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Short communication

Angular momentum of free variable mass systems is partially conserved

Angadh Nanjangud*, Fidelis O. Eke

Department of Mechanical and Aerospace Engineering, University of California Davis, Davis, CA 95616, USA

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ABSTRACT

Variable mass systems are a classic example of open systems in classical mechanics with rockets being a standard practical example. Due to the changing mass, the angular momentum of these systems is not generally conserved. Here, we show that the angular momentum vector of a free variable mass system is fixed in inertial space and, thus, is a partially conserved quantity. It is well known that such conservation rules allow simpler approaches to solving the equations of motion. This is demonstrated by using a graphical technique to obtain an analytic solution for the second Euler angle that characterizes nutation in spinning bodies.

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1. Introduction

This brief note presents developments central to attitude motion analysis of free variable mass systems such as space rockets. Whereas the rigid-body motion of constant mass systems [1] has been scrutinized since the times of Euler, similar studies on variable mass systems are more recent. Contemporary researchers [2,3] consider the mid-20th century document of Rosser et al. [4] on the rotational behavior of rockets to be the first to address the topic of rigid-body motion of systems with changing mass. Rosser's study inspired work on rocket flight dynamics [5,6] and general variable mass systems [7] which utilized discrete models for mass loss. A control volume approach [8,9] to account for continuous mass variation subsequently emerged which has since become the modeling standard amongst the community of researchers on rocket flight dynamics. Recent work using the control volume formulation has focused on equation of motion formulation for general variable mass systems [10,11], modeling and analysis of rocket-type systems [12–15] and an abstraction of the rocket problem [16], and stability of transverse rotational motion in solid rocket motors [17]. The developments presented here on angular momentum also utilize this control volume formulation.

Since the Explorer-1 anomaly [18], the angular momentum property of spinning bodies has received significant attention in the field of spacecraft dynamics and controls as it provides a platform for attitude stability analysis [19,20] and informs innovative

attitude control strategies, such as dual-spin satellites [21]. Angular momentum conservation of freely spinning bodies is the backbone that permits explicit solutions to the second Euler angle, or the nutation angle, of torque-free axisymmetric systems [22]. However, this is only true in the case of constant mass systems. In this paper, we show that the angular momentum vector of variable mass systems possesses a similar useful property.

We begin by showing that the angular momentum vector of any torque-free variable mass system has a fixed direction in space and, thus, is a partially conserved quantity. In comparison to previous analytical studies, the presentation here does not assume axial symmetry in the internal mass flow or system geometry. In other words, the developments here are kept completely general so as to be applicable to a broad set of systems including, but not limited to, rockets. Following this conservation result, we demonstrate its utility in graphically determining the second Euler angle of mass-varying systems. We conclude by briefly discussing the availability of analytic and closed-form solutions in the case of axisymmetric rockets. Thus, our work provides a basis to analyze the rotational motion of variable mass systems.

The work presented here creates avenues for future work. The conservation result provides a foundation for developing criteria for motion stability for a variety of mass-varying systems. For example, one can imagine parallels to Poincaré's geometric interpretations of motion stability in constant mass systems. Such stability analyses will naturally lead to informing attitude control system design and development for a variety of vehicles. Further, mass-varying systems transcend aerospace applications such as rockets; they are also found in marine engineering where vehicles utilize mass variation for propulsion. More broadly, the conservation property will interest researchers concerned with the identification

* Corresponding author.

E-mail addresses: asnanjangud@ucdavis.edu (A. Nanjangud), foeke@ucdavis.edu (F.O. Eke).<https://doi.org/10.1016/j.ast.2018.03.003>

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$$\mathbf{0} = \frac{d\mathbf{H}_0^*}{dt} + \int_{S_0} \rho \mathbf{h} u dS + \boldsymbol{\omega} \times \mathbf{H}_0^*, \quad (11)$$

where $\mathbf{H}_0^*$ is the angular momentum of the variable mass system and is

$$\mathbf{H}_0^* = \int_{V_0} \rho \mathbf{h} dV. \quad (12)$$

Since $V = V_0$ at a particular instant, $\mathbf{H}^*$ and $\mathbf{H}_0^*$ are identical but their time derivatives are generally not identical since their evolution in time is associated with changing sets of matter. Since our interest is in understanding the behavior of the variable mass system's angular momentum from an inertial frame, we revert the time derivative in Equation (11) to N

$$\mathbf{0} = \frac{d\mathbf{H}_0^*}{dt} + \int_{S_0} \rho \mathbf{h} u dS. \quad (13)$$

