On the Existence of Limit Admissible Equilibria in Discontinuous Games*

Guilherme Carmona[†]

University of Surrey

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Abstract

We consider the existence of limit admissible equilibria, i.e. Nash equilibria in which each player assigns zero probability to the interior of the set of his weakly dominated strategies, in (possibly) discontinuous games. We show that standard sufficient conditions for the existence of Nash equilibrium, such as better-reply security, fail to imply the existence of limit admissible equilibria. We then modify better-reply security to obtain a new condition, admissible security, and show that admissible security is sufficient for the existence of limit admissible equilibria. This result implies the existence of limit admissible equilibria in a Bertrand-Edgeworth competition setting with convex costs analogous to that of Maskin (1986).

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[†]Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: g.carmona@surrey.ac.uk.

1 Introduction

The motivation of the refinement literature (see van Damme (1991) for a survey) is that the requirements of Nash equilibrium need to be strengthened to obtain a more appealing solution concept in normal-form games. Along this lines, and following Kohlberg and Mertens (1986), it would be desirable to have a solution concept whose existence is generally guaranteed and in undominated strategies.

For games with finite action spaces, Selten's (1975) perfect equilibrium is a way of obtaining the above properties. However, in games with infinite action spaces, several difficulties arise. First, even when players' payoff functions are continuous, it is possible that all Nash (and, therefore, all perfect) equilibria are in weakly dominated strategies (see Simon and Stinchcombe (1995, Example 2.1)). For this reason, one can only hope to obtain general existence results for limit admissible equilibria, i.e. of Nash equilibrium in which each player assigns zero probability to the interior of the set of his weakly dominated strategies.

Furthermore, when some player's payoff function is discontinuous, existence of perfect equilibria requires conditions considerably stronger than the standard ones guaranteeing existence of Nash equilibrium (such as Reny's (1999) better-reply security). Moreover, it is not always the case that perfect equilibria are limit admissible (see Carbonell-Nicolau (2011b)). Other solution concepts have been proposed such as Andersson, Argenton, and Weibull's (2014) robust equilibrium to strategic uncertainty and Bich's (2016) prudent equilibrium, but again their existence and limit admissibility have only been established under assumptions stronger than the standard ones.

In this paper we directly address the existence of limit admissible equilibria. We first show that a condition different from better-reply security is needed. Indeed, we present two examples of better-reply secure games that fail to have limit admissible equilibria: One of these examples fails to have limit admissible strategies and the other, while having limit admissible strategies, fails to have one that is also a Nash equilibrium.

We then introduce the notion of admissible security and use it to establish the existence of limit admissible equilibrium in mixed strategies.¹ In contrast with the complicated task of establishing the existence of perfect equilibrium (and, thus, of establishing indirectly and under some additional assumptions the existence of limit admissible equilibria), we show that the task of establishing directly that a limit admissible equilibrium exists is a relatively simple one. Indeed, besides having to assume that limit admissible strategies exist, all it takes is that better-reply security be “appropriately relativized” to the set of limit admissible strategies. In this sense, admissible security is the natural modification of better-reply security to address the existence of limit admissible equilibria.

It is important to note that admissible security is neither stronger nor weaker than better-reply security. It is stronger than assuming that the game obtained by restricting players to choose limit admissible strategies is better-reply secure; this latter condition is, as we show, not sufficient for the existence of limit admissible equilibria. An additional advantage of our approach as compared to analysing the game restricted to limit admissible strategies is that the former does not require a complete characterization of limit admissible strategies; the general auction setting of Jackson and Swinkels (2005) provides an example where this is useful.

The paper is organized as follows. Section 2 presents our notation and general definitions. Some motivating examples, including those showing that better-reply security is not sufficient for the existence of limit admissible equilibrium are in Section 3. Our new condition, admissible security, is introduced in Section 4 and used to establish the existence of limit admissible equilibrium.

Several remarks in Section 4 complement this result, by showing that: (1) More general existence results can be obtained by relativizing Reny’s (2016) correspondence security and finite deviation property; (2) an extension of our result for pure strategies is possible in own-strategy quasi-concave games when the set of limit admissible pure strategies to be convex; (3) the relativization of Reny’s (1999) payoff security and of

¹Admissible security also yields the existence of pure-strategy limit admissible equilibria when the set of limit admissible pure strategies is convex for each player.

Bagh and Jofre's (2006) weak reciprocal upper semicontinuity together provide sufficient conditions for admissible security; (4) better-reply secure games are admissible secure provided that each non-limit admissible strategy be weakly dominated by a limit admissible strategy (a weaker version of this property suffices).

In Section 4 we relate our existence result with those that follow from Simon and Stinchcombe (1995) for continuous games and from Carbonell-Nicolau (2011b) for discontinuous games; we also briefly compare our approach with that of Harris, Stinchcombe, and Zame (2005). Our results are applied in Section 5 to a Bertrand-Edgeworth duopoly game with strictly convex costs, modelled along the lines of Maskin (1986). In addition to being an interesting game in its own right, this game is useful to illustrate our results because it is not difficult to characterize its set of weakly dominated strategies and, once such characterization is obtained, applying our results is a relatively simple task.

2 Notation and Definitions

Consider a normal-form game $G = (X_i, u_i)_{i \in N}$ consisting of a finite set of players $N = \{1, \dots, n\}$, and, for each player $i \in N$, a compact metric pure strategy set X_i and a bounded and measurable real-valued payoff function u_i on $X = \prod_{i=1}^n X_i$.

For all $i \in N$, X_i is endowed with its Borel σ -algebra. The set of mixed strategies for player i is the set of (Borel) probability measures M_i on X_i , endowed with the narrow topology. For each $x_i \in X_i$, we write δ_{x_i} for the Dirac measure at x_i . Let $M = \prod_{i=1}^n M_i$. Given a profile $\sigma \in M$ of mixed strategies, we write τ_σ for the corresponding product measure on X . Define, for all $i \in N$ and $\sigma \in M$, $u_i(\sigma) = \int_X u_i(x) d\tau_\sigma(x)$.

Throughout the paper, the product of any number of sets is endowed with the product topology. Given a player $i \in N$, the symbol $-i$ denotes "all players but i ". Also, $X_{-i} = \prod_{j \neq i} X_j$.

A pure strategy $x_i \in X_i$ is *weakly dominated* for $i \in N$ if there exists a mixed strategy $\mu_i \in M_i$ such that $u_i(x_i, x_{-i}) \leq u_i(\mu_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $u_i(x_i, x'_{-i}) < u_i(\mu_i, x'_{-i})$ for some $x'_{-i} \in X_{-i}$. Let O_i be the interior of the set of weakly dominated

strategies for i and $C_i = X_i \setminus O_i$ be its complement.

We say that $\sigma \in M$ is a *Nash equilibrium* of G if $u_i(\sigma) \geq u_i(\sigma'_i, \sigma_{-i})$ for each $i \in N$ and $\sigma'_i \in M_i$; the set of Nash equilibria of G is denoted by $E(G)$. A mixed strategy profile $\sigma \in M$ is *limit admissible* if $\sigma_i(O_i) = 0$ for each $i \in N$. Let $L(G)$ denote the set of Nash equilibria of G that are limit admissible, its elements being referred to as *limit admissible equilibria*.

The existence of Nash equilibria is guaranteed under better-reply security, a notion introduced in Reny (1999) and which we now recall; we shall see that better-reply security is not sufficient for the existence of limit admissible equilibria. A game $G = (X_i, u_i)_{i \in N}$ is *better-reply secure* in mixed strategies if, whenever $(\sigma^*, u^*) \in \text{cl}(\text{graph}(u))$ and σ^* is not a Nash equilibrium of G , there exist $i \in N$, $\mu_i \in M_i$, an open neighborhood U of σ^* and a real number $\alpha_i > u_i^*$ such that $u_i(\mu_i, \sigma_{-i}) \geq \alpha_i$ for each $\sigma \in U$.

