Generalized Equalization Algorithm Utilizing Improper ISI

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Abstract—The improper nature of the intersymbol interference (ISI) for signals transmitted over frequency selective channels is investigated in this paper, and our analysis reveals that for real signals, the improperness originates from both improper signal modulation and the interference cancellation process; whereas for most complex signals, the improperness is only a characteristic of the residual ISI due to interference cancellation. In order to utilize the improperness of ISI, a multistage widely linear equalization algorithm is introduced, and it is generally applicable for both real and complex signal constellations. The results reveal that accounting for the improper nature of the ISI at both the input and output of the equalizer leads to a noticeable performance gain compared to conventional equalization schemes.

I. INTRODUCTION

For a complex random vector $\mathbf{r}$, its second-order statistics are completely characterized by its autocorrelation matrix $\mathbf{R} = E[\mathbf{r}\mathbf{r}^H]$ as well as the pseudo-autocorrelation matrix $\tilde{\mathbf{R}} = E[\mathbf{r}\mathbf{r}^T]$ [1]. Note that throughout this paper, the superscript operators $(\cdot)^H$, $(\cdot)^*$, $(\cdot)^T$ denote the conjugate transpose, conjugate, and transpose operations, respectively; and $E[\cdot]$ denotes expectation operation. Most existing studies on receiver algorithms only exploit the information contained in the autocorrelation function of the observed signal. The pseudo-autocorrelation matrix $\tilde{\mathbf{R}}$ is usually not considered and is implicitly assumed to be zero. While this is the optimum strategy when dealing with proper complex random processes (i.e., when the pseudo-autocorrelation $\tilde{\mathbf{R}}$ is vanishing) [2], it turns out to be sub-optimum in situations where the transmitted signals and/or interference are improper complex random processes (i.e., $\tilde{\mathbf{R}}$ is non-vanishing), for which the performance of a linear receiver can generally be improved by the use of widely linear processing (WLP) [3]¹. It was shown in [1],

[5], [6] that significant performance gain can be achieved by applying WLP compared to conventional processing.

The application of WLP in communication systems was first developed for improving the performance of direct sequence code division multiple access (DS-CDMA) systems with improper data modulation, such as pulse-amplitude modulation (PAM), offset quadrature phase shift keying (OQPSK), offset quadrature amplitude modulation (OQAM), and binary phase-shift-keying (BPSK), etc.. For example, it was shown in [7] that for systems employing BPSK modulation, full exploitation of the available information on the second-order statistics of the observations entails the use of WLP and yields a scheme that outperforms the other schemes currently known in the literature. It was shown in [8] that OQPSK modulation generates improper complex multiple access interference (MAI) and the application of a widely linear (WL) receiver to the OQPSK modulated CDMA system yields remarkable output signal-to-noise ratio (SNR) gains over the strictly linear receiver. A WL minimum-output-energy (MOE) receiver is derived based on a modified cost function in [9]. By exploiting the additional information contained in pseudo-covariance matrix of observations, a performance gain can be attained for improper DS-CDMA signals. In [10], a new WL zero forcing (ZF) receiver was proposed for multi-carrier transmission systems to combat narrow-band interference (NBI). It was designed under the MOE criterion, and resulted in a substantial improvement over the conventional linear ZF and minimum mean square error (MMSE) receivers, and in the meantime, led to improved blind channel identification capabilities by exploiting the non-circularity property of the desired signal. In [11], [12], WLP was applied to direct sequence spreading ultra-wideband (DS-UWB) systems with BPSK and 4-ary bi-orthogonal keying (4BOK) modulation schemes, respectively. Simulation and analytical results showed that the proposed multiple-input, multiple-output (MIMO) WL equalizers allow for power-efficient DS-UWB transmission close to the matched filter

¹There are some exceptions, e.g., as pointed out in [4], coherent detection of BPSK signals with known phase shifts and signature waveforms, for which the use of widely linear filters is unnecessary, since the real part of the signal is a sufficient statistic after phase correction.

bound with moderate computational complexity. Widely linear reception strategies were extended to layered space-time wireless communications in [13], where improved versions of the linear decorrelating, MMSE and nonlinear nulling receivers were developed and analyzed. It was concluded that the proposed receivers not only achieve better performance compared to the conventional receivers, but are also less sensitive to the channel estimation errors. The results also indicate that widely linear detection permits operation even when the number of transmit antennas exceeds the number of receive antennas, and also that widely linear reception of an M -ary real constellation outperforms linear reception of an M -ary complex constellation.

The application of WLP to complex modulation schemes has been addressed in several papers, e.g., in [14], [15] where WLP was applied to complex signals which become improper due to the use of space-time block coding or widely linear space-time mapping. A novel iterative multiuser detector for DS-CDMA systems with complex modulation schemes was proposed in [16]. Owing to the fact that the residual MAI becomes improper when soft decision feedback is used to cancel the MAI, the use of WLP achieves significant gains in power efficiency and improves convergence speed.

The concept of WLP has been applied in several papers for enhancing the performance of equalizers for combating the ISI induced by frequency-selective multipath channels. For example, the equalization of real-valued data transmitted over ISI channels having complex-valued channel coefficients was considered in [5]; and the equalization of space-time block encoded transmissions over MIMO channels was presented in [14] and [15]. An MMSE equalizer and a decision-feedback equalizer (DFE) employing WLP and implemented via finite impulse response (FIR) filters for a MIMO frequency selective channel have been proposed in [17] and [18]. It was concluded that the use of WLP yields considerable performance improvements at the cost of only a limited increase in complexity compared to conventional linear processing.

Linear MMSE filter based turbo equalization which combines equalization and decoding in an iterative fashion has previously been studied, e.g., in [19]–[21] where only the information contained in the autocorrelation function of the observations was employed. In this paper, we further develop the algorithm presented in [19]–[21] by utilizing the improperness of the ISI and exploiting the information contained in the pseudo-autocorrelation function of the observations. The algorithm introduced in [16] approximates the WL filter output as a proper random process. Our investigation reveals that this might be an appropriate assumption for multiuser detection in presence of MAI, it, however, leads to a sub-optimum solution when dealing with ISI introduced by frequency-selective channels, and our results show that exploitation of the improperness of the interference-plus-noise at the filter output further improves the system performance. The proposed equalization algorithms can be applied to systems with real or complex modulation schemes, and they are shown to outperform the WL MMSE and WL DFE introduced in [5] for real signals; and outperform the linear MMSE scheme introduced in [19]–[21] for complex signals.

The remainder of this paper is organized as follows. Section II describes how improper ISI is utilized in the equalization design; while in Section III, the proposed schemes are evaluated and compared to the conventional schemes over some static and time-varying ISI channels. Finally, in Section IV, conclusions are drawn based on the simulation results.

II. EQUALIZATION USING IMPROPER ISI

The transmission system under study will now be briefly described. For a system without channel coding, the information bit sequence is directly mapped into PSK/QAM symbols $\{s_n\}$, which are transmitted over a multipath frequency selective fading channel with L resolvable paths, having complex channel gains $h_0, h_1, \dots, h_{L-1}$. The received signal can be expressed as

$$r_n = \sum_{l=0}^{L-1} s_{n-l} h_l + v_n, \quad (1)$$

where h_l is the complex channel coefficient and is assumed to remain constant during the transmission of one block of data. The complex additive white Gaussian noise (AWGN) v_n with zero mean and variance N_0 , is assumed to be proper. The proposed equalization algorithms work for both real and complex signals. For simplicity, we have selected BPSK/QPSK as the example real/complex-valued modulation schemes to use in this work. However, the extension of the proposed schemes to higher level amplitude-shift keying (ASK) and PSK/QAM schemes is straightforward. The transmitted symbol at time instant n is denoted as s_n . For BPSK modulation, $s_n = \pm 1$ is real-valued; for QPSK modulation, we denote $s_n = x_n + jy_n$, where $x_n, y_n = \frac{\pm 1}{\sqrt{2}}$.

