSs in wireless networks could lead to limitless overall network performance improvements. Their conclusions are based on the assumptions of simple power-law unbounded path-loss model, noise-less networks and random BS deployment with BS locations following a Poisson point process. Under these assumptions, as BS density increases, the change in aggregated interference power is counter-balanced by the change in signal power, and thus the SINR remains unchanged regardless of network density [5]. Following the same assumptions, Dhillon et al. [11] shows SINR invariance property for HetNets.

The applicability of the above conclusions has been questioned in [10], [12], [13], [14] where they show that under a multi-slope path-loss model, and random BS deployment in which BS locations follow a Poisson point process, the coverage probability and mean data rate per cell are dependent on BS density. Besides, for unbounded path-loss model including Line Of Sight (LOS) and Non Line of Sight (NLOS) consideration, Ding et al. show in [15] that the network performance depends on BS density. This is in contrast to the earlier SINR invariance property that suggested potentially infinite aggregated data rate of the network resulting from BS densification. Moreover, it has been shown that for certain critical near-field path-loss exponents an optimal network density exists to maximize the coverage probability and mean data rate per cell. Beyond this optimal network density, the coverage probability decreases as BS density increases. These works indicate that the path-loss model is a critical factor affecting the network performance of a UDN.

Most recently, Ding and Perez have investigated the UDN performance (in which BS locations following a Poisson point process) under a path-loss model that includes antenna height setting. Apart from dismissing the SINR invariance property in their considered scenario, they claim that lowering antenna height can improve the network performance but the area spectral efficiency will reach zero as BS density goes to infinity despite the antenna height setting [3], [4]. Similar results have been also presented by Atzeni et. al in [16], [17]. In parallel, we have presented a work in [6] showing the invalidation of SINR invariance property for UDN performance with random BS deployment under path-loss model with antenna height setting. We establish a relationship between network density and antenna height for a particular coverage probability. The relationship shows that while increasing network density leads to a drop in coverage probability, the drop can be totally offset by lowering antenna height to a specific level. We call this the *density countering condition*, and this condition allows maintenance of SINR invariance property.

Motivated by its tractability, most of the existing works focus on random BS deployment ignoring the impact of

BS locations. This deployment arranges the BSs in random location based on Poisson Point Process. Considering the complexity of analysis, networks with regular BS deployment have not been well investigated, except for simulation based studies (e.g. [18], [19], [20]). In [18] the deployment gain defined as the SIR difference between the probability coverage curves for irregular and regular networks was analysed using simulations by Guo and Haenggi. Their work showed that regular network provides the highest average deployment gain from all the considered networks. Interestingly, their work illustrated that performance of regular network can be approximated by shifting the coverage probability curve to the right by value of the average deployment gain determined through simulations. However, as their analysis was based on the use of the unbounded pathloss model, the impact of BS density on the deployment gain has not been addressed. In [19] Guo and Haenggi extend their work by introducing the asymptotic deployment gain, which does not depend on a target success probability, thus allowing for better approximation of performance of networks in which BS locations follow a non Poisson point process. Their findings were then validated using simulations, assuming a bounded path-loss model. Similar to their previous work, their analysis do not consider the impact of BS density on the deployment gain. Chen et al. in [20] made an attempt to investigate the impact of BS deployment using comparative simulation-based study. For both random and regular BS deployments, their study shows that the area network performance peaks at a certain BS density and then starts to decline. Interestingly, their results also indicate that for certain network densities, the difference in performance between irregular and regular BS deployments is constant approximately. Additional work is necessary to further investigate this finding and study the applicability of this behaviour in the UDN scenario. Results for regular networks have been also presented in [5] which showed, using simulations, that regular network provides an upper bound, whilst irregular network provides a lower bound of the SINR experienced by users, compared to the actual BS deployments. In [21] Błaszczyszyn et al. provided results for regular network to illustrate that, given strong shadowing environment, the SINR distribution in a regular network converges to that of irregular network where BSs are arranged randomly based on Poisson point process. Both [5] and [21] works do not provide a comprehensive analysis of regular networks behaviour and, similar to [18], use unbounded pathloss model which as shown in [3], [4], [16], [17], [6] does not provide realistic results for dense network deployments.

B. Contributions

Despite new and interesting insights provided in recent works, there are still many questions related to UDN performance which need to be answered. In this paper, we focus on the impact of BS deployment. Specifically, we study the behaviour of regular networks and compare them to the irregular networks to understand the performance benefit of careful site selection for BS deployment. The following describes the main contributions of this paper.

- As shown in [5], SINR invariance property for irregular networks holds under an unbounded path-loss model and does not hold when a more realistic, bounded path-loss model with antenna height is considered [3], [4], [6], [16], [17]. In our previous work in [6], a density countering condition for irregular network was identified which allows for SINR invariance to be maintained at a cost of adjusting antenna height. The main contribution of our work is the extension of our previous effort to develop an analytical model which provides mathematically tractable results to study the performance of regular networks. Using this model, we found that regular networks share the same performance behaviour as irregular network in many aspects (see Section III-C and Section IV-C). In particular, similar to that of the irregular networks, under unbounded path-loss model, SINR invariance property holds for regular networks too. Likewise, under bounded path-loss model with antenna height setting, SINR invariance property does not hold for regular networks. Additionally, regular networks exhibits the same density counter condition as that of the irregular networks under the bounded path-loss with antenna height setting.
- In [18], [19] it was observed that regular networks provide the highest average deployment gain from all the considered non-PPP based networks for a considered network density. In [5] results for regular networks with square grid were used to demonstrate that regular networks provide an upper performance bound for actual BS deployments for a given network density. Our main contribution is a more thorough analysis of benefits of proper site selection. We found that proper site selection for BS deployment may improve network performance to some extent. We first derive the deployment gain from careful site selection under the SINR invariance property, showing the level of deployment gain for a network with regular deployment. With our derivation, when SINR invariance property does not hold, we demonstrate that the deployment gain can be higher than that of the gain under SINR invariance property. We further demonstrate that for sparse networks, the gain converges to the same level as that under SINR invariance property, and when network density approaches infinity, performance gain from careful site selection vanishes (see Theorem 7).
- In [6] it was observed that Area Spectral Efficiency (ASE, in bps/Hz/m²) for the irregular networks does not necessarily go to zero when BS density approaches infinity, despite per cell rate converging to zero. This was later mathematically confirmed in [22]. In this aspect, the main contribution of this work is that we derive closed form ASE limit expressions to study this behaviour and show that it also appears in regular networks (see Theorem 8).

II. SYSTEM MODEL

A. Network model

We consider a single-tier cellular network utilizing a multiple access technique which ensures orthogonal resource

allocation within a cell. All BSs in the network transmit with the same, constant power (i.e. no downlink power control). We further assume that a mobile user always connects to the BS with the closest BS to the mobile user. The mobile user density λ_u is assumed to be sufficiently higher than the BS density λ such that each BS always has at least one user to serve (i.e. all BSs are active)¹. A regular BS deployment model for one-dimensional (1D) and two-dimensional (2D) Euclidean space is considered in this work. An irregular BS deployment model is also introduced and investigated with the main purpose of providing a baseline for comparison with the regular model.

In network of regular BS deployment, BSs are arranged in a regular geometrical structure within the network. In the case of 1D regular deployment, BSs are deployed regularly every fixed distance along a line. This can represent a scenario where a series of road-side small cells are placed along a motorway. In the case of 2D, BSs are deployed in a regular hexagonal layout with each BS placed at the center of a hexagon. In other words, the 2D locations of BSs are

$$\Phi^{HEX} = \{(\Upsilon(m + n/2), \Upsilon(n\sqrt{3}/2)) | m, n \in \mathbb{Z}\} \in \mathbb{R}^2$$

and the 1D locations of BSs are

$$\Phi^{LINE} = \{\Upsilon i | i \in \mathbb{Z}\} \in \mathbb{R}$$

where $\mathbb{Z}$ and $\mathbb{R}$ are sets of integer and real numbers, respectively. The quantity Υ is the inter-site distance between two adjacent BSs. The spatial density of BSs in the regular deployment is $\lambda = \frac{2}{\Upsilon^2\sqrt{3}}$ and $\lambda = 1/\Upsilon$ for 2D and 1D cases respectively. Mobile users are located uniformly in the network. An illustration of both deployments is given in Figure 1.

In the irregular deployment, the locations of BSs are modelled according to a homogeneous Poisson point process (PPP) Φ^{PPP} with a spatial density of λ . The mean inter-site distances for the 2D and 1D cases are $\Upsilon = \sqrt{\frac{2}{\lambda\sqrt{3}}}$ and $\Upsilon = 1/\lambda$, respectively. Mobile users are uniformly distributed in the Voronoi cell of its corresponding BS.

B. Channel model

As discussed in the literature, the channel model, particularly the path-loss plays an important role in the performance of the UDN. In this work, we consider a path-loss model function that can describe antenna height, $l_1(r) = (h^2 + r^2)^{-\alpha/2}$, where the parameter h is the difference between BS and mobile user antenna heights, r is the horizontal distance between the mobile user and the BS, and α is the path-loss exponent with $\alpha > 1$ for 1D and $\alpha > 2$ for 2D. A special case of this path-loss model is $l_0(r) = r^{-\alpha}$ when the antenna height is set to zero. The path-loss model $l_0(r)$ is a commonly used model in network performance analysis (see e.g. [5] and [11]). In this paper, we mainly focus on $l_1(r)$ but also use $l_0(r)$ for comparison.