3 Motivating examples

We illustrate with two simple examples that best-reply secure games may fail to have a limit admissible equilibrium. The second example is also an example of a game whose restriction to its set of limit admissible strategies is continuous but, nevertheless fails to have a limit admissible equilibrium.

Example 1. Let $G = (X_i, u_i)_{i \in N}$ be a 2-player game on the unit square (i.e. $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$) with the following payoff functions: For each $x \in X$,

$$u_1(x) = \begin{cases} 1 & \text{if } x_2 \leq x_1 < 1, \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_2(x) = \begin{cases} 1 & \text{if } x_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the set of Nash equilibria of G is $\{(\sigma_1, \delta_0) : \sigma_1 \in M_1\}$.

We now show that G is better-reply secure in mixed strategies. Let $(\sigma^*, u^*) \in \text{cl}(\text{graph}(u))$ be such that σ^* is not a Nash equilibrium of G . Then $\sigma_2^* \neq \delta_0$, i.e. $\sigma_2^*(\{0\}) < 1$. Letting $\{\sigma_k\}_{k=1}^\infty \subseteq M$ be such that $(\sigma_k, u(\sigma_k)) \rightarrow (\sigma^*, u^*)$ and using Portmanteau theorem (e.g. Billingsley (1999, Theorem 2.1, p. 16)), we obtain that

$$u_2^* = \lim_k u_2(\sigma_k) = \lim_k \sigma_{k,2}(\{0\}) \leq \sigma_2^*(\{0\}) < 1.$$

Since $u_2(\sigma_1, \delta_0) = 1$ for each $\sigma_1 \in M_1$, it follows that the condition in the definition of better-reply security is satisfied with $i = 2$, $\mu_2 = \delta_0$, $U = M$ and $\alpha_2 = 1$.

Despite being better-reply secure, G has no limit admissible equilibrium since $C_1 = \emptyset$. Indeed, each $x_1 < 1$ is weakly dominated for $i = 1$ by $x'_1 \in (x_1, 1)$ and $x_1 = 1$ is weakly dominated for $i = 1$ by $x'_1 = 1/2$.

The previous example shows that better-reply security is not sufficient for the non-emptiness of C_i for each $i \in N$. The following example shows that better-reply secure games with $C_i \neq \emptyset$ for each $i \in N$ may fail to have limit admissible equilibria.

Example 2. Let $G = (X_i, u_i)_{i \in N}$ be as in Example 1 except that now

$$u_1(1/2, 1/2) = 2 \text{ and } u_1(x_1, 0) = 0 \text{ for each } 1/4 < x_1 < 3/4.$$

It is clear that $C_2 = \{0\}$ and that $C_1 = \{1/2\}$; for the latter, each $x_1 < 1$ different than $1/2$ is weakly dominated for $i = 1$ by $x'_1 \in (\max\{x_1, 3/4\}, 1)$ and $x_1 = 1$ is weakly dominated for $i = 1$ by $x'_1 = 3/4$. It is also easy to see that the set of Nash equilibria of G is $\{(\sigma_1, \delta_0) : \sigma_1((1/4, 3/4)) = 0\}$.

We now show that G is better-reply secure in mixed strategies. Let $(\sigma^*, u^*) \in \text{cl}(\text{graph}(u))$ be such that σ^* is not a Nash equilibrium of G . Arguing as in Example 1, we may assume that $\sigma_2^* = \delta_0$ and $u_2^* = 1$. Hence, $\sigma_1^*((1/4, 3/4)) > 0$. Letting $\{\sigma_k\}_{k=1}^\infty \subseteq M$ be such that $(\sigma_k, u(\sigma_k)) \rightarrow (\sigma^*, u^*)$, we have that $\lim_k \sigma_{k,2}(\{0\}) = \lim_k u_2(\sigma_k) = u_2^* = 1$. Thus, by the Portmanteau theorem (e.g. Billingsley (1999, Theorem 2.1, p. 16)) and by $\int_{X_1} u_1(x_1, x_2) d\sigma_{k,1}(x_1) \leq 2$ for each $x_2 > 0$ and $k \in \mathbb{N}$,

we obtain that

$$\begin{aligned} u_1^* &= \lim_k u_1(\sigma_k) \leq \overline{\lim}_k (\sigma_{k,2}(\{0\})\sigma_{k,1}(X_1 \setminus (1/4, 3/4)) + 2\sigma_{k,2}((0, 1])) \\ &= \overline{\lim}_k \sigma_{k,1}(X_1 \setminus (1/4, 3/4)) \leq \sigma_1^*(X_1 \setminus (1/4, 3/4)) < 1. \end{aligned}$$

Let $\alpha_1 \in (u_1^*, 1)$ and O_2 be an open neighborhood of δ_0 such that $\int_{X_2} 1_{[0,3/4)} d\sigma_2 > \int_{X_2} 1_{[0,3/4)} d\delta_0 - (1 - \alpha_1)$ for each $\sigma_2 \in O_2$.² Let $i = 1$, $\mu_1 = \delta_{3/4}$ and $U = M_1 \times O_2$; since, for each $\sigma \in U$, $u_1(\mu_1, \sigma_2) = \sigma_2([0, 3/4]) \geq \int_{X_2} 1_{[0,3/4)} d\sigma_2 > \int_{X_2} 1_{[0,3/4)} d\delta_0 - (1 - \alpha_1) = \alpha_1 > u_1^*$, it follows that G is better-reply secure.

Recall that C_i is a singleton for each player i . This implies that the set of limit admissible strategies of each player is also a singleton; hence, the game obtained by restricting players to choose limit admissible strategies is (trivially) continuous.

However, many discontinuous games have limit admissible equilibria. An easy example is provided by a symmetric Bertrand duopoly with constant marginal costs.

Example 3. Suppose that each firm has a constant marginal cost $0 < c < 1$ and there is one commodity whose demand is $d(x) = 1 - x$, where x is the lowest price in the market. Each of the two firms sets a price in the unit interval and gets half of the demand when these prices are equal. Thus, this situation can be described by a game $G = (X_i, u_i)_{i \in N}$ with $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$ and, for each $x \in X$,

$$u(x) = \begin{cases} ((x_1 - c)(1 - x_1), 0) & \text{if } x_1 < x_2, \\ (0, (x_2 - c)(1 - x_2)) & \text{if } x_1 > x_2, \\ ((x_1 - c)(1 - x_1)/2, (x_1 - c)(1 - x_1)/2) & \text{if } x_1 = x_2. \end{cases}$$

It is easy to check that $C_i = [c, (1 + c)/2]$ for each $i \in N$ and that (δ_c, δ_c) is a Nash equilibrium of G and, thus, a limit admissible equilibrium of G . As we shown in the next section, the existence of a limit admissible equilibrium of G also follows from our existence result.

²The function $1_{[0,3/4)}$ is defined by $1_{[0,3/4)}(x_2) = 1$ if $x_2 \in [0, 3/4)$ and $1_{[0,3/4)}(x_2) = 0$ if $x_2 \notin [0, 3/4)$; it is lower semi-continuous. Thus, the existence of O_2 follows by Aliprantis and Border (2006, Theorem 15.5, p. 511).

4 Existence of limit admissible equilibria

We obtain the existence of limit admissible equilibria by relativizing the notion of better-reply security as follows. For each $i \in N$, let $M_i^0 = \{\mu_i \in M_i : \mu_i(O_i) = 0\}$ and let $M^0 = \prod_{i \in N} M_i^0$. We say that a game $G = (X_i, u_i)_{i \in N}$ is *admissibly secure* if

- (a) $C_i \neq \emptyset$ for each $i \in N$ and
- (b) If $(\sigma^*, u^*) \in \text{cl}(\{(\sigma, u(\sigma)) : \sigma \in M^0\})$ and σ^* is not a Nash equilibrium of G , then there exist $i \in N$, $\mu_i \in M_i^0$, an open neighborhood U of σ^* and a real number $\alpha_i > u_i^*$ such that $u_i(\mu_i, \sigma_{-i}) \geq \alpha_i$ for each $\sigma \in U \cap M^0$.