The task of the receiver is to detect the transmitted information bits given the received observation $\{r_n\}$. To this end, we need first to detect the transmitted symbols $\{s_n\}$ which are corrupted with ISI and AWGN. An equalizer is required to reduce the detrimental effect of ISI. Here, we consider the equalization algorithm presented in [19]–[21] and develop an enhanced scheme by modifying the MMSE filter design criterion and applying WLP. Let us define the channel matrix

$$\mathbf{H} = \begin{bmatrix} h_{L-1} & h_{L-2} & \dots & h_0 & 0 & 0 & \dots & 0 \\ 0 & h_{L-1} & \dots & h_1 & h_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_{L-1} & h_{L-2} & \dots & h_0 \end{bmatrix}.$$

and the vectors

$$\begin{aligned} \mathbf{s}_n &= [s_{n-L+1} \quad \dots \quad s_{n-1} \quad s_n \quad s_{n+1} \quad \dots \quad s_{n+L-1}]^T; \\ \bar{\mathbf{s}}_n &= [\bar{s}_{n-L+1} \quad \dots \quad \bar{s}_{n-1} \quad 0 \quad \bar{s}_{n+1} \quad \dots \quad \bar{s}_{n+L-1}]^T; \\ \mathbf{r}_n &= [r_n \quad r_{n+1} \quad \dots \quad r_{n+L-1}]^T; \\ \mathbf{v}_n &= [v_n \quad v_{n+1} \quad \dots \quad v_{n+L-1}]^T, \end{aligned} \quad (2)$$

where $\mathbf{r}_n$, $\mathbf{v}_n$ denote the received vector and the noise vector, respectively; and $\bar{\mathbf{s}}_n$ contains the estimate of the interference symbols from the previous iteration. The derivation of $\bar{\mathbf{s}}_n$ will

be given later on. According to (1), the received vector after interference cancellation is given as [19], [20]

$$\mathbf{r}'_n = \mathbf{r}_n - \mathbf{H}\bar{\mathbf{s}}_n = \mathbf{H}[\mathbf{s}_n - \bar{\mathbf{s}}_n] + \mathbf{v}_n, \quad (3)$$

where $\mathbf{r}'_n$ is the ISI canceled version of $\mathbf{r}_n$. Note that (3) represents a decision-directed iterative scheme, where the detection procedure at the n^{th} stage uses the symbol estimates from the $(n-1)^{\text{th}}$ stage. The performance is improved in an iterative manner owing to the fact that the symbols are more accurately estimated (meaning better interference cancellation), as the iteration procedure goes on. For simplicity, the iteration index is omitted, whenever no ambiguity arises.

In order to further suppress the residual interference in $\mathbf{r}'_n$, an instantaneous linear MMSE filter is applied to $\mathbf{r}'_n$, to obtain $z_n = \mathbf{w}_n^H \mathbf{r}'_n$, where the filter coefficient vector $\mathbf{w}_n$ is chosen to minimize $e_n^L = E\{|\mathbf{w}_n^H \mathbf{r}'_n - s_n|^2\}$. Refer to [19]–[21] for a detailed description of the conventional MMSE algorithm. Next, we shall discuss how the performance can be improved by applying WLP, the principle of which is not only to process $\mathbf{r}'_n$, but also its conjugated version $\mathbf{r}'_n^*$ in order to derive the filter output, i.e.,

$$z_n = \omega_n[0]\mathbf{r}'_n + \omega_n[1]\mathbf{r}'_n^* = \omega_n^H \mathbf{y}_n,$$

where $\omega_n = [\omega_n[0] \ \omega_n[1]]^H$ and $\mathbf{y}_n = [\mathbf{r}'_n^T \ \mathbf{r}'_n^{H^T}]^T$. Substituting (3) into the above equation yields

$$z_n = \omega_n[0]\mathbf{H}[\mathbf{s}_n - \bar{\mathbf{s}}_n] + \omega_n[0]\mathbf{v}_n + \omega_n[1]\mathbf{H}^*[\mathbf{s}_n^* - \bar{\mathbf{s}}_n^*] + \omega_n[1]\mathbf{v}_n^* = \omega_n[0]\mathbf{h}s_n + \omega_n[1]\mathbf{h}^*s_n^* + \mathbf{v}'_n,$$

where $\mathbf{v}'_n = \omega_n[0]\mathbf{v}_n + \omega_n[1]\mathbf{v}_n^*$, and $\mathbf{h} = [h_0 \ h_1 \ \dots \ h_{L-1}]^T$. The decision statistic z_n contains the scaled version of the symbol s_n and its conjugate s_n^* as well as the combined interference cancellation residual and noise denoted as $\mathbf{v}'_n$. Accounting for this changes at the filter output z_n , the cost function needs to be revised accordingly as

$$e_n^{\text{WL}} = E[|z_n - s_n - \gamma s_n^*|^2] = E[|\omega_n^H \mathbf{y}_n - s_n - \gamma s_n^*|^2]. \quad (4)$$

For real-valued constellations (the symbol s_n is real-valued), $\gamma = 1$ is appropriate so that $e_n^{\text{WL}} = E\{|\omega_n^H \mathbf{y}_n - s_n - s_n^*|^2\} = E\{|\omega_n^H \mathbf{y}_n - 2s_n|^2\}$. In this case, $\omega_n[2] = \omega_n^*[1]$, and therefore, $e_n^{\text{WL}} = E\{[2 \operatorname{Re}\{\omega_n[1]\mathbf{r}'_n\} - 2s_n]^2\} = E\{4[\operatorname{Re}\{\omega_n[1]\mathbf{r}'_n\} - s_n]^2\}$ (where $\operatorname{Re}\{\cdot\}$ and $\operatorname{Im}\{\cdot\}$ denote the real and imaginary part of a complex variable, respectively), which is a better cost function than $e_n^L = E\{|\mathbf{w}_n^H \mathbf{r}'_n - s_n|^2\}$, since a conventional MMSE filter yields a complex-valued filter output; however, only the real part of this output is relevant for the decision for systems with a real-valued constellation. It was shown in [5] that $e_n^{\text{WL}} < e_n^L$, leading to an equalizer with enhanced performance. For complex-valued constellations, the role of the parameter γ (the value of which should not be 1) will become apparent in Section III. It is worth noticing that the conventional linear MMSE equalizer is a special case of the WL equalizer, when $\omega_n[1] = \mathbf{w}_n^H$ and $\omega_n[2] = \mathbf{0}$. The WL equalizers are expected to exhibit better performance than their linear counterparts. In particular, the conditions under which

WL estimator can yield significant improvements over linear ones are detailed in [3]. By expanding (4), we obtain

$$\begin{aligned} e_n^{\text{WL}} &= E[|\omega_n^H \mathbf{y}_n - s_n - \gamma s_n^*|^2] \\ &= E\{(\omega_n^H \mathbf{y}_n - s_n - \gamma s_n^*)(\mathbf{y}_n^H \omega_n - s_n^* - \gamma s_n)\} \\ &= \begin{cases} \omega_n^H \mathbf{C}_n \omega_n - \gamma \omega_n^H \tilde{\mathbf{C}}_{ys} - \omega_n^H \mathbf{C}_{ys} - \mathbf{C}_{sy} \omega_n \\ \quad - \gamma \tilde{\mathbf{C}}_{sy}^* \omega_n + (1 + 2\gamma + \gamma^2)\sigma_s^2; & \text{for BPSK} \\ \omega_n^H \mathbf{C}_n \omega_n - \gamma \omega_n^H \tilde{\mathbf{C}}_{ys} - \omega_n^H \mathbf{C}_{ys} - \mathbf{C}_{sy} \omega_n \\ \quad - \gamma \tilde{\mathbf{C}}_{sy}^* \omega_n + (1 + \gamma^2)\sigma_s^2; & \text{for QPSK} \end{cases} \end{aligned} \quad (5)$$