¹This can be justified by the increasing number of Internet of Things (IoT) devices. According to Ericsson's IoT forecast, the number of IoT devices has surpassed the number of mobile phones in 2018 and is foreseen to grow further [23]. The growth in IoT popularity means that the density of devices which use cellular networks is no longer limited by a human population.

Fig. 1. Regular BS deployment for 1D (top) and 2D (bottom) Euclidean space

Unless otherwise stated, the random channel effects are modelled as Rayleigh fading with mean 1, thus the received power (for the desired or interference signal) at a user is $g \cdot l_i(r)$, where $g \sim \exp(\mu)$ is an exponentially distributed random variable with a mean value of $1/\mu$ which is equal to the BS transmit power, and $l_i(r)$ is one of our considered two path-loss model functions ($i \in \{0, 1\}$).

It is worth highlighting here that due to high density of BSs in UDNs, desired signal power as well as interference power are significantly higher compared to Gaussian noise. We also show that when the BS density approaches infinity, the UDN with noise in the path-loss model gives exactly the same performance as that without noise (see Theorem 8). This allows us to neglect noise in order to simplify expressions and conduct our investigations.

III. COVERAGE PROBABILITY ANALYSIS

The coverage probability is a measure of chances that a particular user can be served by the network [5]. Formally, it is defined as $p_c \triangleq \mathbb{P}(\text{SINR} > T)$ which is the complementary cumulative distribution function (CCDF) of SINR. For completeness, the results in this section include regular as well as irregular deployments (even though some of the considered cases do not lead to tractable expressions).

Assuming that a user is always associated with the closest BS, the SINR of a user can be written as

$$\text{SINR} = \frac{g \cdot l(r)}{I_r + \mu \sigma^2}, \quad (1)$$

where r is the distance between the considered user and its serving BS, σ^2 is constant receiver noise power, μ is the inverse of BS transmit power and

$$I_r = \sum_{x \in \Phi \setminus b_o} g_x l(\|x - u_0\|), \quad (2)$$

is the interference power which is the sum of received power from all surrounding BSs except the user's serving BS at b_o and $\|x - u_0\|$ is the distance between each BS and the user located at u_0 .

It is worth highlighting here that (given the assumption that all BSs transmit with the same power) the BS transmit power $1/\mu$ in the SINR expression above impacts only the noise factor σ^2 . Interestingly, this means that lowering the BS transmit power whilst increasing network density (which is a common practice) would not necessarily lead to improvement in the SINR. This means that a downlink power control which tunes downlink transmit power uniformly across all BSs may not lead to system performance improvement.

Depending on the considered BS deployment the distance r separating a typical user from its serving BS can have different distributions. For regular BS deployment, r is uniformly distributed leading to following expressions (for 1D and 2D Euclidean space, respectively)

$$f_r(r) = \frac{2}{\Upsilon}, r \leq \frac{\Upsilon}{2} \quad (3)$$

$$f_r(r) = \frac{6 \cdot 2r}{\Upsilon^2 \sqrt{3}}, 0 \leq r \leq \frac{\Upsilon}{2 \sin(\theta + \frac{\pi}{3})} \quad (4)$$

where Υ is the inter-site distance and $0 \leq \theta \leq \frac{\pi}{3}$.

In case of irregular BS deployment, as shown in [5], r is shown to be Rayleigh distributed. The following expression presents a distribution function of r for a d -dimensional Euclidean space

$$f_r(r) = c_d d r^{d-1} \lambda e^{-c_d \lambda r^d}, r \geq 0 \quad (5)$$

where $d \in \{1, 2\}$ is the dimension of the Euclidean space and $c_d \triangleq \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is volume of the d -dimensional unit ball (see [24]). Given 1D and 2D space, $c_1 = 2$, $c_2 = \pi$.

Definition 1. Based on Theorem 1 in [5], the coverage probability in the downlink for a generic path-loss model function, assuming Rayleigh fading, can be represented as

$$p_c^{\{d,i\}}(T, \lambda, \alpha) = \int_{\mathbb{R}^d} e^{-\frac{\mu \sigma^2 T}{l(r)}} \mathcal{L}_{I_r} \left(\frac{\mu T}{l(r)} \right) f_r(r) dr, \quad (6)$$

where $\mathcal{L}_{I_r}(s)$ is the Laplace transform of the random variable I_r , d is the dimension of the Euclidean space and $i \in \{0, 1\}$ indicates the path-loss model function.

In the following subsections, we derive formulas for coverage probability for regular network and study the extreme condition where BS density approaches infinity. Next, we provide simplified expressions for irregular network based on the formulas found in the literature to draw a parallel with regular network.

A. Regular 1D network

We begin by defining a network where the locations of BSs are generated using a deterministic points process Φ^{LIN} . We study the coverage probability of a user located at a particular point in this network. Without loss of generality we assume that this user is located in the origin and thus the BS locations form the following point process on $\mathbb{R}$

$$\hat{\Phi}_r^{LIN} = \{x - y \mid x \in \Phi^{LIN}\}, \quad (7)$$

where $r = \|y\|$ is a distance between a user and its serving (closest) BS at point y .

We now formulate two lemmas for the coverage probability in 1D regular network under our considered path-loss models $l_0(r)$ and $l_1(r)$.

Lemma 1. *The coverage probability for 1D regular network under $l_0(r)$ path-loss model function can be expressed as*

$$p_c^{\{1,0\}}(T, \lambda, \alpha) = \frac{2}{\Upsilon} \int_0^{\frac{\Upsilon}{2}} e^{-\mu T \sigma^2 r^\alpha} \mathcal{L}_{I_r}(\mu T r^\alpha) dr, \quad (8)$$

where $\Upsilon = \frac{1}{\lambda}$, and

$$\mathcal{L}_{I_r}(s) = 2 \prod_{i=1}^{\infty} \left(\frac{1}{1 + (r + i \Upsilon)^{-\alpha} \frac{s}{\mu}} \right). \quad (9)$$

Proof. The proof of Lemma 1 is based on the proof of Theorem 1 provided in [5]. As the desired signal is assumed to be exponentially distributed, by substituting $f_r(r) = \frac{2}{\Upsilon}$ in (6), the coverage probability in 1D regular network under a generic path-loss model function can be expressed as

$$p_c^{\{1,i\}}(T, \lambda, \alpha) = \frac{2}{\Upsilon} \int_0^{\frac{\Upsilon}{2}} e^{-\frac{\mu T \sigma^2}{l(r)}} \mathcal{L}_{I_r}\left(\frac{\mu T}{l(r)}\right) dr. \quad (10)$$

Using the definition of the Laplace transform we can show that

$$\begin{aligned} \mathcal{L}_{I_r}(s) &= \mathbb{E}_{I_r}[e^{-s I_r}] \\ &= \mathbb{E}_{\{g_x\}} \left[\exp\left(-s \sum_{x \in \hat{\Phi}_r^{LIN} \setminus \{b_o\}} g_x l(\|x\|)\right) \right] \\ &\stackrel{(a)}{=} \prod_{x \in \hat{\Phi}_r^{LIN} \setminus \{b_o\}} \mathbb{E}_g[\exp(-s g l(\|x\|))] \\ &\stackrel{(b)}{=} \prod_{x \in \hat{\Phi}_r^{LIN} \setminus \{b_o\}} \frac{\frac{\mu}{l(\|x\|)}}{\frac{\mu}{l(\|x\|)} + s}, \end{aligned} \quad (11)$$

where (a) follows from the i.i.d distribution of g_x and its independence from the point process $\hat{\Phi}_r^{LIN}$ and (b) from the Laplace transform of an exponential random variable with mean $\frac{\mu}{l(\|x\|)}$. Using the above result, and taking into account that locations of BSs in 1D regular network follow Φ^{LIN} , the Laplace transform can be further expressed as

$$\mathcal{L}_{I_r}(s) = \prod_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \left(\frac{1}{1 + l(\|r + i \Upsilon\|) \frac{s}{\mu}} \right), \quad (12)$$

where $\|r + i \Upsilon\|$ is the distance from the typical user to the i -th base station.

By substituting $l(r) = l_0(r)$ into (10) and removing the absolute value expression from the product in the expression for the Laplace transform presented above, we immediately obtain the final result. Note that, as Φ^{LIN} is deterministic, (11) does not involve computation of the probability generating functional (PGFL). ■

Lemma 2. *The coverage probability for 1D regular network under $l_1(r)$ path-loss model function can be expressed as*

$$p_c^{\{1,1\}}(T, \lambda, \alpha) = \frac{2}{\Upsilon} \int_0^{\frac{\Upsilon}{2}} e^{-\mu T \sigma^2 (r^2 + h^2)^{\alpha/2}} \cdot \mathcal{L}_{I_r}(\mu T (r^2 + h^2)^{\alpha/2}) dr, \quad (13)$$

where $\Upsilon = \frac{1}{\lambda}$, and

$$\mathcal{L}_{I_r}(s) = 2 \prod_{i=1}^{\infty} \left(\frac{1}{1 + ((r + i \Upsilon)^2 + h^2)^{-\alpha/2} \frac{s}{\mu}} \right). \quad (14)$$

Proof. The proof follows that of Lemma 1. ■

It is worth noting here that setting $h = 0$ in $l_1(r)$ reduces $p_c^{\{1,1\}}(T, \lambda, \alpha)$ to $p_c^{\{1,0\}}(T, \lambda, \alpha)$.