Analogously to Reny's (1999) description of better-reply security (the differences are emphasized), admissible security requires that for each non-Nash equilibrium *limit admissible* strategy σ^* and every payoff vector limit u^* resulting from *limit admissible* strategies approaching σ^* , some player i has a *limit admissible* strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from σ^* to another *limit admissible* strategy.

Admissible security is a relativization of better-reply security in mixed strategies and, in this way, is neither stronger nor weaker than better-reply security in mixed strategies.³ A further relativization, namely changing (b) by requiring it to hold only for non-equilibrium strategies of the game obtained by restricting players to choose limit admissible strategies, is not possible when the goal is to obtain a condition that is sufficient for the existence of limit admissible equilibria. This is shown by Example 2 which has no limit admissible equilibrium.

Note further that Example 1 satisfies (b) (vacuously) but not (a) and that Example 2 satisfies (a) but not (b). Thus, both conditions in the definition of admissible security are needed for this condition to imply the existence of limit admissible equi-

³A sufficient condition for admissible security which is stronger than better-reply security in mixed strategies is the following: If $(\sigma^*, u^*) \in \text{cl}(\text{graph}(u))$ and σ^* is not a Nash equilibrium of G , then there exist $i \in N$, $\mu_i \in M_i^0$, an open neighborhood U of σ^* and a real number $\alpha_i > u_i^*$ such that $u_i(\mu_i, \sigma_{-i}) \geq \alpha_i$ for each $\sigma \in U$.

libria. Our main result states that admissible security is indeed sufficient for the existence of limit admissible equilibria.

Theorem 1. *If $G = (X_i, u_i)_{i \in N}$ is admissibly secure, then $L(G) \neq \emptyset$.*

Proof. Let G be admissibly secure and consider $G^0 = (M_i^0, u_i|_{M^0})_{i \in N}$, where $u_i|_{M^0}$ is the restriction of u_i to M^0 for each $i \in N$. Note that $\text{cl}(\{(\sigma, u(\sigma)) : \sigma \in M^0\}) = \text{cl}(\text{graph}(u|_{M^0}))$ and that $U \cap M^0$ is open in M^0 whenever U is open in M . Furthermore, if $\sigma^* \in M^0$ is not a Nash equilibrium of G^0 , then σ^* is not a Nash equilibrium of G . Hence, G^0 is better-reply secure. It then follows by Reny (1999, Theorem 3.1) (G^0 is clearly a compact and own-strategy quasi-concave game) that G^0 has a Nash equilibrium $\sigma^* \in M^0$.

Suppose, in order to reach a contradiction, that $L(G) = \emptyset$. Then σ^* is not a Nash equilibrium of G . Thus, as $(\sigma^*, u(\sigma^*)) \in \text{cl}(\{(\sigma, u(\sigma)) : \sigma \in M^0\})$ and G is admissibly secure, there exists $i \in N$ and $\mu_i \in M_i^0$ such that $u_i(\mu_i, \sigma_{-i}^*) > u_i(\sigma^*)$. But this is a contradiction since σ^* is a Nash equilibrium of G^0 . This contradiction shows that $L(G)$ is nonempty and completes the proof. $\square$

Extensions as well as particular cases of Theorem 1 are discussed in the following remarks.

Remark 1 (Admissible correspondence security). An extension of Theorem 1 can be obtained by weakening admissible security along the lines of Reny's (2016) correspondence security. A game $G = (X_i, u_i)_{i \in N}$ is *admissible correspondence secure* if $C_i \neq \emptyset$ for each $i \in N$ and if whenever $\sigma^* \in M^0$ is not a Nash equilibrium of G , then there exists an open neighborhood U of σ^* and a closed correspondence $\psi : U \rightrightarrows M^0$ with nonempty and convex values such that, for each $\sigma \in U \cap M^0$, there is $i \in N$ for whom $u_i(\mu_i, \sigma'_{-i}) > u_i(\sigma)$ for each $\sigma' \in U \cap M^0$ and $\mu_i \in \psi_i(\sigma')$. We then have, by an argument analogous to that of Theorem 1, that if $G = (X_i, u_i)_{i \in N}$ is admissibly correspondence secure, then $L(G) \neq \emptyset$.

Remark 2 (Admissible finite deviation property). Another extension of Theorem 1 can also be obtained by weakening admissible security along the lines of Reny's (2016)

finite deviation property. We say that $G = (X_i, u_i)_{i \in N}$ satisfies the *admissible finite deviation property* if, for each $\sigma \in M^0$ which is not a Nash equilibrium of G , there exists $\mu^1, \dots, \mu^K \in M^0$ and a neighborhood U of σ such that, for each $\sigma' \in U \cap M^0$, there exists $i \in N$ and $k \in \{1, \dots, K\}$ such that $u_i(\mu_i^k, \sigma'_{-i}) > u_i(\sigma')$. Adapting the proof of Theorem 1, one can show that $L(G) \neq \emptyset$ for each game with the admissible finite deviation property.

Remark 3 (Pure strategies). Our existence result extends to the case of pure strategies in games $G = (X_i, u_i)_{i \in N}$ that are own-strategy quasi-concave (i.e. X_i is convex and $u_i(\cdot, x_{-i})$ is quasi-concave for each $i \in N$ and $x_{-i} \in X_{-i}$) and are such that C_i is convex for each $i \in N$. In this case, letting $C = \prod_{i \in N} C_i$, part (b) of the definition of admissible security is specialized to the case of pure strategies as follows: If $(x^*, u^*) \in \text{cl}(\{(x, u(x)) : x \in C\})$ and x^* is not a Nash equilibrium of G , then there exist $i \in N$, $\bar{x}_i \in C_i$, an open neighborhood U of x^* and a real-number $\alpha_i > u_i^*$ such that $u_i(\bar{x}_i, x_{-i}) \geq \alpha_i$ for each $x \in U \cap C$.

The Bertrand duopoly of Example 3 illustrates. Indeed, it is a quasi-concave game in which C_i is convex for each $i = 1, 2$, namely $C_i = [c, (1+c)/2]$ for each $i = 1, 2$ (note that $(1+c)/2$ is the monopoly quantity). Furthermore, it is admissibly secure: When $x^* \in C$ is not a Nash equilibrium (i.e. $x^* \neq (c, c)$) and $x_i^* = x_j^* > c$, pick player i , $\bar{x}_i \in [c, x_j^*)$ sufficiently close to x_i^* and $U = X_i \times (\bar{x}_i, x_j^*)$; if $x_i^* > x_j^* \geq c$, pick player j , $\bar{x}_j \in (c, x_i^*)$ and $U = X_j \times (\bar{x}_j, x_i^*)$. Thus, in accordance with the above, the Bertrand duopoly of Example 3 has a pure-strategy limit admissible equilibrium, namely $x^* = (c, c)$.

In general, however, own-strategy quasi-concave games satisfying admissible security may fail to have a pure-strategy limit admissible equilibrium when C_i fails to be convex for some $i \in N$. To illustrate, we present an example of an own-strategy quasi-concave game with continuous payoff functions that fails to have a limit admissible equilibrium.⁴ The game $G = (X_i, u_i)_{i \in N}$ is a variation on the matching pennies.

⁴Games with continuous payoff functions have pure-strategy limit admissible equilibria provided that X_i is convex and $u_i(\cdot, x_{-i})$ is *concave* for each $i \in N$ and $x_{-i} \in X_{-i}$; see Carbonell-Nicolau (2011a) for this and related results.