where σ_s^2 denotes the average symbol energy and

$$\begin{aligned} \mathbf{C}_n &= E\{\mathbf{y}_n \mathbf{y}_n^H\} = E\left\{\begin{bmatrix} \mathbf{r}'_n \\ \mathbf{r}'_n^* \end{bmatrix} \begin{bmatrix} \mathbf{r}'_n^H & \mathbf{r}'_n^{HT} \end{bmatrix}\right\} = \begin{bmatrix} \mathbf{R}_n & \tilde{\mathbf{R}}_n \\ \tilde{\mathbf{R}}_n^* & \mathbf{R}_n^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{H}\mathbf{V}_n\mathbf{H}^H + N_0\mathbf{I} & \mathbf{H}\tilde{\mathbf{V}}_n\mathbf{H}^T \\ \mathbf{H}^*\tilde{\mathbf{V}}_n^*\mathbf{H}^H & \mathbf{H}^*\mathbf{V}_n\mathbf{H}^T + N_0\mathbf{I} \end{bmatrix}; \\ \mathbf{V}_n &= \operatorname{diag}\{[1 - |\bar{s}_{n-L+1}|^2 \ \dots \ 1 - |\bar{s}_{n-1}|^2 \ 1 \\ 1 - |\bar{s}_{n+1}|^2 \ \dots \ 1 - |\bar{s}_{n+L-1}|^2]\}; \\ \mathbf{C}_{ys} &= \mathbf{C}_{sy}^H = E\{\mathbf{y}_n s_n^*\} = E\left\{\begin{bmatrix} \mathbf{r}'_n \\ \mathbf{r}'_n^* \end{bmatrix} s_n^*\right\} = E\left\{\begin{bmatrix} \mathbf{r}'_n s_n^* \\ \mathbf{r}'_n^* s_n^* \end{bmatrix}\right\} \\ &= \begin{bmatrix} \mathbf{C}_{r's} \\ \tilde{\mathbf{C}}_{r's}^* \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{h} \\ \mathbf{h}^* \end{bmatrix}; & \text{for BPSK} \\ \begin{bmatrix} \mathbf{h} \\ \mathbf{0} \end{bmatrix}; & \text{for QPSK} \end{cases} \\ \mathbf{C}_{sy} &= E[s_n \mathbf{y}_n^H] = E\{s_n [\mathbf{r}'_n^H \ \mathbf{r}'_n^{HT}]\} = \begin{cases} \begin{bmatrix} \mathbf{h}^H & \mathbf{h}^T \end{bmatrix}; & \text{for BPSK} \\ \begin{bmatrix} \mathbf{h}^H & \mathbf{0} \end{bmatrix}; & \text{for QPSK} \end{cases} \\ \tilde{\mathbf{C}}_{sy} &= E[s_n \mathbf{y}_n^T] = E\{s_n [\mathbf{r}'_n^T \ \mathbf{r}'_n^H]\} = \begin{cases} \begin{bmatrix} \mathbf{h}^T & \mathbf{h}^H \end{bmatrix}; & \text{for BPSK} \\ \begin{bmatrix} \mathbf{0} & \mathbf{h}^H \end{bmatrix}; & \text{for QPSK} \end{cases} \\ \tilde{\mathbf{C}}_{ys} &= \tilde{\mathbf{C}}_{sy}^T = E[\mathbf{y}_n s_n] = E\left\{\begin{bmatrix} \mathbf{r}'_n \\ \mathbf{r}'_n^* \end{bmatrix} s_n\right\} = \begin{cases} \begin{bmatrix} \mathbf{h} \\ \mathbf{h}^* \end{bmatrix}; & \text{for BPSK} \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{h}^* \end{bmatrix}; & \text{for QPSK.} \end{cases} \end{aligned} \quad (6)$$

For a real-valued constellation, $\tilde{\mathbf{V}}_n = E\{[\mathbf{s}_n - \bar{\mathbf{s}}_n][\mathbf{s}_n - \bar{\mathbf{s}}_n]^T\} = \mathbf{V}_n$; for a complex-valued constellation,

$$\begin{aligned} \tilde{\mathbf{V}}_n &= \operatorname{diag}\{[\operatorname{Im}^2\{\bar{s}_{n-L+1}\} - \operatorname{Re}^2\{\bar{s}_{n-L+1}\} \ \dots \\ &\quad \operatorname{Im}^2\{\bar{s}_{n-1}\} - \operatorname{Re}^2\{\bar{s}_{n-1}\} \ 0 \ \operatorname{Im}^2\{\bar{s}_{n+1}\} - \operatorname{Re}^2\{\bar{s}_{n+1}\} \dots \\ &\quad \operatorname{Im}^2\{\bar{s}_{n+L-1}\} - \operatorname{Re}^2\{\bar{s}_{n+L-1}\}]\}. \end{aligned}$$

Each main diagonal element of $\tilde{\mathbf{V}}_n$ is derived as

$$\begin{aligned} E[(s_n - \bar{s}_n)^2] &= E\{[(\operatorname{Re}\{s_n\} + j \operatorname{Im}\{s_n\}) - (\operatorname{Re}\{\bar{s}_n\} + j \operatorname{Im}\{\bar{s}_n\})]^2\} \\ &= E\{(\operatorname{Re}\{s_n\} + j \operatorname{Im}\{s_n\})^2 - 2(\operatorname{Re}\{s_n\} + j \operatorname{Im}\{s_n\}) \\ &\quad \cdot (\operatorname{Re}\{\bar{s}_n\} + j \operatorname{Im}\{\bar{s}_n\}) + (\operatorname{Re}\{\bar{s}_n\} + j \operatorname{Im}\{\bar{s}_n\})^2\} \\ &= E\{(\operatorname{Re}\{s_n\} + j \operatorname{Im}\{s_n\})^2\} - 2E\{(\operatorname{Re}\{s_n\} + j \operatorname{Im}\{s_n\}) \\ &\quad \cdot (\operatorname{Re}\{\bar{s}_n\} + j \operatorname{Im}\{\bar{s}_n\}) + (\operatorname{Re}\{\bar{s}_n\} + j \operatorname{Im}\{\bar{s}_n\})^2\} \\ &= 2jE\{\operatorname{Re}\{s_n\} \operatorname{Im}\{s_n\}\} - (\operatorname{Re}^2\{\bar{s}_n\} - \operatorname{Im}^2\{\bar{s}_n\} + 2j \operatorname{Re}\{\bar{s}_n\} \operatorname{Im}\{\bar{s}_n\}) \\ &= \operatorname{Im}^2\{\bar{s}_n\} - \operatorname{Re}^2\{\bar{s}_n\} + 2jE\{\operatorname{Re}\{s_n\} \operatorname{Im}\{s_n\}\} - \operatorname{Re}\{\bar{s}_n\} \operatorname{Im}\{\bar{s}_n\} \\ &= \operatorname{Im}^2\{\bar{s}_n\} - \operatorname{Re}^2\{\bar{s}_n\}. \end{aligned}$$

The above equations are obtained based on the fact that for QPSK signals, $\text{Re}^2\{s_n\} - \text{Im}^2\{s_n\} = 1/2 - 1/2 = 0$, and the real and imaginary parts of $\bar{s}_n$ correspond to two independent bits, and are thus uncorrelated², i.e., $\text{E}[\text{Re}\{s_n\} \text{Im}\{s_n\}] = \text{Re}\{\bar{s}_n\} \text{Im}\{\bar{s}_n\}$.

Denoting the log-likelihood ratio (LLR) value of s_n as $\lambda(s_n) = \lambda(x_n) + j\lambda(y_n)$ for QPSK signals, the soft estimate of s_n is computed according to its LLR value $\lambda(s_n)$ as

$$\bar{s}_n = \begin{cases} \tanh(\lambda(s_n)/2); & \text{for BPSK} \\ \tanh[\lambda(x_n)/2]/\sqrt{2} + j \tanh[\lambda(y_n)/2]/\sqrt{2}; & \text{for QPSK} \end{cases} \quad (7)$$

At the initial stage, no priori information about s_n is available, its LLR value is thus assumed to be zero. Therefore $\bar{s}_n = 0$, and consequently, $\tilde{\mathbf{V}}_n = \mathbf{0}$ for QPSK signals, and the pseudo-autocorrelation matrix is vanishing, i.e., $\tilde{\mathbf{R}}_n = \text{E}[\mathbf{r}'_n \mathbf{r}'_n{}^T] = \mathbf{H} \tilde{\mathbf{V}}_n \mathbf{H}^T = \mathbf{0}$. Note that the ISI is always improper for BPSK modulated systems since $\tilde{\mathbf{V}}_n = \mathbf{V}_n$, and $\tilde{\mathbf{R}}_n = \mathbf{H} \mathbf{V}_n \mathbf{H}^T \neq \mathbf{0}$ also holds at the initial stage. At the subsequent stages, the ISI terms become improper for both real and complex-valued constellations since $\tilde{\mathbf{R}}_n$ is non-vanishing due to a non-vanishing matrix $\tilde{\mathbf{V}}_n$. In summary, we conclude that the improperness of ISI is a characteristic of the residual ISI generated at the output of the ISI cancellation process for both improper (e.g., BPSK) and proper signal constellations (e.g., QPSK), and in this case, WLP can be applied to the ISI canceled signal vector $\mathbf{r}'_n$; it is also a characteristic of improper signal constellations, and WLP can be applied to the original signal vector $\mathbf{r}_n$ if modulation schemes such as BPSK are employed. It is worth noticing that improperness may also stem from space-time coding [14], [15]. However, this is not our concern in this work.