Based on Lemma 1 and Lemma 2, we have the following theorems (i.e. Theorem 1 - 3).

Theorem 1. *The coverage probability of noise-less 1D regular network under the standard power-law unbounded path-loss model function $l_0(r)$ does not depend on BS density λ .*

Proof. The proof is based on Lemma 1. By substituting $t = \frac{r}{\Upsilon}$ and assuming $\sigma^2 \rightarrow 0$ we obtain

$$p_c^{\{1,0\}}(T, \lambda, \alpha) = 4 \int_0^{\frac{1}{2}} \prod_{i=1}^{\infty} \left(\frac{1}{1 + T \frac{t^\alpha}{(t+i)^\alpha}} \right) dt. \quad (15)$$

From the above formula it can be easily seen that the probability of coverage is independent of λ . ■

Theorem 2. *The coverage probability of 1D regular networks under path-loss model $l_1(r)$ tends to 0 as BS density $\lambda \rightarrow \infty$, that is*

$$\lim_{\lambda \rightarrow \infty} p_c^{\{1,1\}}(T, \lambda, \alpha) = 0. \quad (16)$$

Proof. The proof is based on Lemma 2. By substituting $t = \frac{r}{\Upsilon}$, and given that $\Upsilon = \frac{1}{\lambda}$ we then obtain

$$p_c^{\{1,1\}}(T, \lambda, \alpha) = 4 \int_0^{\frac{1}{2}} e^{-\mu T \sigma^2 ((\frac{t}{\lambda})^2 + h^2)^{\alpha/2}} \cdot \prod_{i=1}^{\infty} \left(\frac{1}{1 + T \left(\frac{t^2 + (h\lambda)^2}{(t+i)^2 + (h\lambda)^2} \right)^{\alpha/2}} \right) dt \quad (17)$$

By taking $\lambda \rightarrow \infty$ and applying the Monotone Convergence Theorem to above formula, we further get

$$\lim_{\lambda \rightarrow \infty} p_c^{\{1,1\}}(T, \lambda, \alpha) = 4 \int_0^{\frac{1}{2}} \lim_{\lambda \rightarrow \infty} e^{-\mu T \sigma^2 ((\frac{t}{\lambda})^2 + h^2)^{\alpha/2}} \cdot \prod_{i=1}^{\infty} \lim_{\lambda \rightarrow \infty} \left(\frac{1}{1 + T \left(\frac{t^2 + (h\lambda)^2}{(t+i)^2 + (h\lambda)^2} \right)^{\alpha/2}} \right) dt. \quad (18)$$

Simplifying the above expression yields

$$\lim_{\lambda \rightarrow \infty} p_c^{\{1,1\}}(T, \lambda, \alpha) = 4 \int_0^{\frac{1}{2}} e^{-\mu T \sigma^2 h^\alpha} \prod_{i=1}^{\infty} \left(\frac{1}{1 + T} \right) dt. \quad (19)$$

Next, given that $T > 0$,

$$\prod_{i=1}^{\infty} \frac{1}{1+T} = 0, \quad (20)$$

which brings the expression in (19) to zero. ■

Theorems 1 and 2 show that 1D regular network exhibits SINR invariance property for $l_0(r)$ path-loss model. However, it does not exhibit SINR invariance property for $l_1(r)$ path-loss model. This is in line with the recent works for irregular networks [10], [12], [13], [15], [16], [17] which dismiss the SINR invariance property for non-standard path-loss models. In other words, for $h > 0$, the increase in the interference power is not counter-balanced by the increase in the signal power.

In the following, we develop the density countering condition using antenna height for 1D regular network.

Theorem 3. *The coverage probability of noise-less 1D regular networks under path-loss model $l_1(r)$ is constant when $\lambda h = c$ and c is constant.*

Proof. The proof is based on the proof of Lemma 2. By substituting $\sigma^2 = 0$ and $\lambda h = c$ in (17) we obtain the following expression which is constant for a constant c

$$p_c^{\{1,1\}}(T, \lambda, \alpha) = 4 \int_0^{\frac{1}{2}} \prod_{i=1}^{\infty} \left(\frac{1}{1 + T \left(\frac{t^2 + c^2}{(t+i)^2 + c^2} \right)^{\alpha/2}} \right) dt. \quad (21)$$

Lemma 1 allows us to numerically calculate coverage probability for arbitrary $\alpha > 1$ and a generic path-loss model function $l(r)$. This can be practically achieved by truncating the infinite product in expressions (12), which is equivalent to limiting the number of interfering BSs around the targeted cell². However, by considering some integer α values we can obtain a simpler expression which allow us to gain additional insight. In the following propositions, we derive coverage probability expressions for $\alpha = 2$ for the considered path-loss model functions.

Proposition 1. *The coverage probability for 1D regular network using the standard path-loss model function $l_0(r) = r^{-\alpha}$, when $\alpha = 2$ is*

$$p_c^{\{1,0\}}(T, \lambda, 2) = \frac{(1+T)}{\pi} \int_0^{\pi} e^{-\frac{\mu T \sigma^2 x^2}{(2\pi\lambda)^2}} \cdot \frac{\cos(x) - 1}{\cos(x) - \cosh(x\sqrt{T})} dx. \quad (22)$$

²To select an appropriate number of interfering BSs to be included for numerical computation, one needs to ensure that the impact of interfering BSs excluded from the computation on interference I_r is below certain small ϵ value.

Proof. Using Lemma 1, we start by expressing the Laplace transform of I_r as

$$\mathcal{L}_{I_r}(s) = \prod_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \left(\frac{1}{1 + \frac{\frac{s}{\mu T^2}}{(\frac{r}{T} + i)^2}} \right). \quad (23)$$

By using the following expression for an infinite product $\prod_{k=-\infty}^{\infty} \left(1 + \frac{z}{(k+a)^2 + b^2} \right) = \frac{\cos(2\pi a) - \cos(2\pi\sqrt{-b^2 - z})}{\cos(2\pi a) - \cosh(2\pi b)}$,

and with the setting of $b = 0$, $z = \frac{-s}{\mu T^2}$, $a = \frac{r}{T}$ and the exclusion of $k = 0$, we obtain

$$\mathcal{L}_{I_r}(s) = \frac{(r^2 + \frac{s}{\mu})(\cos(2\pi\frac{r}{T}) - 1)}{r^2(\cos(2\pi\frac{r}{T}) - \cosh(2\pi\frac{r}{T}\sqrt{T}))}. \quad (25)$$

Plugging in $s = \mu T r^2$ and substituting $x = 2\pi\lambda r$ to the above result concludes the proof. ■

Proposition 2. *The coverage probability for 1D regular network using the path-loss model function $l_1(r)$, when $\alpha = 2$ can be expressed as*

$$p_c^{\{1,1\}}(T, \lambda, 2) = \frac{1+T}{\pi} \int_0^{\pi} e^{-\frac{\mu T \sigma^2 x^2}{(2\pi\lambda)^2}} \cdot \frac{\cos(x) - \cosh(2\pi\lambda h)}{\cos(x) - \cosh(\sqrt{(2\pi\lambda h)^2 + T(x^2 + (2\pi\lambda h)^2))} dx. \quad (26)$$

Proof. The proof follows that of Lemma 1 but in (24), the substitution $b = \frac{h}{d}$ is used instead. ■

B. Regular 2D network

Similar to the 1D regular network, we consider a network generated by the deterministic point process $\Phi^{H\!E\!X}$, and we study the coverage probability of a user located at the origin. We first see that the BS locations form the following point process on $\mathbb{R}^2$

$$\hat{\Phi}_{r,\theta}^{H\!E\!X} = \{(x - r \cos \theta, y - r \sin \theta) \mid (x, y) \in \Phi^{H\!E\!X}\}, \quad (27)$$

where r and θ are the distance and angle between the user, and its serving BS, respectively.

Following a similar approach as in 1D regular network, we first provide the following lemmas for the coverage probability in 2D regular network under our considered path-loss models.