Let $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$, where x_i is interpreted as the probability of player i playing heads, $i = 1, 2$. Player 1 want to match his choice with that of player 2 and, thus, let

$$u_1(x) = -|x_1 - x_2|$$

for each $x \in X$. As for player 2's payoff function, let, for each $x \in X$,

$$w_2(x) = (1 - 2x_1)(2x_2 - 1)$$

as in the standard matching pennies, and

$$u_2(x) = \max\{w_2(x), 0\}.$$

It is easy to see that G is own-strategy quasi-concave with continuous payoff functions and that it has a unique pure-strategy Nash equilibrium $x^* = (1/2, 1/2)$. Moreover, $C_2 = \{0, 1\}$ since any $x_2 \in [1/2, 1)$ is weakly dominated by $x'_2 = 1$, any $x_2 \in (0, 1/2)$ is weakly dominated by $x'_2 = 0$ and neither $x_2 = 0$ nor $x_2 = 1$ are weakly dominated. Therefore, G has no limit admissible equilibria in pure strategies.

Remark 4 (Sufficient conditions 1). We say that a game $G = (X_i, u_i)_{i \in N}$ is *admissibly payoff secure* if for each $i \in N$, $\varepsilon > 0$ and $\sigma \in M_i \times M_{-i}^0$, there is a $\mu_i \in M_i^0$ and an open neighborhood U of σ such that

$$u_i(\mu_i, \sigma'_{-i}) > u_i(\sigma) - \varepsilon$$

for each $\sigma' \in U \cap M^0$. Moreover, we say that G is *admissibly reciprocal upper semi-continuous* if for each $(\sigma^*, \alpha) \in \text{cl}(\{(\sigma, u(\sigma)) : \sigma \in M^0\}) \setminus \{(\sigma, u(\sigma)) : \sigma \in M^0\}$, there exists $i \in N$ and $\mu_i \in M_i$ such that $u_i(\mu_i, \sigma_{-i}^*) > \alpha_i$. Analogously to Bagh and Jofre (2006, Proposition 1), we obtain that if G is admissibly payoff secure and admissibly reciprocal upper semi-continuous, then G is admissibly secure.

Remark 5 (Continuous games). Games with continuous payoff functions, considered in Simon and Stinchcombe (1995), are easily seen to be admissible secure. Indeed, admissible reciprocal upper semi-continuity is trivially satisfied as $\text{cl}(\{(\sigma, u(\sigma)) : \sigma \in M^0\}) \setminus \{(\sigma, u(\sigma)) : \sigma \in M^0\}$ is empty. Furthermore, for admissible payoff security,

it suffices to show that, against any $\sigma \in M$, each player i has a best-reply in M_i^0 . To see that this property holds, let $i \in N$ and $\sigma \in M$ be given. Letting $j \neq i$ and $\{x_{k,j}\}_{k=1}^\infty$ be dense in X_j , define $\bar{\sigma}_j = \sum_{k=1}^\infty \frac{1}{2^k} \delta_{x_{k,j}} \in M_j$ and $\sigma_{k,j} = \frac{1}{k} \bar{\sigma}_j + (1 - \frac{1}{k}) \sigma_j$ for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and, using the continuity of u_i , let $x_{k,i} \in X_i$ be such that $u_i(x_{k,i}, \sigma_{k,-i}) = \max_{\sigma_i \in M_i} u_i(\sigma_i, \sigma_{k,-i})$. If $x_{k,i} \notin C_i$, then letting $\mu_i \in M_i$ weakly dominate $x_{k,i}$, we have that $u_i(x_{k,i}, \sigma_{k,-i}) < u_i(\mu_i, \sigma_{k,-i})$ as $\tau_{\sigma_{k,-i}}$ gives strictly positive probability to each open subset of X_{-i} ; but this is a contradiction to $u_i(x_{k,i}, \sigma_{k,-i}) = \max_{\sigma_i \in M_i} u_i(\sigma_i, \sigma_{k,-i})$. Thus, $x_{k,i} \in C_i$ for each $k \in \mathbb{N}$. Since C_i is compact, we may assume that $\{x_{k,i}\}_{k=1}^\infty$ converges and let $x_i = \lim_k x_{k,i}$. Then $x_i \in C_i$ and the continuity of u_i , together with $\sigma_{k,-i} \rightarrow \sigma_{-i}$, implies that $u_i(x_i, \sigma_{-i}) = \max_{\sigma_i \in M_i} u_i(\sigma_i, \sigma_{-i})$.

Remark 6 (A related approach). One way of obtaining continuous games is to endow players' strategy spaces with Fudenberg and Levine's (1986) "most utility difference it can make to anyone" pseudo-metric, defined by

$$p(x_i, x'_i) = \max_{j \in N} \sup_{x_{-i} \in X_{-i}} |u_j(x_i, x_{-i}) - u_j(x'_i, x_{-i})|$$

for each $x_i, x'_i \in X_i$ in a game $G = (X_i, u_i)_{i \in N}$. In general, however, players' strategy spaces may fail to be compact.⁵ This difficulty can sometimes be overcome in a way that allows for existence of Nash equilibrium by compactification as shown by Harris, Stinchcombe, and Zame (2005).

We leave as open the question of under what conditions can this approach yield an existence results for limit admissible equilibria. Here we just remark that there are games to which our approach can be applied but not that of Harris, Stinchcombe, and Zame (2005). Such an example is provided by their Example 2: Let $G = (X_i, u_i)_{i \in N}$ be a 2-player game on the unit square (i.e. $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$) with the following payoff functions: For each $x \in X$ and $i \in N$,

$$u_i(x) = \begin{cases} x_1 & \text{if } x_1 = x_2, \\ 0 & \text{otherwise,} \end{cases}$$

⁵These observations provide an alternative view on Examples 1 and 2.

for each $x \in X$. In this game, $C_i = X_i$ for each $i \in N$, hence the set of limit admissible equilibria coincides with the set of Nash equilibria. As Harris, Stinchcombe, and Zame (2005) show, this game is not nearly compact and continuous, hence their results do not apply. In contrast, the condition in Remark 1 holds in pure strategies (because $C_i = X_i$ for each $i \in N$, this is just Reny's (2016) correspondence security) and, as discussed in Remark 3 hence, G has a (limit admissible) equilibrium in pure strategies.

Remark 7 (Related results). The existence of a limit admissible equilibrium can be obtained from some results in Carbonell-Nicolau (2011b), namely from Theorem 4 and also by combining Theorem 7 and Remark 6 in that paper. Under the assumptions of Theorem 7 in Carbonell-Nicolau (2011b), both admissible payoff security and admissible reciprocal upper semicontinuity hold, hence, the existence of a limit admissible equilibrium follows by Remark 4 and Theorem 1 above (this argument makes Remark 6 in Carbonell-Nicolau (2011b) redundant). Under the assumptions of Theorem 4 in Carbonell-Nicolau (2011b), better-reply security in mixed strategies holds and so does admissible reciprocal upper semicontinuity. However, it is unclear whether or not admissible payoff security holds. This is the case even under the assumptions of Carbonell-Nicolau's (2011b) Theorem 4 which imply payoff security, i.e., it is the case that for each $i \in N$, $\varepsilon > 0$ and $\sigma \in M$, there is $\mu_i \in M_i$ and an open neighborhood U of σ such that $u_i(\mu_i, \sigma'_{-i}) > u_i(\sigma) - \varepsilon$ for each $\sigma' \in U$; the difficulty is to conclude that μ_i can be taken to be an element of M_i^0 as required by admissible payoff security. On the other end, our Theorem 1 does not require any of the conditions imposed in Carbonell-Nicolau's (2011b) results.

Remark 8 (Sufficient conditions 2). Carbonell-Nicolau's (2011b) Theorem 4 described in the previous remark can be seen as providing conditions that, when added to better-reply security in mixed strategies, imply the existence of Nash equilibria with additional structure. Here we ask: What additional conditions need to be added to better-reply security in mixed strategies for a game to have a limit admissible equilibrium? One such condition is: For each $i \in N$ and $\sigma_i \in M_i \setminus M_i^0$, there exists $\sigma_i^0 \in M_i^0$ such that $u_i(\sigma_i^0, x_{-i}) \geq u_i(\sigma_i, x_{-i})$ for each $x_{-i} \in X_{-i}$; this is a weakening of

the requirement that each strategy outside M_i^0 be weakly dominated by a strategy in M_i^0 . It is easy to see that any game satisfying this condition and better-reply security in mixed strategies is admissibly secure.