In the above equation, $N d/dt$ is a time derivative taken in the inertial reference frame N . Any vector can be expressed as a combination of a scalar and a unit vector directed along the vector itself. So, $\mathbf{h}$ is rewritten as $\mathbf{h} = h\mathbf{n}_h$, where $\mathbf{n}_h$ is a unit vector directed along $\mathbf{h}$ whose magnitude is h . As a result, Equation (12) can be written as

$$\mathbf{H}_0^* = \left(\int_{V_0} \rho h dV \right) \mathbf{n}_h \quad (14)$$

and Equation (13) as

$$\frac{d\mathbf{H}_0^*}{dt} = \left(- \int_{S_0} \rho h u dS \right) \mathbf{n}_h \quad (15)$$

Equation (14) asserts that $\mathbf{H}_0^*$ is generally not of constant magnitude while Equations (14) and (15) assert that it is always directed along the $\mathbf{n}_h$ vector, which it will now be proved is an inertially fixed vector.

Let $\mathbf{n}_f$ and $\mathbf{n}_g$ be two unit vectors which form a dextral set with $\mathbf{n}_h$ such that $\mathbf{n}_f \times \mathbf{n}_g = \mathbf{n}_h$ and so on. This dextral set of unit vectors are attached to an imaginary reference frame Q whose inertial angular velocity is expressed as

$$\boldsymbol{\omega}^Q = \Omega_1 \mathbf{n}_f + \Omega_2 \mathbf{n}_g + \Omega_3 \mathbf{n}_h. \quad (16)$$

The time rate of change of $\mathbf{n}_h$ in the inertial frame is

$$\frac{d\mathbf{n}_h}{dt} = \frac{d\mathbf{n}_h}{dt} + \boldsymbol{\omega}^Q \times \mathbf{n}_h \quad (17)$$

where the first term on the right-hand side of Equation (17) evaluates to zero since $\mathbf{n}_h$ is fixed in Q . Then, substituting for $\boldsymbol{\omega}^Q$ from Equation (16) in Equation (17) gives

$$\frac{d\mathbf{n}_h}{dt} = \Omega_2 \mathbf{n}_f - \Omega_1 \mathbf{n}_g. \quad (18)$$

Further, Equation (15) is rewritten as

$$\frac{d}{dt} \int_{V_0} \rho h dV \mathbf{n}_h = \left(- \int_{S_0} \rho h u dS \right) \mathbf{n}_h \quad (19)$$

or

Fig. 2. Graphical evaluation of nutation angle.

$$\begin{aligned} & \frac{d}{dt} \left(\int_{V_0} \rho h dV \right) \mathbf{n}_h + \left(\int_{V_0} \rho h dV \right) \frac{d\mathbf{n}_h}{dt} \\ &= \left(- \int_{S_0} \rho h u dS \right) \mathbf{n}_h. \end{aligned} \quad (20)$$

The result from Equation (18) is substituted in Equation (20) to give

$$\begin{aligned} & \frac{d}{dt} \left(\int_{V_0} \rho h dV \right) \mathbf{n}_h + \left(\int_{V_0} \rho h dV \right) (\Omega_2 \mathbf{n}_f - \Omega_1 \mathbf{n}_g) \\ &= \left(- \int_{S_0} \rho h u dS \right) \mathbf{n}_h. \end{aligned} \quad (21)$$

The above equation, when rewritten in component form, leads to $\Omega_1 = \Omega_2 = 0$. Using these values for Ω_1 and Ω_2 in Equation (18) gives $\frac{d\mathbf{n}_h}{dt} = \mathbf{0}$, which explains that $\mathbf{n}_h$ is an inertially fixed unit vector thus, also making Q an inertial frame. By extension, it is also evident that the angular momentum of a variable mass system is also an inertially fixed vector as it is directed along $\mathbf{n}_h$.