Some examples of improper signal constellations are 1) BPSK, which is used in the DS-UWB system [22], [23] for wireless personal area network (WPANs); 2) Gaussian minimum-shift keying (GMSK) modulation for the Global System for Mobile Communication (GSM) [24]; 3) offset quadrature amplitude modulation (OQAM) and 4) offset quadrature phase-shift keying (OQPSK), which have been adopted by the CDMA 2000 standard [25] and the mobile communication standard Universal Wireless Communications (UWC-136) [26], respectively. In comparison, proper signal constellations (complex modulations), such as PSK/QAM, are more widely used in the current as well as in the future communication systems, e.g., in the W-CDMA [27] and broadband fixed wireless access systems [28], [29], just to name a few. In all these scenarios, utilizing the improperness of the received and/or interference canceled signals in the receiver will lead to a better performance at the expense of some increase in complexity.

Differentiating ϵ_n^{WL} in (5) with respect to ω_n results in $\frac{\partial \epsilon}{\partial \omega_n} = (\mathbf{C}_n \omega_n)^* - \mathbf{C}_{sy}^T - \gamma \tilde{\mathbf{C}}_{sy}^H$, which is set to zero to

²This is obviously true for uncoded systems. For coded systems, the interleaver breaks the bit dependency introduced by channel coding, the interleaved bits can thus be modeled as statistically independent random variables. Consequently, the real and imaginary parts of $\bar{s}_n$ which correspond to the consecutive coded and interleaved bits can also be regarded as uncorrelated.

yield the optimum vector of ω_n

$$\omega_n = \mathbf{C}_n^{-1} (\mathbf{C}_{sy}^H + \gamma \tilde{\mathbf{C}}_{sy}^T) = \mathbf{C}_n^{-1} (\mathbf{C}_{ys} + \gamma \tilde{\mathbf{C}}_{ys}) = \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix}. \quad (8)$$

For the proposed WL equalizer, the augmented autocorrelation matrix $\mathbf{C}_n$ expressed in (6) which gives a complete second order description of $\mathbf{r}'_n$ is used for deriving the equalizer filter coefficient vector ω_n ; whereas for the conventional linear MMSE algorithm, the filter coefficient vector $\mathbf{w}_n$ is calculated using only the autocorrelation of the observation $\mathbf{R}_n = \text{E}[\mathbf{r}'_n \mathbf{r}'_n{}^H]$ and the cross-correlation between the observation and desired signal $\mathbf{C}_{r's} = \text{E}[\mathbf{r}'_n s_n^*]$, i.e., $\mathbf{w}_n = \mathbf{R}_n^{-1} \mathbf{C}_{r's}$. The pseudo-autocorrelation matrix $\tilde{\mathbf{R}}_n$ is implicitly assumed to be zero. However, as shown previously, $\tilde{\mathbf{R}}_n$ is non-vanishing except at the initial stage for QPSK signals, hence omitting $\tilde{\mathbf{R}}_n$ would lead to sub-optimum solutions.

The WL MMSE filter output can be expressed as $z_n = \mu_n s_n + \mu'_n s_n^* + \eta_n$, where the combined noise and residual interference η_n can be approximated as a Gaussian random variable [20], [30]. In [16], η_n is regarded as a proper random process since $\text{E}[\eta_n^2] = 0$ for a sufficiently large spreading factor. However, as will become evident later on, this treatment is sub-optimum for the equalization algorithm under study. Next, we present two schemes to derive the LLR values for x_n and y_n for QPSK systems based on the assumption that the interference-plus-noise term η_n at the output of the WL filter output is either a proper or an improper random process.

A. Scheme 1

The first scheme is based on the common assumption that η_n is proper. In this case, the second order statistic of the zero-mean Gaussian random variable η_n is completely characterized by its variance $N_\eta = \text{E}[|\eta_n|^2]$. The parameters μ_n, μ'_n, N_η can be determined by taking expectation with respect to the interfering symbols and the channel noise

$$\begin{aligned} \mu_n &= \text{E}\{z_n s_n^*\} = \omega_n^H \text{E}\{\mathbf{y}_n s_n^*\} = \omega_n^H \mathbf{C}_{ys}; \\ \mu'_n &= \text{E}\{z_n s_n\} = \omega_n^H \text{E}\{\mathbf{y}_n s_n\} = \omega_n^H \tilde{\mathbf{C}}_{ys}; \\ N_\eta &= \text{E}[|\eta_n|^2] = \text{E}[|z_n - \mu_n s_n - \mu'_n s_n^*|^2] \\ &= \text{E}[(z_n - \mu_n s_n - \mu'_n s_n^*)(z_n^* - \mu_n^* s_n^* - \mu_n'^* s_n)] \\ &= \text{E}\{|z_n|^2\} - |\mu_n|^2 - |\mu'_n|^2 = \mu_n^* + \gamma \mu_n'^* - |\mu_n|^2 - |\mu'_n|^2. \end{aligned} \quad (9)$$

The above equation holds since $z_n = \omega_n^H \mathbf{y}_n$ and $\omega_n = \mathbf{C}_n^{-1} (\mathbf{C}_{ys} + \gamma \tilde{\mathbf{C}}_{ys})$, therefore,

$$\begin{aligned} \text{E}\{|z_n|^2\} &= \text{E}\{\omega_n^H \mathbf{y}_n \mathbf{y}_n^H \omega_n\} = \omega_n^H \mathbf{C}_n \omega_n \\ &= (\mathbf{C}_{ys}^H + \gamma \tilde{\mathbf{C}}_{ys}^H) \mathbf{C}_n^{-1} \mathbf{C}_n \omega_n \\ &= (\mathbf{C}_{ys}^H + \gamma \tilde{\mathbf{C}}_{ys}^H) \omega_n = \mu_n^* + \gamma \mu_n'^*. \end{aligned}$$

After computing the values of μ_n, μ'_n and N_η , the conditional PDF of the equalizer output can be obtained as

$$f(z_n | s_n = s_M) = \frac{1}{\pi N_\eta} \exp \left(-\frac{|z_n - \mu_n s_M - \mu'_n s_M^*|^2}{N_\eta} \right),$$

Fig. 1. QPSK constellation and bit-to-symbol mapping.

and the LLR value of x_n can thus be computed as

$$\begin{aligned} \lambda(x_n) &= \ln \frac{f(z_n|x_n=0)}{f(z_n|x_n=1)} = \ln \frac{f(z_n|s_n=s_0) + f(z_n|s_n=s_3)}{f(z_n|s_n=s_1) + f(z_n|s_n=s_2)} \\ &\approx \ln \frac{\exp(-|z_n - \mu_n s_n^+ - \mu'_n s_n^{+*}|^2/N_\eta)}{\exp(-|z_n - \mu_n s_n^- - \mu'_n s_n^{-*}|^2/N_\eta)} \\ &= \frac{2}{N_\eta} \text{Re}\{(\mu_n s_n^{+*} z_n + \mu'_n s_n^+ z_n) - (\mu_n s_n^{-*} z_n + \mu'_n s_n^- z_n)\}, \end{aligned} \quad (10)$$

where s_n^+ denotes the QPSK symbol corresponding to $\max\{f(z_n|s_n=s_0), f(z_n|s_n=s_3)\}$, and s_n^- denotes the QPSK symbol corresponding to $\max\{f(z_n|s_n=s_1), f(z_n|s_n=s_2)\}$ since the real part of the symbols s_0, s_3 corresponds to 0, and the real part of the symbols s_1, s_2 corresponds to 1 as shown in Fig. 1. The dual maxima rule [31] is used in (10) utilizing the fact that one term usually dominates each sum.

Similarly,

$$\begin{aligned} \lambda(y_n) &= \ln \frac{f(z_n|s_n=s_0) + f(z_n|s_n=s_1)}{f(z_n|s_n=s_2) + f(z_n|s_n=s_3)} \\ &\approx \frac{2}{N_\eta} \text{Re}\{(\mu_n s_n^{+*} z_n + \mu'_n s_n^+ z_n) - (\mu_n s_n^{-*} z_n + \mu'_n s_n^- z_n)\}, \end{aligned}$$

where s_n^+ denotes the QPSK symbol corresponding to $\max\{f(z_n|s_n=s_0), f(z_n|s_n=s_1)\}$, and s_n^- denotes the QPSK symbol corresponding to $\max\{f(z_n|s_n=s_2), f(z_n|s_n=s_3)\}$ since the imaginary part of the symbols s_0, s_1 corresponds to 0, and the imaginary part of the symbols s_2, s_3 corresponds to 1 as shown in Fig. 1.