Remark 9 (Bertrand duopoly). The conditions in the previous remark apply to the Bertrand duopoly of Example 3. It is well known that better-reply security in mixed strategies holds for this example. Moreover, for each $i \in N$ and $\sigma_i \in M_i \setminus M_i^0$, there exists $\sigma_i^0 \in M_i^0$ such that $u_i(\sigma_i^0, x_{-i}) \geq u_i(\sigma_i, x_{-i})$ for each $x_{-i} \in X_{-i}$. To see this, recall that $C_i = [c, (1+c)/2]$, define $f : X_i \rightarrow C_i$ by setting, for each $x_i \in X_i$, $f(x_i) = c$ when $x_i < c$, $f(x_i) = (1+c)/2$ when $x_i > (1+c)/2$ and $f(x_i) = x_i$ otherwise and set $\sigma_i^0 = \sigma_i \circ f^{-1} \in M_i^0$. For each $x_{-i} \in X_{-i}$, we have that $u_i(x_i, x_{-i}) \leq 0 = u_i(c, x_{-i})$ for each $x_i < c$ and $u_i(x_i, x_{-i}) \leq u_i((1+c)/2, x_{-i})$ for each $x_i > (1+c)/2$. Hence, $u_i(\sigma_i^0, x_{-i}) - u_i(\sigma_i, x_{-i}) = \int_{[0,c)} (u_i(c, x_{-i}) - u_i(x_i, x_{-i})) d\sigma_i(x_i) + \int_{((1+c)/2, 1]} (u_i((1+c)/2, x_{-i}) - u_i(x_i, x_{-i})) d\sigma_i(x_i) \geq 0$. Thus, Theorem 1 applies.

Remark 10 (An extension). In Theorem 1, C_i is the closure of the set of undominated strategies for each $i \in N$. More generally, the conclusion of Theorem 1 holds for any nonempty, closed subset of X_i as follows. Let $G = (X_i, u_i)_{i \in N}$ be a game and, for each $i \in N$, $\tilde{C}_i$ be a nonempty, closed subset of X_i . Define $\tilde{M}_i = \{\sigma_i \in M_i : \sigma_i(\tilde{C}_i) = 1\}$ for each $i \in N$ and $\tilde{M} = \prod_{i \in N} \tilde{M}_i$. Then G has a Nash equilibrium in $\tilde{M}$ provided that the following condition analogous to admissible security holds: If $(\sigma^*, u^*) \in \text{cl}(\{(\sigma, u(\sigma)) : \sigma \in \tilde{M}\})$ and σ^* is not a Nash equilibrium of G , then there exist $i \in N$, $\mu_i \in \tilde{M}_i$, an open neighborhood U of σ^* and a real number $\alpha_i > u_i^*$ such that $u_i(\mu_i, \sigma_{-i}) \geq \alpha_i$ for each $\sigma \in U \cap \tilde{M}$.

5 Applications: Bertrand-Edgeworth duopoly with strictly convex costs

We consider a Bertrand-Edgeworth duopoly with strictly convex costs as an economic application of our main result, Theorem 1. We depart from the standard formalization of Bertrand competition considered in Andersson, Argenton, and Weibull (2014) and

Bich (2016) because the assumption that the firm posting the lowest price serves the entire demand leads to difficulties in the presence of strictly convex costs. This is so because firms may prefer to tie to reduce the quantity produced. For this reason, it is more appropriate to allow firms to choose the quantity they want to supply. We allow for this by allowing each firm to choose a price and the maximum production level it is willing to produce, i.e. an endogenous capacity. Our formalization is thus analogous to that of Dasgupta and Maskin (1986, Section 2.2) where each firm has a capacity but which is exogenously given. Our formalization is closer to that of Maskin (1986) where firms choose both prices and quantities, and firms produce to order, i.e., produce only after the entire price profile has been observed.

As in Dasgupta and Maskin (1986), there is a market for a single commodity with a continuum of consumers represented by the unit interval $[0, 1]$. Consumers are identical, and the representative consumer's demand for the commodity is a function $d : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that there exists $\bar{p} > 0$ satisfying $d(p) > 0$, $d'(p) < 0$ and $d''(p) \leq 0$ for all $p \leq \bar{p}$, $d(p) = 0$ for all $p \geq \bar{p}$ and d is C^2 in $[0, \bar{p}]$. There is a C^2 cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $c(0) = 0$, $\bar{p} > c'(0)$ and, for each $q > 0$, $c'(q) > 0$ and $c''(q) > 0$.

There are two firms. Each firm $i = 1, 2$ chooses a price p_i and a capacity s_i , the latter being the maximum amount the firm is willing to produce. Let $P = [0, \bar{p}]$ and $S = [0, d(0)]$.

If one firm offers a price p lower than the price p' offered by the other firm, then it serves the entire market up to its capacity s . A fraction $\frac{(d(p)-s)^+}{d(p)} = \frac{\max\{d(p)-s, 0\}}{d(p)}$ of consumers is not served and each of these consumers demands $d(p')$ from the firm offering the highest price. When both firms set the same price, then the demand at the common price is split by each firm up to its capacity.

In the latter case, several allocations are possible and we will focus on allocations that maximize the total profits in the market. Formally, the quantity produced by firms is described by the correspondence $\Phi : (P \times S)^2 \rightrightarrows \mathbb{R}_+^2$ defined by setting, for

each $x \in (P \times S)^2$,

$$\Phi(x) = \begin{cases} (\min\{d(p_1), s_1\}, \min\{\frac{(d(p_1)-s_1)^+}{d(p_1)}d(p_2), s_2\}) & \text{if } p_1 < p_2, \\ \{\phi \in \mathbb{R}_+^2 : \phi_1 + \phi_2 \leq d(p_1), \phi_i \leq s_i \text{ and} \\ [d(p_1) - \phi_1 - \phi_2][s_i - \phi_i] = 0 \text{ for each } i = 1, 2\} & \text{if } p_1 = p_2, \\ (\min\{\frac{(d(p_2)-s_2)^+}{d(p_2)}d(p_1), s_1\}, \min\{d(p_2), s_2\}) & \text{if } p_1 > p_2. \end{cases}$$

When prices are different, these quantities are the same as in both Dasgupta and Maskin (1986) and Maskin (1986) (with the proportional rationing rule in the latter). However, we rule out the possibility that a firm produces less than its capacity when there is unfulfilled demand (i.e. $\phi_1 + \phi_2 < d(p_1)$ and $\phi_i < s_i$ for some i is not possible); in contrast this is allowed in both Dasgupta and Maskin (1986) and Maskin (1986).

We then let $\phi : (P \times S)^2 \rightrightarrows \mathbb{R}_+^2$ be measurable and such that, for each $x \in (P \times S)^2$, $\phi(x)$ solves

$$\max_{\phi \in \Phi(x)} \sum_{i=1}^2 (p_i \phi_i - c(\phi_i)).$$

Players' payoffs are then

$$u_i(x) = p_i \phi_i(x) - c(\phi_i(x)) \quad (1)$$

for each $i \in \{1, 2\}$ and $x \in (P \times S)^2$. Our results, however, extend to a broad class of Bertrand-Edgeworth games obtained by changing the function ϕ ; see Remark 11 below.

The function ϕ and, therefore, players' payoff functions can be explicitly computed.