Lemma 3. *The coverage probability for 2D regular network under $l_0(r)$ path-loss model function can be expressed as*

$$p_c^{\{2,0\}}(T, \lambda, \alpha) = \frac{12}{\Upsilon^2 \sqrt{3}} \int_0^{\frac{\pi}{3}} \int_0^{\frac{\Upsilon}{2 \sin(\theta + \frac{\pi}{3})}} e^{-\mu \sigma^2 T r^\alpha} \cdot \mathcal{L}_{I_{r,\theta}}(\mu T r^\alpha) r dr d\theta, \quad (28)$$

where $\Upsilon = \sqrt{\frac{2}{\lambda \sqrt{3}}}$, and

$$\mathcal{L}_{I_{r,\theta}}(s) = \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left[1 + \left(\left(\Upsilon \left(m + \frac{n}{2} \right) - r \cos \theta \right)^2 + \left(\Upsilon n \frac{\sqrt{3}}{2} - r \sin \theta \right)^2 \right)^{-\frac{\alpha}{2}} \frac{s}{\mu} \right]^{-1}. \quad (29)$$

$$\lim_{\lambda \rightarrow \infty} p_c^{\{2,1\}}(T, \lambda, \alpha) = 2\sqrt{3} \int_0^{\frac{\pi}{3}} \int_0^{\frac{1}{2\sin(\theta+\frac{\pi}{3})}} \lim_{\lambda \rightarrow \infty} e^{-\mu\sigma^2 T(\frac{2}{\lambda\sqrt{3}}t+h^2)^{\alpha/2}} \cdot \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \lim_{\lambda \rightarrow \infty} \left[1 + T \cdot \left(\frac{t + \frac{\sqrt{3}}{2}\lambda h^2}{(m + \frac{n}{2} - \sqrt{t}\cos\theta)^2 + (n\frac{\sqrt{3}}{2} - \sqrt{t}\sin\theta)^2 + \frac{\sqrt{3}}{2}\lambda h^2} \right)^{\frac{\alpha}{2}} \right]^{-1} dt d\theta \quad (35)$$

Proof. The proof of Lemma 3 is analogous to that of Lemma 1 and is based on the proof of Theorem 1 provided in [5]. As the desired signal is assumed to be exponentially distributed, by substituting (4) in (6), the coverage probability can be expressed as

$$p_c^{\{2,i\}}(T, \lambda, \alpha) = \frac{12}{\Upsilon^2\sqrt{3}} \int_0^{\frac{\pi}{3}} \int_0^{\frac{\Upsilon}{2\sin(\theta+\frac{\pi}{3})}} e^{-\frac{\mu\sigma^2 T}{l(r)}} \cdot \mathcal{L}_{I_{r,\theta}} \left(\frac{\mu T}{l(r)} \right) r dr d\theta. \quad (30)$$

Similar to Proposition 3, by using the definition of the Laplace transform we can show that

$$\begin{aligned} \mathcal{L}_{I_{r,\theta}}(s) &= \mathbb{E}_{I_{r,\theta}}[e^{-sI_{r,\theta}}] \\ &= \mathbb{E}_{\{g_u\}} \left[\exp(-s \sum_{u \in \hat{\Phi}_{r,\theta}^{H\text{EX}} \setminus \{b_o\}} g_u l(\|u\|)) \right] \\ &\stackrel{(a)}{=} \prod_{u \in \hat{\Phi}_{r,\theta}^{H\text{EX}} \setminus \{b_o\}} \mathbb{E}_g[\exp(-s g l(\|u\|))] \\ &\stackrel{(b)}{=} \prod_{u \in \hat{\Phi}_{r,\theta}^{H\text{EX}} \setminus \{b_o\}} \frac{\frac{\mu}{l(\|u\|)}}{\frac{\mu}{l(\|u\|)} + s}, \end{aligned} \quad (31)$$

where (a) follows from the i.i.d distribution of g_u and it is independence of the point process $\hat{\Phi}_{r,\theta}^{H\text{EX}}$, and (b) follows from the Laplace transform of an exponential random variable with mean $\frac{\mu}{l(\|u\|)}$. Using the above, and taking into account that locations of BSs in 2D regular network follow $\Phi^{H\text{EX}}$, the Laplace transform can be further expressed as

$$\mathcal{L}_{I_{r,\theta}}(s) = \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left[1 + l(\| \Upsilon(m + \frac{n}{2}) - r \cos \theta, \right. \\ \left. \Upsilon n \frac{\sqrt{3}}{2} - r \sin \theta \|) \frac{s}{\mu} \right]^{-1}.$$

By substituting $l(r) = l_0(r)$ into (30), and removing the absolute value expression from the product in the expression for the Laplace transform presented above, we immediately obtain the final result which concludes the proof. ■

Lemma 4. The coverage probability for 2D regular network under $l_1(r)$ path-loss model function can be expressed as

$$p_c^{\{2,1\}}(T, \lambda, \alpha) = \frac{12}{\Upsilon^2\sqrt{3}} \int_0^{\frac{\pi}{3}} \int_0^{\frac{\Upsilon}{2\sin(\theta+\frac{\pi}{3})}} e^{-\mu\sigma^2 T(r^2+h^2)^{\alpha/2}} \cdot \mathcal{L}_{I_{r,\theta}} \left(\mu T(r^2+h^2)^{\alpha/2} \right) r dr d\theta, \quad (32)$$

where

$$\mathcal{L}_{I_{r,\theta}}(s) = \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left[1 + \left(\left(\Upsilon(m + \frac{n}{2}) - r \cos \theta \right)^2 + \left(\Upsilon n \frac{\sqrt{3}}{2} - r \sin \theta \right)^2 + h^2 \right)^{-\frac{\alpha}{2}} \frac{s}{\mu} \right]^{-1}. \quad (33)$$

Proof. The proof follows that of Lemma 3. ■

Theorem 4. The coverage probability for noise-less 2D regular network under the standard path-loss model function $l_0(r) = r^{-\alpha}$ does not depend on BS density λ .

Proof. The proof is based on Lemma 3. By substituting $t = \frac{r^2}{\Upsilon^2}$ and assuming $\sigma^2 \rightarrow 0$, the coverage probability of 2D regular network can be expressed as

$$p_c^{\{2,0\}}(T, \lambda, \alpha) = 2\sqrt{3} \int_0^{\frac{\pi}{3}} \int_0^{\frac{1}{2\sin(\theta+\frac{\pi}{3})}} \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left[1 + \frac{T t^{\frac{\alpha}{2}}}{\left((m + \frac{n}{2} - \sqrt{t}\cos\theta)^2 + (n\frac{\sqrt{3}}{2} - \sqrt{t}\sin\theta)^2 \right)^{\frac{\alpha}{2}}} \right]^{-1} dt d\theta.$$

From the above formula it can be easily seen that the coverage probability is independent of λ . ■

Theorem 5. The coverage probability of 2D regular networks under the path-loss model $l_1(r)$ tends to 0 as BS density $\lambda \rightarrow \infty$, that is

$$\lim_{\lambda \rightarrow \infty} p_c^{\{2,1\}}(T, \lambda, \alpha) = 0. \quad (34)$$

Proof. The proof is based on Lemma 4. We start by substituting $t = \frac{r^2}{\Upsilon^2}$ and $\Upsilon^2 = \frac{2}{\lambda\sqrt{3}}$ into (32). Next, by applying the Monotone Convergence Theorem to the obtained formula and some algebraic manipulation, we obtain the expression in (35) which can be further simplified to

$$\lim_{\lambda \rightarrow \infty} p_c^{\{2,1\}}(T, \lambda, \alpha) = 2\sqrt{3} \int_0^{\pi/3} \int_0^{\frac{1}{2\sin(\theta+\frac{\pi}{3})}} e^{-\mu\sigma^2 T h^\alpha} \cdot \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left(\frac{1}{1+T} \right) dt d\theta. \quad (36)$$

Next, given that $T > 0$,

$$\prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left(\frac{1}{1+T} \right) = 0, \quad (37)$$

and thus bring the expression (36) to zero. ■

In the previous subsection, we showed the density counteracting condition for regular 1D network under $l_1(r)$. As shown in the following theorem, this condition also applies to 2D regular network.

Theorem 6. *The coverage probability of a noise-less 2D regular network for the path-loss model function $l_1(r)$ does not change when $\lambda h^2 = c$ and c is constant.*

Proof. The proof is based on the proof of Theorem 5. By substituting $\sigma^2 = 0$ and $\lambda h^2 = c$ in (32), we obtain the following expression which is constant for a constant c

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} p_c^{\{2,1\}}(T, \lambda, \alpha) \\ = 2\sqrt{3} \int_0^{\pi/3} \int_0^{\frac{1}{2\sin(\theta+\frac{\pi}{3})}} \prod_{\substack{(n,m) \in \\ \mathbb{Z}^2 \setminus \{(0,0)\}}} \left(\frac{1}{1+T\beta(n,m)} \right) dt d\theta, \end{aligned}$$

where

$$\beta(n, m) = \left(\frac{t + \frac{\sqrt{3}}{2}c}{(m + \frac{n}{2} - \sqrt{t} \cos \theta)^2 + (n \frac{\sqrt{3}}{2} - \sqrt{t} \sin \theta)^2 + \frac{\sqrt{3}}{2}c} \right)^{\frac{\alpha}{2}}.$$

■

In contrast to Lemma 1, the Laplace transform of the interference power $\mathcal{L}_{I_r, \theta}(s)$ in Lemma 3 does not have a closed form expression. However, if desirable, the coverage probability under $l_0(r)$, $l_1(r)$ may be calculated numerically.

C. Comparison with Irregular Network

Irregular networks have been well studied in the literature. In the following we provide expressions for irregular network based on formulas found in ([5], [6], [3]) to draw a parallel with regular networks. We revise the existing expressions to focus on our considered path-loss models $l_0(r)$ and $l_1(r)$ for 1D and 2D networks.