Then we use Equ. (7) to convert LLRs to soft symbol estimate $\bar{s}_n$, which is needed by the equalizer for the interference cancellation at the next iteration.

B. Scheme 2

The second scheme takes into account the improperness of η_n , and utilizes the fact that $\tilde{N}_\eta = \text{E}[\eta_n^2] \neq 0$. According to the definition,

$$\begin{aligned} \tilde{N}_\eta &= \text{E}[\eta_n^2] = \text{E}[(z_n - \mu_n s_n - \mu'_n s_n^*)^2] \\ &= \text{E}[(z_n - \mu_n s_n - \mu'_n s_n^*)(z_n - \mu_n s_n - \mu'_n s_n^*)^*] \\ &= \text{E}\{z_n^2\} - 2\mu_n \mu'_n = \text{E}\{\omega_n^H \mathbf{y}_n \mathbf{y}_n^T \omega_n^*\} - 2\mu_n \mu'_n \\ &= \omega_n^H \tilde{\mathbf{C}}_n \omega_n^* - 2\mu_n \mu'_n. \end{aligned} \quad (11)$$

Equ. (11) holds since $\omega_n^H \mathbf{y} = \mathbf{y}^T \omega_n^*$. The matrix $\tilde{\mathbf{C}}_n$ is computed as

$$\begin{aligned} \tilde{\mathbf{C}}_n &= \text{E}\{\mathbf{y}_n \mathbf{y}_n^T\} = \text{E}\left\{\begin{bmatrix} \mathbf{r}'_n \\ \mathbf{r}''_n \end{bmatrix} \begin{bmatrix} \mathbf{r}'_n{}^T & \mathbf{r}''_n{}^H \end{bmatrix}\right\} = \begin{bmatrix} \tilde{\mathbf{R}}_n & \mathbf{R}_n \\ \mathbf{R}_n^* & \tilde{\mathbf{R}}_n^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{H} \tilde{\mathbf{V}}_n \mathbf{H}^T & \mathbf{H} \mathbf{V}_n \mathbf{H}^H + N_0 \mathbf{I} \\ \mathbf{H}^* \mathbf{V}_n \mathbf{H}^T + N_0 \mathbf{I} & \mathbf{H}^* \tilde{\mathbf{V}}_n^* \mathbf{H}^H \end{bmatrix}. \end{aligned}$$

Let us denote $z_n = z_n^r + jz_n^i$, $s_n = s_n^r + js_n^i$, and $\eta_n = \eta_n^r + j\eta_n^i$. The filter output $z_n = \mu_n s_n + \mu'_n s_n^* + \eta_n$ can be re-written as

$$\underbrace{\begin{bmatrix} z_n^r \\ z_n^i \end{bmatrix}}_{\mathbf{t}_n} = \underbrace{\begin{bmatrix} (\mu_n + \mu'_n) s_n^r \\ (\mu_n - \mu'_n) s_n^i \end{bmatrix}}_{\mathbf{d}_n} + \underbrace{\begin{bmatrix} \eta_n^r \\ \eta_n^i \end{bmatrix}}_{\mathbf{i}_n}.$$

Since the probability distribution of a complex random variable or vector is a joint distribution of its real and imaginary part, we have

$$\begin{aligned} f(z_n|s_n) &= f(\mathbf{t}_n|\mathbf{d}_n) \\ &= \frac{1}{2\pi\sqrt{\det \mathbf{W}_n}} \exp\left(-\frac{1}{2}(\mathbf{t}_n - \mathbf{d}_n)^H \mathbf{W}_n^{-1}(\mathbf{t}_n - \mathbf{d}_n)\right), \end{aligned} \quad (12)$$

where the covariance matrix of the Gaussian noise is $\mathbf{W}_n = \text{E}[\mathbf{i}_n \mathbf{i}_n^H]$. Let us define

$$\begin{aligned} \tilde{\mathbf{t}}_n &= \begin{bmatrix} z_n \\ z_n^* \end{bmatrix}; & \tilde{\mathbf{d}}_n &= \begin{bmatrix} \mu_n s_n + \mu'_n s_n^* \\ (\mu_n s_n + \mu'_n s_n^*)^* \end{bmatrix}; \\ \tilde{\mathbf{i}}_n &= \begin{bmatrix} \eta_n \\ \eta_n^* \end{bmatrix}; & \tilde{\mathbf{W}}_n &= \text{E}[\tilde{\mathbf{i}}_n \tilde{\mathbf{i}}_n^H], \end{aligned}$$

and a mapping matrix [4], [32]

$$\mathbf{J} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix},$$

which is an unitary matrix since $\mathbf{J}\mathbf{J}^H = \mathbf{J}^H\mathbf{J} = \mathbf{I}$. It can be easily shown that $\mathbf{t}_n - \mathbf{d}_n = \frac{1}{\sqrt{2}}\mathbf{J}^H(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)$, and $\mathbf{i}_n = \sqrt{2}\mathbf{J}\tilde{\mathbf{i}}_n$. The PDF expressed by (12) can thus be reformed as

$$\begin{aligned} f(z_n|s_n) &= \frac{1}{2\pi\sqrt{\det \mathbf{W}_n}} \exp\left(-\frac{1}{4}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)^H \mathbf{J} \mathbf{W}_n^{-1} \mathbf{J}^H (\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)\right) \\ &= \frac{1}{2\pi\sqrt{\det \mathbf{W}_n}} \exp\left(-\frac{1}{4}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)^H (\mathbf{J} \mathbf{W}_n \mathbf{J}^H)^{-1} (\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)\right) \\ &= \frac{1}{2\pi\sqrt{\det \mathbf{W}_n}} \exp\left(-\frac{1}{2}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)^H \tilde{\mathbf{W}}_n^{-1} (\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n)\right), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \tilde{\mathbf{W}}_n &= \text{E}[\tilde{\mathbf{i}}_n \tilde{\mathbf{i}}_n^H] = \text{E}\left\{\begin{bmatrix} \eta_n \\ \eta_n^* \end{bmatrix} \begin{bmatrix} \eta_n^* & \eta_n \end{bmatrix}\right\} \\ &= \text{E}\left\{\begin{bmatrix} \eta_n \eta_n^* & \eta_n \eta_n \\ \eta_n^* \eta_n^* & \eta_n^* \eta_n \end{bmatrix}\right\} = \begin{bmatrix} N_\eta & \tilde{N}_\eta \\ \tilde{N}_\eta^* & N_\eta \end{bmatrix}. \end{aligned}$$

The third equality in (13) follows from the fact that

$$\begin{aligned} \mathbf{J} \mathbf{W}_n \mathbf{J}^H &= \mathbf{J} \text{E}[\mathbf{i}_n \mathbf{i}_n^H] \mathbf{J}^H = \text{E}[(\mathbf{J} \mathbf{i}_n)(\mathbf{J} \mathbf{i}_n)^H] \\ &= \frac{1}{2} \text{E}[\tilde{\mathbf{i}}_n \tilde{\mathbf{i}}_n^H] = \frac{1}{2} \tilde{\mathbf{W}}_n. \end{aligned} \quad (14)$$

The LLR value of x_n can thus be computed as

$$\begin{aligned}\lambda(x_n) &= \ln \frac{f(z_n|x_n=0)}{f(z_n|x_n=1)} = \ln \frac{f(z_n|s_n^r = +1/\sqrt{2})}{f(z_n|s_n^r = -1/\sqrt{2})} \\ &\approx \ln \frac{\exp\left(-\frac{1}{2}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+)\right)}{\exp\left(-\frac{1}{2}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)\right)} \\ &= -\frac{1}{2}[(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+) \\ &\quad - (\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)],\end{aligned}\quad (15)$$

where $\tilde{\mathbf{d}}_n^+$ denotes the vector $\tilde{\mathbf{d}}_n$ corresponding to

$$\max\{f(\tilde{\mathbf{t}}_n|s_n^r = \frac{+1}{\sqrt{2}}, s_n^i = \frac{+1}{\sqrt{2}}), f(\tilde{\mathbf{t}}_n|s_n^r = \frac{+1}{\sqrt{2}}, s_n^i = \frac{-1}{\sqrt{2}})\};$$

which is equivalent to $\max\{f(\tilde{\mathbf{t}}_n|s_n = s_0), f(\tilde{\mathbf{t}}_n|s_n = s_3)\}$, and $\tilde{\mathbf{d}}_n^-$ denotes the vector $\tilde{\mathbf{d}}_n$ corresponding to

$$\max\{f(\tilde{\mathbf{t}}_n|s_n^r = \frac{-1}{\sqrt{2}}, s_n^i = \frac{+1}{\sqrt{2}}), f(\tilde{\mathbf{t}}_n|s_n^r = \frac{-1}{\sqrt{2}}, s_n^i = \frac{-1}{\sqrt{2}})\}$$

which is equivalent to $\max\{f(\tilde{\mathbf{t}}_n|s_n = s_1), f(\tilde{\mathbf{t}}_n|s_n = s_2)\}$.