Lemma 1. *For each $x \in (P \times S)^2$ with $p_1 = p_2$,*

$$\phi(x) = \begin{cases} (s_1, s_2) & \text{if } s_1 + s_2 < d(p_1), \\ (s_1, d(p_1) - s_1) & \text{if } 2 \min\{s_1, s_2\} < d(p_1) \leq s_1 + s_2 \text{ and } s_1 < s_2, \\ (d(p_1) - s_2, s_2) & \text{if } 2 \min\{s_1, s_2\} < d(p_1) \leq s_1 + s_2 \text{ and } s_1 > s_2, \\ (d(p_1)/2, d(p_1)/2) & \text{if } d(p_1) \leq 2 \min\{s_1, s_2\}. \end{cases}$$

Proof. Let $x \in (P \times S)^2$ be such that $p_1 = p_2$ and let p denote this common value. The definition of $\Phi(x)$ implies that $\phi(x) = (s_1, s_2)$ if $s_1 + s_2 < d(p)$. Moreover, the

definition of $\Phi(x)$ also implies that $\phi_1 + \phi_2 = d(p)$ when $s_1 + s_2 \geq d(p)$. Hence, in this case, $\phi(x)$ is the solution to

$$\begin{aligned} & \max_{\phi_1, \phi_2} p(\phi_1 + \phi_2) - c(\phi_1) - c(\phi_2) \\ & \text{subject to } \phi_1 + \phi_2 = d(p), 0 \leq \phi_1 \leq s_1, 0 \leq \phi_2 \leq s_2. \end{aligned}$$

This problem can be rewritten as

$$\begin{aligned} & \max_{\phi_1} p d(p) - c(\phi_1) - c(d(p) - \phi_1) \\ & \text{subject to } \max\{0, d(p) - s_2\} \leq \phi_1 \leq \min\{s_1, d(p)\}. \end{aligned}$$

The unconstrained solution is ϕ_1 such that $c'(\phi_1) = c'(d(p) - \phi_1)$, i.e. $\phi_1 = d(p)/2$. Thus, $\phi(x) = (d(p)/2, d(p)/2)$ when $\max\{0, d(p) - s_2\} \leq d(p)/2 \leq \min\{s_1, d(p)\}$, i.e. when $d(p) \leq 2 \min\{s_1, s_2\}$.

Finally, consider the case $2 \min\{s_1, s_2\} < d(p) \leq s_1 + s_2$. Clearly, $s_1 \neq s_2$; for concreteness, assume that $s_1 < s_2$. Then $\min\{s_1, d(p)\} = s_1$. If $d(p) = s_1 + s_2$ or $s_1 = 0$, then $\max\{0, d(p) - s_2\} = s_1$ and, hence, $\phi(x) = (s_1, d(p) - s_1)$. Finally, if $d(p) < s_1 + s_2$ and $s_1 > 0$, there is $\max\{0, d(p) - s_2\} < \phi_1 < s_1$ and, at any such ϕ_1 , the derivative of the objective function is $-c'(\phi_1) + c'(d(p) - \phi_1) > 0$ since $\phi_1 < s_1 < d(p)/2$. This shows that $\phi(x) = (s_1, d(p) - s_1)$. $\square$

While in the above we allowed firms to choose any capacity in S , some choices are easily seen to be redundant. Indeed, note that for each $i = 1, 2$, $x = (p_i, s_i, p_j, s_j)$ and $\phi \in \Phi(x)$, we have that $\phi_i \leq d(p_i)$. Furthermore, if $s_i > d(p_i)$ and $x' = (p_i, d(p_i), p_j, s_j)$, then $\phi_i(x) = \phi_i(x')$ (this can easily be seen using the definition of Φ and Lemma 1). Hence, we may assume that $s_i \leq d(p_i)$ for each $i = 1, 2$ and $p_i \in P$. In particular, $s_i = 0$ whenever $p_i = \bar{p}$. We then let $X_i = \{(p_i, s_i) \in P \times S : s_i \in [0, d(p_i)]\}$ for each $i \in N$.

Let $G_B = (X_1, X_2, u_1, u_2)$ be the game just defined. The following is our main result of this section.

Theorem 2. *The game G_B is admissibly secure and, thus, $L(G_B) \neq \emptyset$.*

Proceeding to the proof of Theorem 2, we start by characterizing the set C_i for each $i \in \{1, 2\}$. Define, for each $p \in P$, $\phi^*(p)$ to be the unique solution to

$$\max_{0 \leq \phi \leq d(p)} (p\phi - c(\phi)).$$

Note that the uniqueness of the solution follows because c is strictly convex. Define $m : P \rightarrow \mathbb{R}_+$ by setting

$$m(p) = p\phi^*(p) - c(\phi^*(p))$$

for each $p \in P$. It follows by the maximum theorem (e.g. Aliprantis and Border (2006, Theorem 17.31, p. 570)) that m and ϕ^* are continuous. Further properties of m and ϕ^* are described in the following lemma.

Lemma 2. *The following holds:*

1. *There exists a unique $\hat{p} \in P$ such that $c'(d(\hat{p})) = \hat{p}$. Moreover, $c'(0) < \hat{p} < \bar{p}$ and $p - c'(d(p)) > 0$ for each $p > \hat{p}$.*

2. *For each $p \in P$,*

$$\phi^*(p) = \begin{cases} 0 & \text{if } p \leq c'(0), \\ (c')^{-1}(p) & \text{if } c'(0) < p < \hat{p}, \\ d(p) & \text{if } p \geq \hat{p}. \end{cases}$$

3. *The function m has a unique maximizer $p^* \in (\hat{p}, \bar{p})$. Furthermore, m is strictly increasing in $[c'(0), p^*]$ and strictly decreasing in $[p^*, \bar{p}]$.*

For example, if $c(q) = q^2$ for each $q \in \mathbb{R}_+$ and $d(p) = 1 - p$ for each $p \in P$, then $\hat{p} = 2/3$,

$$\phi^*(p) = \begin{cases} p/2 & \text{if } p < 2/3, \\ 1 - p & \text{if } p \geq 2/3, \end{cases}$$

$$m(p) = \begin{cases} p^2/4 & \text{if } p < 2/3, \\ 3p - 1 - 2p^2 & \text{if } p \geq 2/3, \end{cases}$$

and $p^* = 3/4$.

Proof of Lemma 2. We start with the proof of part 1. Define, for each $p \in P$, $f(p) = p - c'(d(p))$. We have that $f'(p) = 1 - c''(d(p))d'(p) \geq 1$ for each $p \in P$; hence f is strictly increasing. Moreover, f is continuous, $f(0) = -c'(d(0)) < 0$ and $f(\bar{p}) = \bar{p} - c'(0) > 0$. Thus, there exists $\hat{p} \in (0, \bar{p})$ such that $f(\hat{p}) = 0$. Since f is strictly increasing, $\hat{p}$ is unique and $p - c'(d(p)) > 0$ for each $p > \hat{p}$. Finally, $\hat{p} < \bar{p}$ implies that $d(\hat{p}) > 0$ and, therefore, $\hat{p} = c'(d(\hat{p})) > c'(0)$.

We next turn to the proof of part 2 of the lemma. Let, for each $p \in P$ and $\phi \in \mathbb{R}_+$, $f(\phi; p) = p\phi - c(\phi)$. If $p \leq c'(0)$, then $f'(\phi; p) = p - c'(\phi) < p - c'(0) \leq 0$ for each $\phi > 0$; hence, $\phi^*(p) = 0$.

If $p \geq \hat{p}$, then $f'(\phi; p) = p - c'(\phi) > p - c'(d(p)) \geq 0$ for each $\phi < d(p)$, where the last inequality follows by part 1; hence, $\phi^*(p) = d(p)$.

Finally, when $p \in (c'(0), \hat{p})$, the first order condition gives $p = c'(\phi^*(p))$.