We start by providing formula for probability of coverage for 1D and 2D networks under $l_0(r) = r^{-\alpha}$ [5]

$$p_c^{\{d,0\}}(T, \lambda, \alpha) = \int_0^\infty e^{-\mu T \sigma^2 (\frac{k}{c_d d \lambda})^{\alpha/d}} \cdot e^{-k(1+\rho_d(T, \alpha))} dk, \quad (38)$$

where

$$c_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}, \quad (39)$$

and

$$\rho_d(T, \alpha) = T^{d/\alpha} \int_{T^{-d/\alpha}}^\infty \frac{1}{1+u^{\alpha/d}} du. \quad (40)$$

By assuming noise-less conditions, the above formula can be simplified to

$$p_c^{\{d,0\}}(T, \lambda, \alpha) = \frac{1}{1+\rho_d(T, \alpha)}. \quad (41)$$

The coverage probabilities for irregular 1D and 2D networks under $l_1(r)$ path-loss model function can be expressed, based on the findings in [3], as shown below. Notice that both formulas for $l_1(r)$ reduce to the coverage probability for $l_0(r) = r^{-\alpha}$, when $h = 0$.

$$p_c^{\{1,1\}}(T, \lambda, \alpha) = 2 \int_0^\infty e^{-2k} e^{-\mu T \sigma^2 (\frac{k^2}{\lambda^2} + h^2)^{\alpha/2}} \cdot e^{-2\xi(T, \alpha, k)} dk \quad (42)$$

where

$$\xi(T, \alpha, k) = \int_k^\infty \frac{1}{1+T^{-1} \left(\frac{t^2 + (h\lambda)^2}{k^2 + (h\lambda)^2} \right)^{\frac{\alpha}{2}}} dt \quad (43)$$

and for 2D irregular network,

$$p_c^{\{2,1\}}(T, \lambda, \alpha) = \pi e^{-\lambda \pi h^2 \rho_2(T, \alpha)} \int_0^\infty e^{-\mu T \sigma^2 (\frac{k}{\lambda} + h^2)^{\alpha/2}} \cdot e^{-\pi k(1+\rho_2(T, \alpha))} dk. \quad (44)$$

The coverage probabilities for irregular 1D and 2D networks under $l_1(r)$ can be computed numerically using the above formulas. Similar to the expression for $l_0(r)$, the above result also permits simple expressions for noise-less condition. For the case of 2D network, we can derive a closed form expression. In case of 1D network, an expression which requires a single numerical integration can be obtained for some specific α values, such as $\alpha = 2$. Both expressions are presented below.

$$p_c^{\{1,1\}}(T, \lambda, 2) = 2 \int_0^\infty e^{-2k} \cdot e^{\frac{T((\lambda h)^2 + k^2)(\arctan(k^{-1}\sqrt{T((\lambda h)^2 + k^2) + (\lambda h)^2}))}{\sqrt{T((\lambda h)^2 + k^2) + (\lambda h)^2}}} dk \quad (45)$$

$$p_c^{\{2,1\}}(T, \lambda, \alpha) = \frac{e^{-\lambda \pi h^2 \rho_2(T, \alpha)}}{1+\rho_2(T, \alpha)}. \quad (46)$$

Based on the above expressions and the findings on the behaviour of coverage probability in [5], [6], [3], it can be easily seen that SINR invariance holds (resp. does not hold) for 1D and 2D irregular networks under $l_0(r)$ and (resp. $l_1(r)$). The above expressions also show that BS density λ in 1D and 2D irregular network can be counter-balanced by the adjustments of the $l_1(r)$ path-loss model parameter h to maintain the same coverage probability (i.e. density counteracting condition). This as well as the findings of the previous section clearly indicate that both regular and irregular networks exhibit

similar behaviour. This surprising result suggests that various conclusions on the widely studied irregular networks could also be extended to regular networks.

IV. AVERAGE ACHIEVABLE RATE ANALYSIS

In the following section, we focus on the analysis of the mean achievable data rate over a cell. More specifically, we compute the ergodic capacity which measures the long-term achievable rate averaged over all channel and network realizations [25].

Definition 2. *The average ergodic rate achievable over a cell in the downlink, assuming Rayleigh fading for desired signal, can be represented as (see Theorem 3 in [5])*

$$\begin{aligned}\tau^{\{d,i\}}(\lambda, \alpha) &\triangleq \frac{1}{\ln 2} \mathbb{E}[\ln(1 + \text{SINR})] \\ &= \frac{1}{\ln 2} \int_{\mathbb{R}^d} f_r(r) \int_{\gamma_0}^{\infty} e^{-\frac{\mu \sigma^2 (e^t - 1)}{l(r)}} \mathcal{L}_{I_r} \left(\frac{\mu(e^t - 1)}{l(r)} \right) dt dr \\ &= \frac{1}{\ln 2} \int_{\gamma_0}^{\infty} p_c^{\{d,i\}}(e^t - 1, \lambda, \alpha) dt\end{aligned}\quad (47)$$

where d is the dimension of the Euclidean space and $i \in \{0, 1\}$ indicates the path-loss model function. The quantity γ_0 is the minimum working SINR. In this paper, we set $\gamma_0 = 0$ to investigate the performance upper bound.

The average achievable rate per cell can be computed numerically using Definition 2 with formulas for the coverage probability. However, since we have derived some closed form expressions for the coverage probability under certain conditions, further insights can be obtained for the per cell average achievable rate. In the following, we focus on deriving the average achievable rate per cell for these conditions.

A. Regular 1D network

In the previous section, we have derived a closed form coverage probability expression for noise-less 1D network under $l_0(r)$ path-loss model with $\alpha = 2$. By substituting (22) into (47) and employing a change of variables $x = 2\pi\lambda r$, $k = \sqrt{e^t - 1}$, we obtain the average rate per cell as

$$\tau^{\{1,0\}}(\lambda, 2) = \frac{2}{\ln 2 \pi} \int_0^{\infty} \int_0^{\pi} \frac{k(\cos(x) - 1)}{\cos(x) - \cosh(xk)} dx dk. \quad (48)$$

The above expression indicates a constant per cell average achievable rate regardless of BS density which is similar to what was shown in [5] for irregular networks. This expression also allows us to numerically compute the per cell average rate.

Next, for path-loss model $l_1(r)$, by substituting (22) into (47) and employing a change of variables $k = e^t - 1$, the average achievable rate per cell with $\alpha = 2$ can be expressed as

$$\begin{aligned}\tau^{\{1,1\}}(\lambda, 2) &= \frac{1}{\ln 2 \pi} \\ &\cdot \int_0^{\infty} \int_0^{\pi} \frac{\cos(x) - \cosh(2\pi\lambda h)}{\cos(x) - \cosh(\sqrt{((2\pi\lambda h)^2(1+k) + x^2 k})})} dx dk.\end{aligned}\quad (49)$$

From the last result, by considering $l_1(r)$ with $\alpha = 2$, we see that the average achievable rate per cell depends on BS density (see also Figure 2). Based on Definition 2 and Lemma 2, we further know the average achievable rate per cell approaches zero when BS density goes to infinity. However, by setting $h = \frac{c}{\lambda}$ where c is some constant positive value, we can maintain a constant average achievable rate per cell regardless of BS density. This argument is also valid for general condition of $\alpha > 1$ by observing Lemmas 2 and 3 in conjunction with Definition 2.

B. Regular 2D network

As the probability of coverage for 2D regular network does not yield any closed form expression, we seek numerical computation to calculate its average achievable rate per cell. We present the numerical results in Figure 3 illustrating the impact of λ and h on the average ergodic rate per cell $\tau^{\{2,1\}}(\lambda, \alpha)$. As can be seen, the rate of decay of $\tau^{\{2,1\}}(\lambda, \alpha)$ for an increasing λ depends on h and $\tau^{\{2,1\}}(\lambda, \alpha)$ tends to zero. This is in line with the results for other network configurations discussed in the previous and next subsections.

Additionally, based on Theorem 6 and Definition 2, it can be easily shown that the average achievable rate per cell in the noise-less 2D regular network under path-loss model $l_1(r)$ does not change when $\lambda h^2 = c$ and c is constant. In other words, it is possible to maintain per cell average achievable rate by countering the increase in BS density through lowering the antenna height accordingly.

Furthermore, Theorem 4 and Definition 2 indicate that the 2D noise-less regular network under $l_0(r)$ exhibits SINR invariance property for all conditions. Consequently, the average rate per cell for $l_0(r)$ does not depend on BS density.

C. Comparison with Irregular Network

Similar to Section III-C, to draw a parallel with regular networks, we revise the expressions for irregular networks provided in the literature (see e.g. [5], [3], [6]) to focus on our considered path-loss models $l_0(r)$ and $l_1(r)$ for 1D and 2D networks.

The average ergodic rate in the downlink of noise-less irregular network for path-loss model $l_1(r)$ for 1D and 2D can be respectively expressed using [3] findings as

$$\begin{aligned}\tau^{\{1,1\}}(\lambda, 2) &= \frac{2\lambda}{\ln 2} \int_0^{\infty} \int_0^{\infty} e^{-2\lambda r} \\ &\cdot e^{\frac{2\lambda (e^t - 1)(h^2 + r^2)(\arctan(r((e^t - 1)(h^2 + r^2) + h^2))^{-1/2}) - \pi/2}{\sqrt{(e^t - 1)(h^2 + r^2) + h^2}}} dr dt\end{aligned}\quad (50)$$

and

$$\tau^{\{2,1\}}(\lambda, \alpha) = \frac{1}{\ln 2} \int_0^{\infty} \frac{e^{-\pi\lambda h^2(\rho_2(e^t - 1, \alpha))}}{1 + \rho_2(e^t - 1, \alpha)} dt. \quad (51)$$

The above results are obtained by substituting (45) into (47) and (46) into (47), as well as setting $\sigma^2 = 0$.

Fig. 2. Impact of BS density λ and h on average ergodic rate τ of noise-less 1D network for different path-loss models with $\alpha = 2$. The path-loss parameter h affects the rate of decay.