Similarly,

$$\begin{aligned}\lambda(y_n) &= \ln \frac{f(z_n|y_n=0)}{f(z_n|y_n=1)} = \ln \frac{f(z_n|s_n^i = +1/\sqrt{2})}{f(z_n|s_n^i = -1/\sqrt{2})} \\ &\approx -\frac{1}{2}[(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^+) \\ &\quad - (\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)^H \tilde{\mathbf{W}}_n^{-1}(\tilde{\mathbf{t}}_n - \tilde{\mathbf{d}}_n^-)],\end{aligned}\quad (16)$$

where $\tilde{\mathbf{d}}_n^+$ denotes the vector $\tilde{\mathbf{d}}_n$ corresponding to $\max\{f(\tilde{\mathbf{t}}_n|s_n = s_0), f(\tilde{\mathbf{t}}_n|s_n = s_1)\}$, and $\tilde{\mathbf{d}}_n^-$ denotes the vector $\tilde{\mathbf{d}}_n$ corresponding to $\max\{f(\tilde{\mathbf{t}}_n|s_n = s_3), f(\tilde{\mathbf{t}}_n|s_n = s_2)\}$.

III. NUMERICAL RESULTS

In this section, we compare the performance of the proposed schemes with that of the WL MMSE and DFE equalizers introduced in [5] and the iterative linear MMSE equalizer introduced in [19]–[21]. During each Monte-Carlo run, the block size is set to 10000 information bits, which corresponds to 5000 QPSK or 10000 BPSK symbols. The noise variance N_0 and the complex channel coefficients are assumed to be known to the receiver. Both time-varying and static channels are tested. For the time-varying channel, we choose the SUI-3 fixed wireless access channel introduced in [29], [33]. The channel coefficients vary from one data block to another, however, they are assumed to remain constant during the transmission of one block of data, due to the slowly time varying nature of the SUI-3 channel³. For the static channel, we use a 5-tap channel with impulse response $h[n] = (2 - 0.4j)\delta[n] + (1.5 + 1.8j)\delta[n-1] + \delta[n-2] + (1.2 - 1.3j)\delta[n-3] + (0.8 + 1.6j)\delta[n-4]$. The total channel gain is normalized so that $P = \sum_{n=0}^4 |h[n]|^2 = 1$.

³In fact, the proposed algorithm only requires the channel to be constant during the transmission of $2L-1$ symbols (where L is the number of channel taps), i.e., during the transmission of symbols $s_{n-L+1}, \dots, s_{n+L-1}$ in the vector $\mathbf{s}_n$ expressed by (2).

Fig. 2 shows the performance of different equalization schemes for BPSK signals transmitted over the SUI-3 channel. The results are averaged over at least 500 channel realizations. The 10-tap WL MMSE equalizer proposed in [5] outperforms its linear counterparts by over 1 dB at BERs between 10^{-2} and 10^{-3} . The use of decision feedback can further improve the performance since it is observed that the 10-tap WL DFE (6 feedforward taps and 4 feedback taps) yields better results than the 10-tap WL MMSE. The performance of the conventional DFE lies between the WL MMSE and the WL DFE. More gain is obtained by applying WLP to linear MMSE filtering than to DFE. Equalizers with length greater than 10 taps are also tested and are shown to have a similar performance to the 10-tap equalizers. The figure also shows that the proposed WL equalizer at the 3rd stage outperforms the WL DFE by 0.8 dB at BER= 10^{-3} . It is observed that most of the gains are obtained at the 2nd and the 3rd stages with the proposed iterative equalization scheme, for which the parameter γ in (8) is set to be 1. As discussed in Section II, the ISI is improper for BPSK systems with and without interference cancellation. This is verified by Fig. 2.b), which compares the performance between the proposed scheme and the iterative linear MMSE scheme introduced in [19]–[21]. The topmost curve represents the initial stage equalization and the bottom-most curve represents the 4th stage equalization. We noticed significant performance improvement by applying WLP at the first iteration (no interference cancellation has taken place yet), as well as at the second iteration (interference cancellation has been performed). However, the performance gap becomes much smaller when the algorithms reach convergence at the 4th iteration.

Fig. 3 shows the impact of the parameter γ on the performance of the proposed equalization schemes for QPSK signals transmitted over the SUI-3 channel. The curve is plotted for different values of γ at the 5th equalization stage, and E_b/N_0 is set to 20 dB. The results are averaged over at least 500 channel realizations so that the proposed procedure would experience a variety of different channel conditions. Both Figs. 3.a) and 3.b) indicate that the performance of the WL scheme 2 is independent of γ . For the WL scheme 1, as illustrated by Fig. 3.a), when $\gamma < 1$, the optimum value is $\gamma = 0$, which is the solution proposed in [16]; when choosing in the region $\gamma > 1$, Fig. 3.b) indicates that there seem to be infinite number of choices because the performance of the scheme 1 becomes insensitive to the choice of γ when it goes beyond a certain value (e.g., $\gamma = 15$). The second scheme outperforms the first one at all values of γ , indicating that ignoring the improperness of the residual interference-plus-noise at the filter output renders performance loss, and the scheme accounting for its improperness is always optimized regardless of the value of γ .

One can see that the choice of $\gamma = 1$ leads to the worst performance for first scheme (it also does not work for the second scheme, which is not shown in the figure). This is in contrast to systems with real-valued constellations, for which $\gamma = 1$ is appropriate. The reason is that $\omega_n[1] = \omega_n^*[0]$ when choosing $\gamma = 1$, and it can be shown that $\mu_n = \mu_n'$ in such a case. Therefore, the equalizer output

Fig. 2. Comparison of different equalization schemes for BPSK signals in SUI-3 channel.

becomes $z_n = \mu_n s_n + \mu'_n s_n^* + \eta_n = 2\mu_n \text{Re}\{s_n\} + \eta_n$, which means the decision statistic does not contain any information about the imaginary part of the symbol s_n . Consequently, the second bit corresponding to the imaginary part of the QPSK symbol cannot be correctly detected. In order to tackle this problem, we can adopt the method suggested by the Max-SINR approach derived in the appendix, i.e., using $\gamma = 1$ and $\gamma = -1$ for making the decision for the first bit and second bit corresponding to the real and imaginary parts of the QPSK symbol, respectively. The points marked '*' in plot a) show the performance for the Max-SINR approach. Since it decodes the two bits of QPSK symbols independently without utilizing the correlation between the real and imaginary parts of the filter output (failing to exploit the improperness), it is therefore a suboptimum solution compared to the scheme exploiting improperness as shown by the result in the figure.

Fig. 4 shows the performance comparison between the proposed schemes and the iterative linear MMSE scheme introduced in [19]–[21] for QPSK signals transmitted over the SUI-3 channel. It takes 4-5 stages for all the algorithms to converge. Upon convergence, the proposed schemes outperform the linear MMSE scheme by 2-3 dB at BER around 10^{-3} . Compared to the initial equalization stage, the subsequent equalization stages achieve much better performance, and most gains are obtained at the 2nd and 3rd stages. The WL scheme 2 outperforms the WL scheme 1 by 1dB at BER= 10^{-3} . In contrast to the systems with BPSK systems, all the algorithms have identical performance at the initial stage due to the fact that the improperness of ISI for QPSK signals is only a characteristic of interference cancellation as shown in Section II ($\tilde{\mathbf{R}}_n = \mathbf{0}$ for QPSK signalling at the initial stage), thus no improvement can be achieved by WLP. At the following cancellation stages, the ISI canceled signal becomes improper, and exploitation of the improperness of the

ISI results in better performance.