3. Discussion

As mentioned in the introductory section, this directional conservation of the angular momentum vector is extremely useful in attitude determination. Fig. 2 shows the setup of the angular momentum vector $\mathbf{H}_0^*$ relative to the body-fixed principal directions $\mathbf{b}_i$ ($i = 1, 2, 3$). The angular momentum vector is seen to lie in the plane made by $\mathbf{b}_3$ and $\mathbf{b}_{12}$, the latter is a unit vector in the $\mathbf{b}_1$ - $\mathbf{b}_2$ plane. The angular momentum vector can then be expressed as a linear combination of two vectors in the $\mathbf{b}_{12}$ - $\mathbf{b}_3$ plane as

$$\mathbf{H}_0^* = \mathbf{H}_{12} + \mathbf{H}_3 = H_{12} \mathbf{b}_{12} + H_3 \mathbf{b}_3 \quad (22)$$

where

$$\mathbf{b}_{12} = \frac{\mathbf{H}_{12}}{H_{12}}. \quad (23)$$

Our interest here is in evaluating θ , the angle between $\mathbf{b}_3$ and $\mathbf{n}_h$. Since $\mathbf{n}_h$ is an inertially fixed vector, θ gives attitude information about the system from an inertial reference frame and, for the 3-1-3 Euler sequence of rotations, it is the second Euler angle.

From Fig. 2, it is clear that

$$\theta = \tan^{-1} \left(\frac{H_{12}}{H_3} \right). \quad (24)$$

In the case of variable mass systems, the difficulty lies in evaluating H_{12} and H_3 because the internal flow pattern of the fluid phase is not known everywhere inside the control volume. However, for systems such as space-rockets, it is reasonable to assume

that the fluid flow relative to the rigid base is axisymmetric. Consequently, the expression to the angular momentum in eq. (12) simplifies to $\mathbf{H}_0^* = \int_{V_0} \rho \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dV$. Further, it is reasonable to assume that the rocket is axisymmetric about $\mathbf{b}_3$. Then, the angular momentum can be expressed as

$$\mathbf{H}_0^* = \mathbf{I}_0^* \cdot \boldsymbol{\omega} = I\omega_{12}\mathbf{b}_{12} + J\omega_3\mathbf{b}_3, \quad (25)$$

where ω_{12} is the angular speed in the $\mathbf{b}_{12}$ direction, ω_3 is the spin rate in the $\mathbf{b}_3$ direction, and I and J are moment of inertia scalars. The angular speeds and moments of inertia are known parameters; in the case of the axisymmetric cylinder [24], these parameters are known explicitly for a variety of idealized models of mass loss. Thus, θ is also known explicitly and is given by

$$\theta = \tan^{-1} \left(\frac{I\omega_{12}}{J\omega_3} \right). \quad (26)$$

The expression for θ in eq. (26) is identical to that for an axisymmetric constant-mass system [22], but the properties of the parameters are inherently different due to mass variation; in the constant mass case all these parameters are constant and, thus, θ is constant.

In the classical mechanics literature, θ is referred to as the nutation angle [1] and $\mathbf{b}_3$ is the spin axis. In the case of spacecraft, growths in this angle have an undesirable effect on its heading direction. A control strategy for this nutation instability, suggested by eq. (26), is to impart a high spin rate to an axisymmetric rocket, thus, increasing the angular momentum in the spin direction. Such nutation instabilities have been observed in solid rocket motors [25] but the cause for the instability remains an open problem.

In summary, it has been shown that the angular momentum of a free variable mass system is inertially fixed and is, thus, a partially conserved quantity. This result can serve as the foundation for analytical and geometric examinations of the rotational motions of variable mass systems. Further, the utility of this result has been demonstrated with a brief discussion on graphically evaluating the second Euler angle without integrating the differential equation of motion. This analytical result provides footing for investigating nutation stability and developing control algorithms for a variety of systems with mass variation.

Conflict of interest statement

None declared.

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