Note that by simply setting $h = 0$ in the equations above, we can obtain (based on [5]) the average ergodic rate for path-loss $l_0(r)$ as

$$\tau^{\{d,0\}}(\lambda, \alpha) = \frac{1}{\ln 2} \int_0^\infty \frac{1}{1 + \rho_d(e^t - 1, \alpha)} dt. \quad (52)$$

From the above, as shown in [5], we can see that the average rate per cell is unaffected by the BS density for $l_0(r)$ but it is affected by the BS density for $l_1(r)$ (see also Figs. 2-3). Using the above expressions, we can numerically compute the per cell average rate. We shall discuss the implication of this numerical results in the next section. A summary of the findings is given in Table I.

V. PERFORMANCE LIMITS IN UDN

In the previous section, we have derived the coverage probability and average achievable rate per cell for various scenarios. In this section, we investigate how extreme densification affects gains from site planning and its impact on ASE, and summarize the findings in Table I.

A. Deployment gain

We define deployment gain as the ratio of average achievable rate per cell for regular network to that of the irregular network. In Section IV, we have presented the average achievable rate per cell for different scenarios in Figures 2 and 3. When comparing the rate performance between the regular and irregular networks, we made the following observations: (i) for the case where SINR invariance property holds, the deployment gain remains constant regardless of BS density, (ii) for the case where SINR invariance property does not hold, the deployment gain depends on BS density and no gain is observed when BS density tends to infinity, (iii) in general, we achieve a performance gain when the deployment of BSs follows a regular pattern. To illustrate our observations, we further plot the deployment gain in Figure 4 for 1D network

Fig. 3. Impact of BS density λ and h on average ergodic rate τ of noise-less 2D network for different path-loss models with $\alpha = 4$. The path-loss parameter h affects the rate of decay.

with $\alpha = 2$ and Figure 5 for 2D network with $\alpha = 4$. To focus on the performance in ultra dense region, our plots use average inter-site distance on x-axis.

From Figure 4, we first see a constant deployment gain of approximately 1.414 for 1D network when SINR invariance property holds. The constant deployment gain of 1.414 is computed by finding the ratio of average rate per cell for regular network (≈ 3.037 bits/sec/Hz) to that of irregular network (≈ 2.148 bits/sec/Hz) based on (48) and (52) respectively. In case of 2D network the deployment gain is slightly higher with value of 1.433. This is due to a slightly higher per cell throughput for regular deployment (≈ 3.079 bits/sec/Hz). Based on the results, we see that regular deployment generally offers better performance than that of the irregular deployment. As irregular deployment permits BSs to be arbitrary close, regardless of BS density λ , some users will experience very high levels of interference, thus degrading the overall system performance. This is not the case with regular deployment in which distance is the same between all BSs across the network.

When SINR invariance property does not hold, the deployment gain varies based on BS density. We see from the figure that the deployment gain increases as the average inter-site distance decreases. After peaking at a certain point, the performance advantage of regular deployment appears to decrease to zero when the inter-site distance approaches zero. In other words, there appears to be no gain when BS density is extremely high. Interestingly, there is a small region of inter-site distance when deployment gain value falls below one, indicating negative impact of site planning. Moreover, the deployment gain when SINR invariance does not hold also appears to tend to the case where SINR invariance holds as the average inter-site distance becomes large. This can be justified by the vanishing impact of the relative antenna height on the overall system performance, and can be observed in Figure 6 that ASE performance for both cases (i.e. $h = 0$ and $h > 0$) converges (see next section for a formal ASE definition and

TABLE I
SUMMARY OF RESULTS AND PROPERTIES FOR 1D/2D REGULAR AND IRREGULAR NETWORKS

Property	Path-loss model	1D		2D	
		Irregular	Regular	Irregular	Regular
SINR invariance	$l_0(r)$	holds	holds	holds	holds
	$l_1(r)$	does not hold	does not hold	does not hold	does not hold
Density countering condition	$l_0(r)$	N/A	N/A	N/A	N/A
	$l_1(r)$	$\lambda h = c$	$\lambda h = c$	$\lambda h^2 = c$	$\lambda h^2 = c$
Average achievable rate per cell [bits/sec/Hz]	$l_0(r)$	≈ 2.148 (indep. of λ)	≈ 3.037 (indep. of λ)	≈ 2.148 (indep. of λ)	≈ 3.079 (indep. of λ)
	$l_1(r)$	0, when $\lambda \rightarrow \infty$ ≈ 2.148 , when $\lambda \rightarrow 0$	0, when $\lambda \rightarrow \infty$ ≈ 3.037 , when $\lambda \rightarrow 0$	0, when $\lambda \rightarrow \infty$ ≈ 2.148 , when $\lambda \rightarrow 0$	0, when $\lambda \rightarrow \infty$ ≈ 3.079 , when $\lambda \rightarrow 0$
Average area spectral efficiency	$l_0(r)$	Unbounded	Unbounded	Unbounded	Unbounded
	$l_1(r)$	Bounded (see (58))	Bounded (see (58))	Bounded (see (59) and [22])	Bounded (see (59))
Deployment gain	$l_0(r)$	≈ 1.414 (indep. of λ)		≈ 1.433 (indep. of λ)	
	$l_1(r)$	1.0 (no gain), when $\lambda \rightarrow \infty$ ≈ 1.414 , when $\lambda \rightarrow 0$		1.0 (no gain), when $\lambda \rightarrow \infty$ ≈ 1.433 , when $\lambda \rightarrow 0$

Fig. 4. Deployment gain for 1D noise-less network with $\alpha = 2$.

Fig. 5. Deployment gain for 2D noise-less network with $\alpha = 4$.

discussion). In Figure 5, we observe similar behavior for the 2D network.

It is interesting to note that the deterioration of deployment gain is directly dependent on the relative antenna height h . This suggests that by lowering antenna heights, network operators could potentially retain the benefits from careful site selection. This finding may be of a particular importance to operators who provide services in large indoor open spaces (e.g. exhibition halls, airports, train stations) where BS deployments often follow a certain planning rather than random. In the following, we show that deployment gain vanishes when BS density approaches infinity.

Theorem 7. *System performance under $l_1(r)$ does not depend on BS deployment strategy (i. e. no gain from site planning) as $\lambda \rightarrow \infty$.*

Proof. We first study the distance between a user at a particular point and an arbitrary interfering BS. In regular network, since the location of the interfering BS is deterministic, the distance between the two is also deterministic. Let this distance be R . In irregular network, the location of the BS follows the

PPP. The distance is a random variable, say $\tilde{R}$, which has the following statistical properties [24] for 1D network

$$\mathbb{E}[\tilde{R}] = \frac{n}{2\lambda} \quad (53)$$

$$\text{var}(\tilde{R}) = \frac{n}{4\lambda^2}, \quad (54)$$

and 2D network

$$\mathbb{E}[\tilde{R}] = \frac{1}{\sqrt{\lambda\pi}} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \quad (55)$$

$$\text{var}(\tilde{R}) = \frac{1}{\lambda\pi} \left(n - \left(\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \right)^2 \right). \quad (56)$$

Considering an interfering BS located at a particular distance away from the user. For regular network, this distance is a fixed value R . For irregular network, this distance is $\tilde{R}$ with mean value of R . Notice that when $\lambda \rightarrow \infty$, the variance of $\tilde{R}$ tends to zero. In other words, when λ continues to increase, the variance of this distance continues to decrease which reduces the deployment difference between irregular and regular networks. For a regular network, the

interference caused by this BS is $g \cdot l(R)$ where g is a random variable describing the channel fading and $l(\cdot)$ is the path-loss expression. For an irregular network, the interference caused by this BS is $g \cdot l(\tilde{R})$. Given that $l_1(r)$ path-loss model is bounded, we have $l_1(r) < \infty$ for an arbitrary $r \geq 0$. When $\lambda \rightarrow \infty$, the variance of $\tilde{R}$ decreases to zero, we see that the interference characteristic caused by the BS in the irregular network converges to that of the regular network, that is $gl(\tilde{R}) \rightarrow gl(R)$. This shows that system performance under $l_1(r)$ for irregular network converges to that of the regular network when $\lambda \rightarrow \infty$. ■

The above proposition can be extended to any bounded path-loss model. For the unbounded path-loss model, we show in Section IV that a constant performance gain is achieved when the deployment of BSs follows a regular pattern.

B. Average Area Spectral Efficiency

The ASE is a measure of the overall rate over a network area and is defined by

$$ASE = \lambda \cdot \tau^{\{d,i\}}(\lambda, \alpha). \quad (57)$$

ASE may increase or decrease as network densifies. This depends on whether the decay of rate per cell as BS density increases, can be countered by spatial reuse, and therefore it is of interest to investigate whether continued network densification will still lead to ASE improvement. The finding may help network operators to optimize their investments in the infrastructure and identify when further densification may not be beneficial.

In the previous section, we showed that when SINR invariance does not hold, the average achievable rate per cell goes to zero as $\lambda \rightarrow \infty$. However, as we show in the following, the average ASE does not necessarily go to zero, as can be observed in Figure 6 and Figure 7. This is in line with the findings reported in [22] for irregular networks. To support this, we formulate the following Theorem.