The three schemes are compared for the 5-tap static channel in Fig. 5. After the system reaches convergence at the 6th stage, performance gains of 1.1 dB and 1.8 dB compared to that of the linear MMSE scheme are observed by applying the WL scheme 1 and scheme 2, respectively, at BER = 10^{-4} . As indicated by Fig. 4 and Fig. 5, the proposed equalizers do not achieve much performance gain at low SNR. This is due to the fact that the proper noise is dominant at low SNR; whereas the advantage of exploiting the improper ISI becomes evident when the SNR increases. Fig. 5.b) also shows clear superiority of the WL scheme 2 over the conventional MMSE (10 taps) and DFE (6 feedforward taps and 4 feedback taps) equalizers. Note that no gain can be obtained by the WL MMSE or the WL DFE since the improperness of ISI only comes from interference cancellation for QPSK signals.

IV. CONCLUSIONS

In this paper, we introduced an iterative widely linear approach to equalization of signals transmitted over frequency selective channels. The proposed algorithm is generic in the sense that it is applicable for both real and complex signalling formats. The improper nature of the ISI is analyzed and different behavior evident for real and complex signals has been revealed by our simulations using BPSK and QPSK modulation. In the latter case, we proposed two variants of the widely linear equalization algorithm, both of which utilize the improperness of the ISI canceled signal at the input of the WL MMSE filter. However, the first scheme is based on the common assumption that the residual interference-plus-noise at the filter output is proper, whereas the second scheme takes into consideration the improperness of the filter output. The results indicate that the algorithm exploiting improperness at both input and output of the WL filter leads to the best

(a) BER as a function of γ ($\gamma \leq 1$).

(b) BER as a function of γ ($\gamma \geq 1$).

Fig. 3. Impact of γ on the performance of the proposed equalization schemes for QPSK signals. In plot (a), the markers with '*' indicate the performance of the Max-SINR approach at $\gamma = 1$ and $\gamma = -1$.

Fig. 4. Performance of the proposed equalization for QPSK signals in SUI-3 channel ($\gamma = 15$). BER for different stages of the iterative process is shown.

performance. This is in contrast to the previously proposed widely multiuser detector in CDMA systems, for which the improperness of the filter output can be neglected.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers whose insightful comments have significantly improved the quality of this work.

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(a)

(b)

Fig. 5. Comparison of different equalization schemes for QPSK signals in 5-tap static channel ($\gamma = 0$). In plot (a), the curves represent the performance at the 6th stage for all the schemes. In plot (b), BER of the WL scheme 2 at the 3rd stage is shown.

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APPENDIX

Maximum SINR solution

One way of choosing an appropriate value of γ is to maximize the signal-to-interference-plus-noise ratio (SINR) at the filter output $z_n = \mu_n s_n + \mu'_n s_n^* + \eta_n$, which can be expressed as

$$\text{SINR} = \frac{|\mu_n(\gamma)|^2 + |\mu'_n(\gamma)|^2}{N_\eta(\gamma)} = \frac{|\mu_n(\gamma)|^2 + |\mu'_n(\gamma)|^2}{\text{E}\{|z_n|^2\} - |\mu_n(\gamma)|^2 - |\mu'_n(\gamma)|^2},$$

where

$$\begin{aligned} \mu_n &= \text{E}\{z_n s_n^*\} = \omega_n^H \text{E}[\mathbf{y} s_n^*] = \omega_n^H \mathbf{C}_{ys}; \\ \mu'_n &= \text{E}\{z_n s_n\} = \omega_n^H \text{E}[\mathbf{y} s_n] = \omega_n^H \tilde{\mathbf{C}}_{ys}; \\ \omega_n &= \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ \mathbf{h}^* \end{bmatrix}; \quad \mathbf{C}_{ys} = \begin{bmatrix} \mathbf{h} \\ \mathbf{0} \end{bmatrix}; \quad \tilde{\mathbf{C}}_{ys} = \begin{bmatrix} \mathbf{0} \\ \mathbf{h}^* \end{bmatrix}. \end{aligned}$$

The solutions of this approach can be obtained by differentiating SINR with respect to γ and setting it to zero. The matrix $\mathbf{C}_n$ is a hermitian matrix, i.e., $\mathbf{C}_n^H = \mathbf{C}_n$. Therefore, $(\mathbf{C}_n^{-1})^H = (\mathbf{C}_n^H)^{-1} = \mathbf{C}_n^{-1}$. Let us denote $\mathbf{C}_n^{-1} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix}$. Using the block matrix inverse formula

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}.$$

For the matrix $\mathbf{C}_n = \begin{bmatrix} \mathbf{R} & \tilde{\mathbf{R}} \\ \tilde{\mathbf{R}}^* & \mathbf{R}^* \end{bmatrix}$ defined in (6), we have

$$\begin{aligned} \mathbf{C}_n^{-1} &= \begin{bmatrix} \mathbf{R} & \tilde{\mathbf{R}} \\ \tilde{\mathbf{R}}^* & \mathbf{R}^* \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\mathbf{R} - \tilde{\mathbf{R}}\mathbf{R}^{*-1}\tilde{\mathbf{R}}^*)^{-1} & -(\mathbf{R} - \tilde{\mathbf{R}}\mathbf{R}^{*-1}\tilde{\mathbf{R}}^*)^{-1}\tilde{\mathbf{R}}\mathbf{R}^{*-1} \\ -(\mathbf{R}^* - \tilde{\mathbf{R}}^*\mathbf{R}^{-1}\tilde{\mathbf{R}})^{-1}\tilde{\mathbf{R}}^*\mathbf{R}^{-1} & (\mathbf{R}^* - \tilde{\mathbf{R}}^*\mathbf{R}^{-1}\tilde{\mathbf{R}})^{-1} \end{bmatrix} \\ \mathbf{C}_{00} &= (\mathbf{R} - \tilde{\mathbf{R}}\mathbf{R}^{*-1}\tilde{\mathbf{R}}^*)^{-1}; \quad \mathbf{C}_{01} = -(\mathbf{R} - \tilde{\mathbf{R}}\mathbf{R}^{*-1}\tilde{\mathbf{R}}^*)^{-1}\tilde{\mathbf{R}}\mathbf{R}^{*-1}; \\ \mathbf{C}_{10} &= -(\mathbf{R}^* - \tilde{\mathbf{R}}^*\mathbf{R}^{-1}\tilde{\mathbf{R}})^{-1}\tilde{\mathbf{R}}^*\mathbf{R}^{-1}; \quad \mathbf{C}_{11} = (\mathbf{R}^* - \tilde{\mathbf{R}}^*\mathbf{R}^{-1}\tilde{\mathbf{R}})^{-1}. \end{aligned} \quad (17)$$

It is obvious to see from the above equations that $\mathbf{C}_{11} = \mathbf{C}_{00}^*$ and $\mathbf{C}_{10} = \mathbf{C}_{01}^*$. The exact expressions of $|\mu_n(\gamma)|^2$, $|\mu'_n(\gamma)|^2$ and $E\{|z_n|^2\}$ can be obtained as

$$\begin{aligned} |\mu_n(\gamma)|^2 &= \omega_n^H \mathbf{C}_{ys} \mathbf{C}_{ys}^H \omega_n \\ &= [\mathbf{h}^H \quad \gamma \mathbf{h}^T] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \mathbf{0} \end{bmatrix} [\mathbf{h}^H \quad \mathbf{0}] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix} \\ &= \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} + \gamma (\mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} + \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^*) \\ &\quad + \gamma^2 \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^*; \\ |\mu'_n(\gamma)|^2 &= \omega_n^H \tilde{\mathbf{C}}_{ys} \tilde{\mathbf{C}}_{ys}^H \omega_n \\ &= [\mathbf{h}^H \quad \gamma \mathbf{h}^T] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{h}^* \end{bmatrix} [\mathbf{0} \quad \mathbf{h}^T] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix} \\ &= \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \\ &\quad + \gamma (\mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* + \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{10} \mathbf{h}) \\ &\quad + \gamma^2 \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^*; \\ E\{|z_n|^2\} &= E\{\omega_n^H \mathbf{y}_n \mathbf{y}_n^H \omega_n\} = \omega_n^H \mathbf{C}_n \omega_n \\ &= (\mathbf{C}_{ys}^H + \gamma \tilde{\mathbf{C}}_{ys}^H) \mathbf{C}_n^{-1} \mathbf{C}_n \omega_n = (\mathbf{C}_{ys}^H + \gamma \tilde{\mathbf{C}}_{ys}^H) \omega_n \\ &= [\mathbf{h}^H \quad \gamma \mathbf{h}^T] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} \\ \mathbf{C}_{10} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix} \\ &= \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} + \gamma (\mathbf{h}^T \mathbf{C}_{10} \mathbf{h} + \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^*) + \gamma^2 \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^*, \end{aligned} \quad (18)$$