We now establish part 3 of the lemma. Note first that $m(p) = 0$ for each $p \leq c'(0)$ by part 2. Next, note that m is strictly increasing in $[c'(0), \hat{p}]$. Indeed, for each $p \in (c'(0), \hat{p})$, m is differentiable at p and, using part 2,

$$m'(p) = \phi^*(p) + \frac{p - c'(\phi^*(p))}{c''(\phi^*(p))} = \phi^*(p) > 0.$$

For each $p \in (\hat{p}, \bar{p})$, $\phi^*(p) = d(p)$ by part 2. Hence,

$$\begin{aligned} m'(p) &= d(p) + [p - c'(d(p))]d'(p), \text{ and} \\ m''(p) &= d'(p)[2 - c''(d(p))d'(p)] + d''(p)[p - c'(d(p))]. \end{aligned}$$

Hence, using part 1, it follows that $\lim_{p \rightarrow \hat{p}} m'(p) = d(\hat{p}) + [\hat{p} - c'(d(\hat{p}))]d'(\hat{p}) = d(\hat{p}) > 0$, $\lim_{p \rightarrow \bar{p}} m'(p) = [\bar{p} - c'(d(\bar{p}))]d'(\bar{p}) < 0$ and $m''(p) < 0$ for each $p \in (\hat{p}, \bar{p})$. Thus, there exists a unique $p^* \in (\hat{p}, \bar{p})$ such that $m'(p^*) = 0$, $m'(p) > 0$ for each $p \in (\hat{p}, p^*)$ and $m'(p) < 0$ for each $p \in (p^*, \bar{p})$. The conclusion follows. $\square$

We use the above lemma to characterize the interior of the set of weakly dominated strategies.

Lemma 3. *For each $i = 1, 2$,*

$$C_i = \{(p_i, s_i) \in X_i : p_i \in [c'(0), p^*], s_i = \phi^*(p_i)\}.$$

Lemma 3 is obtained by combining Lemmas 4–8 below.

Lemma 4. *If $i \in \{1, 2\}$, $x = (p_1, s_1, p_2, s_2) \in (P \times S)^2$ and $x' = (p'_1, s'_1, p'_2, s'_2) \in (P \times S)^2$ are such that $p'_1 = p_1$, $p'_2 = p_2$, $s'_j = s_j$ and $s'_i > s_i$, then either $\phi_i(x) = \phi_i(x')$ or $\phi_i(x) = s_i < \phi_i(x')$.*

Proof. We have that:

$$\begin{aligned}\phi_i(x) &= \min\{d(p_i), s_i\} \leq \min\{d(p_i), s'_i\} = \phi_i(x') \text{ if } p_i < p_j, \\ \phi_i(x) &= \min\left\{\frac{(d(p_j) - s_j)^+}{d(p_j)}d(p_i), s_i\right\} \leq \min\left\{\frac{(d(p_j) - s_j)^+}{d(p_j)}d(p_i), s'_i\right\} = \phi_i(x') \text{ if } p_i > p_j, \\ \phi_i(x) &= s_i < s'_i = \phi_i(x') \text{ if } p_i = p_j \text{ and } d(p_1) \geq s'_i + s_j.\end{aligned}$$

In fact, in the above three cases, we have that $\phi_i(x) = s_i$ whenever $\phi_i(x) < \phi_i(x')$.

We next consider the two remaining cases. If $p_i = p_j$ and $s_1 + s_2 \leq d(p_1) < s'_i + s_j$, then $\phi_i(x) = s_i$ and $\phi_i(x') = d(p_1) - \phi_j(x') \geq d(p_1) - s_j \geq s_i = \phi_i(x)$.

Suppose next that $d(p_1) < s_1 + s_2$. If $d(p_1) \leq 2 \min\{s_1, s_2\}$, then $\phi_i(x) = \phi_i(x') = d(p_1)/2$. If $2 \min\{s_1, s_2\} < d(p_1) \leq 2 \min\{s'_i, s_j\}$, it cannot be that $s_j = \min\{s_1, s_2\}$. Hence, $s_i < s_j$ and, thus, $s_i < d(p_1)/2$. As $\phi_i(x) = s_i$ and $\phi_i(x') = d(p_1)/2$, the conclusion of the lemma follows. Finally, if $2 \min\{s'_i, s_j\} < d(p_1)$, then $s_1 \neq s_2$, $s'_i \neq s_j$ and it cannot be that $s_i < s_j < s'_i$ (otherwise, $s_j = \min\{s'_i, s_j\} < d(p_1)/2$ and $s_1 + s_2 < 2s_j < d(p_1)$ contradicting $d(p_1) < s_1 + s_2$). But then $\phi_i(x) = \phi_i(x')$. This completes the proof of the lemma. $\square$

We note that the set C_i is independent of the sharing rule provided that the conclusion of the previous lemma holds, i.e. from this point onwards we shall only use Lemma 4 and not the explicit characterization of ϕ given in Lemma 1 above.

The next result shows that a capacity choice different from $\phi^*(p_i)$ when choosing a price level p_i is weakly dominated for i .

Lemma 5. *For each $i \in \{1, 2\}$ and $(p_i, s_i) \in X_i$, if $s_i \neq \phi^*(p_i)$, then (p_i, s_i) is weakly dominated for i by $(p_i, \phi^*(p_i))$.*

Proof. Let $i \in \{1, 2\}$, $(p_i, s_i) \in X_i$ and $s_i \neq \phi^*(p_i)$. Let $j \neq i$, $(p_j, s_j) \in X_j$, $x = (p_i, \phi^*(p_i), p_j, s_j)$ and $x' = (p_i, s_i, p_j, s_j)$. We clearly have $u_i(x) = u_i(x')$ if $\phi_i(x) =$

$\phi_i(x')$. Assuming $\phi_i(x) \neq \phi_i(x')$, Lemma 4 implies that $\phi_i(x) = \phi^*(p_i) < \phi_i(x')$ if $s_i > \phi^*(p_i)$ and hence $u_i(x) > u_i(x')$ by the definition of $\phi^*(p_i)$; furthermore, in the case $s_i < \phi^*(p_i)$, we have that $\phi_i(x) = s_i < \phi_i(x') \leq \phi^*(p_i)$ and thus $u_i(x) > u_i(x')$ by the definition of $\phi^*(p_i)$ and the strict concavity of $\phi \mapsto p_i\phi - c(\phi)$.

Recall that $s_i \leq d(p_i)$ and $d(\bar{p}) = 0$; thus, $s_i \neq \phi^*(p_i)$ implies that $p_i < \bar{p}$. Letting $p_j = \bar{p}$, it then follows that $p_i < p_j$ and, hence, $\phi_i(x) = \min\{d(p_i), \phi^*(p_i)\} = \phi^*(p_i)$ and $\phi_i(x') = \min\{d(p_i), s_i\} = s_i$ (recall also that $\phi^*(p_i) \leq d(p_i)$). Thus, $u_i(x) > u_i(x')$ by the definition of $\phi^*(p_i)$. This completes the proof. $\square$

It follows from Lemma 5 that $C_i \subseteq \{(p, \phi^*(p)) : p \in P\}$ (recall that ϕ^* is continuous). The following lemma shows that $C_i \subseteq \{(p_i, s_i) \in X_i : p_i \geq c'(0)\}$.

Lemma 6. *For each $i \in \{1, 2\}$, if $p_i < c'(0)$, then $(p_i, \phi^*(p_i))$ is weakly dominated for i by $(\hat{p}, \phi^*(\hat{p}))$.*

Proof. Let $i = 1$ without loss of generality and $p_1 < c'(0)$. Since $\phi^*(p_1) = 0$, it follows that $u_1(p_1, \phi^*(p_1), p_2, s_2) = 0$ for each $(p_2, s_2) \in X_2$. We have that $u_1(\hat{p}, \phi^*(\hat{p}), p_2, s_2) \geq 0$ for each $(p_2, s_2) \in X_2$ and, when $p_2 > \hat{p}$, $u_1(\hat{p}, \phi^*(\hat{p}), p_2, s_2) = m(\hat{p}) > 0$. $\square$

The following lemma restrict C_i further by showing that $C_i \subseteq \{(p_i, s_i) \in X_i : p_i \leq p^*\}$.