Theorem 8. *Average ASE of regular networks under $l_1(r)$ converges to a non-zero value for $h > 0$ as $\lambda \rightarrow \infty$, given $\alpha > 1$ for 1D or $\alpha > 2$ for 2D.*

Proof. Proof is given in Appendix A. ■

As can be seen in the proof of Theorem 8, the ASE of UDN under $l_1(r)$ has the following lower bounds when $\lambda \rightarrow \infty$, for 1D network

$$\frac{1}{2 \ln 2 \ h \left(1 + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\right)}, \quad (58)$$

and 2D network

$$\frac{2\sqrt{3}}{12 \ln 2 \ h^2 \left(1 + 2 \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-1)}\right)}. \quad (59)$$

These lower bounds depend on h and α . They show that by lowering h , the ASE can be improved. This is very encouraging as network operators could theoretically provide services even with over-densified networks. In the extreme case when $h \rightarrow 0$, we have ASE approaching infinity. This

is true since when $h = 0$, the path loss model $l_1(r)$ reduces to $l_0(r)$, and SINR invariance property holds for this scenario.

VI. CONCLUSION

Motivated by the growing interests in UDNs as key enabler for next generation wireless networks and the challenge in deploying a massive number of BSs, we investigated the performance behaviour of regular ultra dense networks which can be of particular interest to network operators who provide indoor deployments in large open spaces such as exhibition halls, airports and train stations in where a regular grid still often used to determine location of BSs. We first developed an analytical model to describe the performance of regular UDNs. Based on our study, we found that regular networks share many of the same performance behaviour as irregular network. In particular, for both regular and irregular networks, we showed that SINR invariance property holds under unbounded path-loss model and does not hold if the considered path-loss model is bounded. We further formulated the relationship between BS density and relative antenna height for regular networks showing how average per cell rate can be maintained whilst increasing network density. The established relationship confirmed that both regular and irregular networks share the same density countering condition.

In terms of the benefit of proper BS site selection, we compared the average per cell rate of regular networks and that of the irregular networks, and we found that proper BS deployment may improve network performance to some extent. We showed that when SINR invariance property holds, the deployment gain is constant. However, for more realistic bounded path-loss models, the deployment gain from proper site selection can be higher than that of the unbounded path-loss model. As the network density approaches infinity, performance gain vanishes, and the rate of vanishing depends on the relative antenna height. The insights provided in this work may help network operators to optimize their investments in the infrastructure and identify when further densification or careful site selection may no longer be beneficial due to low cost-effectiveness.

We finally studied the ASE performance of the regular networks. Despite the pessimistic conclusion related to the per cell rate converging to zero, we showed that its ASE does not necessarily decay to zero as BS density approaches infinity. A non-negligible ASE can be achieved which is dependent of the path-loss exponent and relative antenna height. This result is in line with the recent finding for the irregular networks.

APPENDIX A PROOF OF THEOREM 8

We start by stating the following Lemma which is then used in the proof of the theorem.

Lemma 5. *Given a path-loss model $l_1(r)$ and $h > 0$, the ASE performance of a network when $\lambda \rightarrow \infty$ converges to the ASE performance of a network where each user is collocated with its serving BS.*

Proof. Let $\tau_0^{\{d,i\}}(\lambda, \alpha)$ be the achievable rate per cell given both a typical user and its serving BS are located in the origin.

Fig. 6. Average Area Spectral Efficiency for noise-less 1D networks for $\alpha = 2$

For 1D regular network, $\tau^{\{1,i\}}(\lambda, \alpha)$ can be expressed as

$$\tau^{\{1,i\}}(\lambda, \alpha) = \mathbb{E}_g \left[\int_0^{\frac{\Upsilon}{2}} \frac{2}{\Upsilon} \cdot \log_2 \left(1 + g_0 l(x) \cdot \left(\sum_{i=1}^{\infty} g_i l(i \Upsilon + x) + \sum_{i=1}^{\infty} g_i l(i \Upsilon - x) + \mu \sigma^2 \right)^{-1} \right) dx \right] \quad (60)$$

where $\Upsilon = \frac{1}{\lambda}$. By substituting $x = k \frac{\Upsilon}{2}$ to the above result, we obtain

$$\tau^{\{1,i\}}(\lambda, \alpha) = \mathbb{E}_g \left[\int_0^1 \log_2 \left(1 + g_0 l\left(\Upsilon \frac{k}{2}\right) \cdot \left(\mathbb{E}[g] \sum_{i=1}^{\infty} l(\Upsilon i + \Upsilon \frac{k}{2}) + \mathbb{E}[g] \sum_{i=1}^{\infty} l(\Upsilon i - \Upsilon \frac{k}{2}) + \mu \sigma^2 \right)^{-1} \right) dk \right]. \quad (61)$$

Next, given $\mathbb{E}[g] = 1$, by assuming a bounded path-loss model function (i.e. $l(0) < \infty$) and applying the Monotone Convergence Theorem twice, we derive the following limit expression

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \tau^{\{1,i\}}(\lambda, \alpha) &= \lim_{\lambda \rightarrow \infty} \mathbb{E}_g \left[\int_0^1 \log_2 \left(1 + \frac{g_0 l(0)}{\sum_{i=1}^{\infty} l(\frac{i}{\lambda}) + \sum_{i=1}^{\infty} l(\frac{i}{\lambda}) + \mu \sigma^2} \right) dk \right] \\ &= \mathbb{E}_g \left[\lim_{\lambda \rightarrow \infty} \log_2 \left(1 + \frac{g_0 l(0)}{\sum_{i=1}^{\infty} l(\frac{i}{\lambda}) + \sum_{i=1}^{\infty} l(\frac{i}{\lambda}) + \mu \sigma^2} \right) \right]. \end{aligned} \quad (62)$$

Now we express $\tau_0^{\{d,i\}}(\lambda, \alpha)$ for 1D regular network under a bounded path-loss model function as

$$\tau_0^{\{1,i\}}(\lambda, \alpha)$$

Fig. 7. Average Area Spectral Efficiency for noise-less 2D networks for $\alpha = 4$

$$= \mathbb{E}_g \left[\log_2 \left(1 + \frac{g_0 l(0)}{\sum_{i=1}^{\infty} l(i \Upsilon) + \sum_{i=1}^{\infty} l(i \Upsilon) + \mu \sigma^2} \right) \right]. \quad (63)$$

Using (62) and (63), we can easily obtain the following

$$\lim_{\lambda \rightarrow \infty} \tau^{\{d,i\}}(\lambda, \alpha) = \lim_{\lambda \rightarrow \infty} \tau_0^{\{d,i\}}(\lambda, \alpha). \quad (64)$$

We can repeat the above approach for 2D networks, thus concluding the proof. ■

In the following we provide the proof of Theorem 8.

Proof. We begin the proof by focusing on the 1D network. Using Lemma 5, we write the limit of ASE for 1D regular network when $\lambda \rightarrow \infty$ as

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{1,i\}}(\lambda, \alpha) &= \\ \lim_{\lambda \rightarrow \infty} \lambda \mathbb{E}_g \left[\log_2 \left(1 + \frac{g_0 l(0)}{\sum_{i=1}^{\infty} l(i \Upsilon) + \sum_{i=1}^{\infty} l(i \Upsilon) + \mu \sigma^2} \right) \right]. \end{aligned} \quad (65)$$

To derive the lower bound of the above expression for $l_1(r)$, we introduce a new path-loss model function $l_2(r) = (\max(h, r))^{-\alpha}$ where $l_2(r) \geq l_1(r)$, for all $r \geq 0, h \geq 0$. By substituting $l(r) = l_1(r)$ and $l(r) = l_2(r)$ in (65) it can be easily seen that

$$\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{1,1\}}(\lambda, \alpha) \geq \lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{1,2\}}(\lambda, \alpha). \quad (66)$$

The above inequality can be further simplified into the following expression

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{1,1\}}(\lambda, \alpha) &\geq \lim_{\lambda \rightarrow \infty} \lambda \mathbb{E}_g \left[\log_2 \left(1 + \frac{g_0 h^{-\alpha}}{\mu \sigma^2 + 2(nh^{-\alpha} + \Upsilon^{-\alpha} \zeta(\alpha, 1 + n))} \right) \right], \end{aligned} \quad (67)$$

where

$$\zeta(\alpha, 1+n) = \sum_{i=n+1}^{\infty} \frac{1}{i^\alpha} \quad (68)$$

is the Hurwitz Zeta function and $n = \lfloor \frac{h}{\Upsilon} \rfloor$.