where $E\{|z_n|^2\}$ is derived using the fact that $z_n = \omega_n^H \mathbf{y}$ and $\omega_n = \mathbf{C}_n^{-1} (\mathbf{C}_{ys} + \gamma \tilde{\mathbf{C}}_{ys})$. Denote

$$\begin{aligned} \text{SINR} &= \frac{|\mu_n(\gamma)|^2 + |\mu'_n(\gamma)|^2}{N_\eta(\gamma)} = \frac{a(\gamma)}{b(\gamma)}; \\ a(\gamma) &= |\mu_n(\gamma)|^2 + |\mu'_n(\gamma)|^2 = a_0 + a_1\gamma + a_2\gamma^2; \\ b(\gamma) &= N_\eta(\gamma) = E\{|z_n|^2\} - |\mu_n(\gamma)|^2 - |\mu'_n(\gamma)|^2 \\ &= E\{|z_n|^2\} - a(\gamma) = b_0 + b_1\gamma + b_2\gamma^2. \end{aligned} \quad (19)$$

From (18), we know that

$$\begin{aligned} a_0 &= \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} + \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} \\ a_1 &= \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} + \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* \\ &\quad + \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* + \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \\ a_2 &= \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* + \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* \\ b_0 &= \mathbf{h}^H \mathbf{C}_{00} \mathbf{h} - a_0 \\ b_1 &= \mathbf{h}^T \mathbf{C}_{10} \mathbf{h} + \mathbf{h}^H \mathbf{C}_{01} \mathbf{h}^* - a_1 \\ b_2 &= \mathbf{h}^T \mathbf{C}_{11} \mathbf{h}^* - a_2. \end{aligned}$$

Since $\mathbf{C}_{11} = \mathbf{C}_{00}^*$ and $\mathbf{C}_{10} = \mathbf{C}_{01}^*$ as shown in (17), consequently, $a_2 = a_0^*$ and $b_2 = b_0^*$. It is observed that they are real-valued parameters, and the imaginary part of these variables is negligible. Therefore, $a_2 = a_0$ and $b_2 = b_0$. Differentiating $\text{SINR} = \frac{a(\gamma)}{b(\gamma)}$ with respect to γ yields

$$\frac{\partial}{\partial \gamma} \left(\frac{a(\gamma)}{b(\gamma)} \right) = \left[b(\gamma) \frac{\partial a(\gamma)}{\partial \gamma} - a(\gamma) \frac{\partial b(\gamma)}{\partial \gamma} \right] / b(\gamma)^2. \quad (20)$$

The optimum values of γ is obtained by setting (20) to zero, leading to the solution

$$b(\gamma) \frac{\partial a(\gamma)}{\partial \gamma} = a(\gamma) \frac{\partial b(\gamma)}{\partial \gamma}.$$

By (19), we have

$$\begin{aligned} b(\gamma) \frac{\partial a(\gamma)}{\partial \gamma} &= (b_0 + \gamma b_1 + \gamma^2 b_2)(a_1 + 2\gamma a_2); \\ a(\gamma) \frac{\partial b(\gamma)}{\partial \gamma} &= (a_0 + \gamma a_1 + \gamma^2 a_2)(b_1 + 2\gamma b_2). \end{aligned}$$

The equation $b(\gamma) \frac{\partial a(\gamma)}{\partial \gamma} = a(\gamma) \frac{\partial b(\gamma)}{\partial \gamma}$ is then equivalent to $(a_1 b_2 - a_2 b_1) \gamma^2 + 2(a_0 b_2 - a_2 b_0) \gamma + a_0 b_1 - a_1 b_0 = 0$. With $a_2 = a_0$, $b_2 =$

b_0 , it reduces to $(a_1 b_2 - a_2 b_1) \gamma^2 - (a_1 b_2 - a_2 b_1) = 0$, leading to the final solution $\gamma_{\text{opt}} = \pm 1$.

In what follows, we explain how the solution can be used for deriving LLR values. It can be easily shown that $\mu'_n = \begin{cases} \mu_n & \text{if } \gamma = 1, \\ -\mu_n & \text{if } \gamma = -1. \end{cases}$. Therefore, $N_\eta = \mu_n^* + \gamma \mu_n'^* - |\mu_n|^2 - |\mu_n'|^2 = 2\mu_n^* - 2|\mu_n|^2$. Let us denote the LLR value of s_n as $\lambda(s_n) = \lambda(x_n) + j\lambda(y_n)$ for QPSK signals, the solution $\gamma = 1$ is used for deriving the LLR value of x_n , i.e.,

$$\begin{aligned} \omega_0 &= \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix} = \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ \mathbf{h}^* \end{bmatrix}; \\ z_n[0] &= \omega_0^H \mathbf{y}; \quad \mu_n[0] = \omega_0^H \mathbf{C}_{ys}. \end{aligned}$$

Denoting $z_n[0] = \mu_n[0] s_n + \mu_n[0] s_n^* + \eta_n = 2\mu_n[0] \text{Re}\{s_n\} + \eta_n$, and approximating η_n as Gaussian random variable [20], [30], yield the conditional PDF

$$f(z_n[0] | s_n = s_M) = \frac{1}{\pi N_\eta} \exp \left(-\frac{|z_n[0] - 2\mu_n[0] \text{Re}\{s_M\}|^2}{N_\eta} \right),$$

and the LLR value of x_n can be computed as

$$\begin{aligned} \lambda(x_n) &= \ln \frac{f(z_n[0] | x_n = 0)}{f(z_n[0] | x_n = 1)} = \ln \frac{f(z_n[0] | s_n = s_0) + f(z_n[0] | s_n = s_3)}{f(z_n[0] | s_n = s_1) + f(z_n[0] | s_n = s_2)} \\ &= \ln \frac{2 \exp(-|z_n[0] - 2\mu_n[0] / \sqrt{2}|^2 / N_\eta)}{2 \exp(-|z_n[0] + 2\mu_n[0] / \sqrt{2}|^2 / N_\eta)} = \frac{8\mu_n[0] \text{Re}\{z_n[0]\}}{\sqrt{2} N_\eta}. \end{aligned}$$

The solution $\gamma = -1$ is used for deriving the LLR value of y_n , i.e.,

$$\begin{aligned} \omega_1 &= \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ \gamma \mathbf{h}^* \end{bmatrix} = \mathbf{C}_n^{-1} \begin{bmatrix} \mathbf{h} \\ -\mathbf{h}^* \end{bmatrix}; \\ z_n[1] &= \omega_1^H \mathbf{y}; \quad \mu_n[1] = \omega_1^H \mathbf{C}_{ys}. \end{aligned}$$

Since $z_n[1] = \mu_n[1] s_n - \mu_n[1] s_n^* + \eta_n = 2\mu_n[1] \text{Im}\{s_n\} j + \eta_n$, the conditional PDF and LLR value of y_n can be derived similarly as

$$\begin{aligned} f(z_n[1] | s_n = s_M) &= \frac{1}{\pi N_\eta} \exp \left(-\frac{|z_n[1] - 2\mu_n[1] \text{Im}\{s_M\} j|^2}{N_\eta} \right); \\ \lambda(y_n) &= \ln \frac{f(z_n[1] | y_n = 0)}{f(z_n[1] | y_n = 1)} \\ &= \ln \frac{2 \exp(-|z_n[1] - 2\mu_n[1] j / \sqrt{2}|^2 / N_\eta)}{2 \exp(-|z_n[1] + 2\mu_n[1] j / \sqrt{2}|^2 / N_\eta)} \\ &= \frac{8\mu_n[1] \text{Im}\{z_n[1]\}}{\sqrt{2} N_\eta}. \end{aligned}$$

Note that this Max-SINR approach suggests independent processing of the real and the imaginary parts of the filtered signal, the correlation between $\text{Re}\{\eta_n\}$ and $\text{Im}\{\eta_n\}$ is not exploited in any way. It is therefore suboptimum, and the scheme described in Section II-B is not applicable in this case.