Next, we apply the Monotone Convergence Theorem and we substitute $t = \frac{h}{\Upsilon}$ and without loss of generality, as $\lambda \rightarrow \infty$ (or $\Upsilon \rightarrow 0$), we can safely assume $t \in \mathbb{N}$ which leads to the following expression

$$\mathbb{E}_g \left[\lim_{t \rightarrow \infty} \frac{t}{h} \log_2 \left(1 + \frac{g_0}{\mu\sigma^2 h^\alpha + 2(t + t^\alpha \zeta(\alpha, 1+t))} \right) \right] \quad (69)$$

By applying Watson's Lemma to the integral representation of the Hurwitz Zeta function we obtain the asymptotic expansion of $\zeta(\alpha, 1+t)$ for $t \rightarrow \infty$ as

$$\begin{aligned} \zeta(\alpha, 1+t) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha-1} e^{-(1+t)x}}{1 - e^{-x}} dx \\ &\sim \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} (1+t)^{-\alpha+1} + \frac{1}{2} (1+t)^{-\alpha} + O(t^{-\alpha-1}). \end{aligned} \quad (70)$$

By substituting $\zeta(\alpha, 1+t)$ with its asymptotic expansion and by applying the L'Hopital rule we can rewrite (69) and simplify it to obtain the following result

$$\begin{aligned} &\mathbb{E}_g \left[\lim_{t \rightarrow \infty} \frac{t}{h} \log_2 \left[1 + g_0 \left(\mu\sigma^2 h^\alpha + 2 \left(t + t^\alpha \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \cdot (1+t)^{-\alpha+1} + \frac{1}{2} (1+t)^{-\alpha} \right) \right)^{-1} \right] \right] \\ &= \mathbb{E}_g \left[\lim_{k \rightarrow 0^+} \frac{1}{kh} \log_2 \left[1 + g_0 k \cdot \left(k\mu\sigma^2 h^\alpha + 2 \left(1 + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \cdot (1+k)^{-\alpha+1} + \frac{k}{2} (1+k)^{-\alpha} \right) \right)^{-1} \right] \right] \\ &= \mathbb{E}_g \left[\frac{g_0 \Gamma(\alpha)}{2 \ln 2 h (\Gamma(\alpha) + \Gamma(\alpha-1))} \right]. \end{aligned} \quad (71)$$

Finally, by taking the expectation of the above result, with respect to g_0 , and given that $\mathbb{E}[g_0] = 1$, we arrive at

$$\frac{\Gamma(\alpha)}{2 \ln 2 h (\Gamma(\alpha) + \Gamma(\alpha-1))}, \quad (72)$$

thus concluding the proof for 1D network.

From the above result it can be easily seen that the lower bound of ASE as $\lambda \rightarrow \infty$ for 1D regular network under $l_1(r)$ is greater than zero and the lower bound depends on h and α .

For 2D network, similar to 1D network, we use Lemma 5 and write the limit of ASE for 2D regular network when $\lambda \rightarrow \infty$ as

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,i\}}(\lambda, \alpha) \\ &= \lim_{\lambda \rightarrow \infty} \lambda \mathbb{E}_g \left[\log_2 \left(1 + \frac{g_0 l(\|b_0\|)}{\mu\sigma^2 + \sum_{\Phi^{HEX} \setminus b_o} l(\|b_i\|)} \right) \right] \end{aligned} \quad (73)$$

Comparing the limit of ASE under the path-loss models between $l_1(r)$ and $l_2(r)$, we get

$$\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,1\}}(\lambda, \alpha) \geq \lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,2\}}(\lambda, \alpha). \quad (74)$$

It can be also easily seen that $I_0 = \sum_{\Phi^{HEX} \setminus b_o} l(\|b_i\|)$ in (73) is the cumulated interference from BSs deployed according to Φ^{HEX} given that a typical user and its serving BS are located at the origin. As this expression does not have a closed-form we use the expressions for lower and upper bounds as presented in [24]

$$\sum_{k=1}^{\infty} 6k \cdot l(\Upsilon k) < I_0 < 6l(\Upsilon) + \sum_{k=2}^{\infty} 6k \cdot l(\Upsilon \sqrt{3}/2k) \quad (75)$$

By substituting $l(r) = l_2(r)$ and using the definition of Hurwitz zeta function, we can simplify the expressions for lower and upper bound as presented below

$$\begin{aligned} &\sum_{k=1}^{n_1} 6k \cdot h^{-\alpha} + 6\Upsilon^{-\alpha} \zeta(\alpha-1, 1+n_1) < I_0 \\ &\sum_{k=1}^{n_2} 6k \cdot h^{-\alpha} + 6\Upsilon^{-\alpha} (\sqrt{3}/2)^{-\alpha} \zeta(\alpha-1, 1+n_2) > I_0 \end{aligned}$$

where $n_1 = \lfloor \frac{h}{\Upsilon} \rfloor$, $n_2 = \lfloor \frac{2h}{\sqrt{3}\Upsilon} \rfloor$ and $n_1 \geq 0$, $n_2 \geq 1$.

By taking the upper bound of the cumulated interference, and applying the Monotone Convergence Theorem we derive the lower bound of $\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,2\}}(\lambda, \alpha)$ as presented below

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,2\}}(\lambda, \alpha) \\ &> \mathbb{E}_g \left[\lim_{\Upsilon \rightarrow 0^+} \frac{2}{\Upsilon^2 \sqrt{3}} \log_2 \left(1 + g_0 h^{-\alpha} \left(\mu\sigma^2 + 3(n_2+1)n_2 h^{-\alpha} \right. \right. \right. \\ &\quad \left. \left. \left. + 6\Upsilon^{-\alpha} (\sqrt{3}/2)^{-\alpha} \zeta(\alpha-1, 1+n_2) \right)^{-1} \right) \right] \\ &= \mathbb{E}_g \left[\lim_{\Upsilon \rightarrow 0^+} \frac{2}{\Upsilon^2 \sqrt{3}} \log_2 \left(1 + g_0 \left(\mu\sigma^2 h^\alpha \right. \right. \right. \\ &\quad \left. \left. \left. + 3 \left(\lfloor \frac{2h}{\sqrt{3}\Upsilon} \rfloor + 1 \right) \lfloor \frac{2h}{\sqrt{3}\Upsilon} \rfloor \right. \right. \right. \\ &\quad \left. \left. \left. + 6 \left(\frac{h}{\Upsilon} \right)^\alpha (\sqrt{3}/2)^{-\alpha} \zeta(\alpha-1, 1 + \lfloor \frac{2h}{\sqrt{3}\Upsilon} \rfloor) \right)^{-1} \right) \right]. \end{aligned} \quad (76)$$

Note that it is also the lower bound of $\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,1\}}(\lambda, \alpha)$. We substitute $k = \frac{2h}{\sqrt{3}\Upsilon}$ in (76). As $\lambda \rightarrow \infty$, we have $\Upsilon \rightarrow 0$ and hence we can safely assume that $k \in \mathbb{N}$ which leads to the following expression

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,1\}}(\lambda, \alpha) \geq \lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,2\}}(\lambda, \alpha) \\ &> \mathbb{E}_g \left[\lim_{k \rightarrow \infty} \frac{k^2 \sqrt{3}}{2h^2} \cdot \log_2 \left(1 + g_0 \left(\mu\sigma^2 h^\alpha + 3k(k+1) \right. \right. \right. \\ &\quad \left. \left. \left. + 6k^\alpha \zeta(\alpha-1, 1+k) \right)^{-1} \right) \right]. \end{aligned} \quad (77)$$

Next, by applying Watson's Lemma to the integral representation of the Hurwitz Zeta function we obtain the asymptotic expansion of $\zeta(\alpha - 1, 1 + k)$ for $k \rightarrow \infty$.

$$\begin{aligned}\zeta(\alpha - 1, 1 + k) &= \frac{1}{\Gamma(\alpha - 1)} \int_0^\infty \frac{x^{\alpha-2} e^{-(1+k)x}}{1 - e^{-x}} dx \\ &\sim \frac{\Gamma(\alpha - 2)}{\Gamma(\alpha - 1)} (1 + k)^{-\alpha+2} + \frac{1}{2} (1 + k)^{-\alpha+1} + O(k^{-\alpha}).\end{aligned}\quad (78)$$

By substituting $\zeta(\alpha - 1, 1 + k)$ for its asymptotic expansion in (77) and by applying the L'Hopital rule $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ we provide the lower bound for $l_1(r)$ as

$$\begin{aligned}&\lim_{\lambda \rightarrow \infty} \lambda \tau_0^{\{2,1\}}(\lambda, \alpha) \\ &> \mathbb{E}_g \left[\lim_{k \rightarrow \infty} \frac{k^2 \sqrt{3}}{2h^2} \log_2 \left[1 + g_0 \left(\mu \sigma^2 h^\alpha + 3k(k+1) \right) \right. \right. \\ &\quad \left. \left. + 6k^\alpha \left(\frac{\Gamma(\alpha - 2)}{\Gamma(\alpha - 1)(1 + k)^{\alpha-2}} + \frac{1}{2(1 + k)^{\alpha-1}} \right) \right]^{-1} \right] \\ &= \mathbb{E}_g \left[\lim_{t \rightarrow 0^+} \frac{\sqrt{3}}{2t^2 h^2} \log_2 \left[1 + g_0 t^2 \cdot \left(\mu \sigma^2 h^\alpha t^2 + 3(1 + t) \right) \right. \right. \\ &\quad \left. \left. + 6 \left(\frac{\Gamma(\alpha - 2)}{\Gamma(\alpha - 1)(1 + t)^{\alpha-2}} + \frac{t}{2(1 + t)^{\alpha-1}} \right) \right]^{-1} \right] \\ &= \mathbb{E}_g \left[\frac{g_0 2\sqrt{3}}{12 \ln 2 h^2 \left(1 + 2 \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-1)} \right)} \right].\end{aligned}\quad (79)$$

Finally, by taking the expectation of the above result, with respect to g_0 , and given that $\mathbb{E}[g_0] = 1$, we arrive at

$$\frac{2\sqrt{3}}{12 \ln 2 h^2 \left(1 + 2 \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-1)} \right)}, \quad (80)$$

thus concluding the proof for 2D network.

From the above result it can be easily seen that the lower bound of ASE as $\lambda \rightarrow \infty$ for 2D regular network under $l_1(r)$ is greater than zero and depends on h and α . ■

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