Lemma 7. *For each $i \in \{1, 2\}$, if $p_i > p^*$, then $(p_i, \phi^*(p_i))$ is weakly dominated for i by $(p^*, \phi^*(p^*))$.*

Proof. Let $i = 1$ without loss of generality and $p_1 > p^*$. Lemma 2.2 implies that $\phi^*(p^*) = d(p^*)$ and $\phi^*(p_1) = d(p_1)$. Fix $(p_2, s_2) \in X_2$ and let $x^* = (p^*, d(p^*), p_2, s_2)$, $x' = (p_1, d(p_1), p_2, s_2)$ and $\gamma = u_1(x^*) - u_1(x')$.

We consider three cases. The first case is when $p_2 < p^*$. In this case, we have that $\phi_1(x^*) = \alpha d(p^*)$ and $\phi_1(x') = \alpha d(p_1)$, where $\alpha = (d(p_2) - s_2)/d(p_2)$ (recall that $s_2 \leq d(p_2)$ and, hence, $(d(p_2) - s_2)^+ = d(p_2) - s_2$). Hence,

$$\gamma = \alpha p^* d(p^*) - c(\alpha d(p^*)) - [\alpha p_1 d(p_1) - c(\alpha d(p^*))].$$

Letting $m_\alpha(p) = \alpha p d(p) - c(\alpha d(p))$ for each $p \in [p^*, \bar{p}]$, we have that p^* is the unique maximizer of m_α in $[p^*, \bar{p}]$. Indeed, for each $p \in (\hat{p}, \bar{p})$,

$$\begin{aligned} m'_\alpha(p) &= \alpha(d(p) + [p - c'(\alpha d(p))]d'(p)), \\ m''_\alpha(p) &= \alpha(d'(p)[2 - \alpha c''(\alpha d(p))d'(p)] + d''(p)[p - c'(\alpha d(p))]) \end{aligned}$$

and, recall,

$$m'(p) = d(p) + [p - c'(d(p))]d'(p).$$

Since $-d'(p^*)c'(\alpha d(p^*)) < -d'(p^*)c'(d(p^*))$, we have that $m'_\alpha(p^*) < \alpha m'(p^*) = 0$. Moreover, for each $p > \hat{p}$, $p > c'(d(p)) > c'(\alpha d(p))$ implies that $m''_\alpha(p) < 0$. Thus, m_α is decreasing in $[p^*, \bar{p}]$ and, thus, p^* is the unique maximizer of m_α in $[p^*, \bar{p}]$. It then follows that $\gamma > 0$.

The second case is when $p_2 = p^*$. Then $\phi_1(x') = \alpha d(p_1)$ where $\alpha = (d(p_2) - s_2)/d(p_2)$ and $\alpha d(p^*) \leq \phi_1(x^*) \leq d(p^*)$. Indeed, if $\phi_1(x^*) < d(p^*)$, then $\phi_1(x^*) + \phi_2(x^*) = d(p^*)$. Hence, using $p_2 = p^*$ and $\phi_2(x^*) \leq s_2$, $\phi_1(x^*) \geq d(p_2) - s_2 = \frac{d(p_2) - s_2}{d(p_2)} d(p^*)$. Thus, $\gamma \geq m_\alpha(p^*) - m_\alpha(p_1) > 0$.

The final case is $p_2 > p^*$. In this case, $\gamma = m(p^*) - u_1(p_1, d(p_1), p_2, s_2) \geq m(p^*) - m(p_1) > 0$. This completes the proof. $\square$

Lemma 8. *For each $i \in \{1, 2\}$ and $p_i \in (c'(0), p^*)$, $(p_i, \phi^*(p_i))$ is not weakly dominated for i .*

Proof. Let $i = 1$ without loss of generality and $p_1 \in (c'(0), p^*)$. Suppose that there exists $\mu_1 \in M_1$ that weakly dominates $(p_1, \phi^*(p_1))$. Consider a sequence $\{p_{2,k}\}_{k=1}^\infty$ such that $p_{2,k} \downarrow p_1$ and $\mu_1(\{p_{2,k}\}) = 0$ for each $k \in \mathbb{N}$; such sequence exists because the set of atoms of μ_1 is countable. Let $s_{2,k} = d(p_{2,k})$ for each $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, $u_1(p_1, \phi^*(p_1), p_{2,k}, s_{2,k}) = m(p_1)$, $u_1(\mu_1, p_{2,k}, s_{2,k}) \leq \int_{[0, p_{2,k})} m(p) d\mu_1(p)$ (equality holds if $\mu_1(\{(p, s) : s = \phi^*(p)\}) = 1$) and

$$\int_{[0, p_{2,k})} m(p) d\mu_1(p) \geq u_1(\mu_1, p_{2,k}, s_{2,k}) \geq u_1(p_1, \phi^*(p_1), p_{2,k}, s_{2,k}) = m(p_1).$$

Letting $f = m1_{[0, p_1]}$ and, for each $k \in \mathbb{N}$, $f_k = m1_{[0, p_{2,k})}$, we have that $f_k \downarrow f$. Hence, by the monotone convergence theorem,

$$\int_{[0, p_1]} m(p) d\mu_1(p) = \lim_k \int_{[0, p_{2,k})} m(p) d\mu_1(p) \geq m(p_1).$$

By Lemma 2.3, we have that $m(p) < m(p_1)$ for each $p < p_1$. This, together with $\int_{[0, p_1]} m(p) d\mu_1(p) \geq m(p_1)$ then implies that $\mu_1(\{p_1\} \times [0, d(p_1)]) = 1$. But then μ_1 does not weakly dominate $(p_1, \phi^*(p_1))$. Indeed, this is clear if $\mu_1 = \delta_{(p_1, \phi^*(p_1))}$ since then $u_1(p_1, \phi^*(p_1), p_2, s_2) = u_1(\mu_1, p_2, s_2)$ for each $(p_2, s_2) \in X_2$; if $\mu_1 \neq \delta_{(p_1, \phi^*(p_1))}$, then $u_1(p_1, \phi^*(p_1), p_2, s_2) \geq u_1(\mu_1, p_2, s_2)$ for each $(p_2, s_2) \in X_2$ by Lemma 5. $\square$

Combining Lemmas 5–8, it follows that

$$C_i = \{(p_i, s_i) \in X_i : p_i \in [c'(0), p^*], s_i = \phi^*(p_i)\}.$$

We turn now to showing that G_B is admissibly secure, which we will do by establishing the conditions in Remark 4.

Lemma 9. *The game G_B is admissibly reciprocal upper semicontinuous.*

Proof. It follows by Aliprantis and Border (2006, Lemma 17.30, p. 569) that $u_1 + u_2$ is upper semicontinuous. Thus, the conclusion follows by Carmona (2013, Theorem 3.20, p. 39). $\square$

Lemma 10. *The game G_B is admissibly payoff secure.*

Proof. We first note that it suffices to show that the following condition (*) holds: For each $i \in \{1, 2\}$, $\varepsilon > 0$ and $\sigma \in M_i \times M_j^0$, there exists $\beta \in [c'(0), p^*]$ such that $\sigma_j(\{(\beta, \phi^*(\beta))\}) = 0$ and $u_i(\beta, \phi^*(\beta), \sigma_j) > u_i(\sigma) - \varepsilon$. Indeed, suppose that condition (*) holds and that G fails to be admissibly payoff secure. Then there is $i \in \{1, 2\}$, $\varepsilon > 0$ and $\sigma \in M_i \times M_j^0$ such that, for each $\mu_i \in M_i^0$ and open neighborhood U of σ , $u_i(\mu_i, \sigma'_j) \leq u_i(\sigma) - \varepsilon$ for some $\sigma' \in U \cap M^0$. By condition (*), let $\beta \in [c'(0), p^*]$ such that $\sigma_j(\{(\beta, \phi^*(\beta))\}) = 0$ and $u_i(\beta, \phi^*(\beta), \sigma_j) > u_i(\sigma) - \varepsilon$. As $\delta_{(\beta, \phi^*(\beta))} \in M_i^0$, for each $k \in \mathbb{N}$, pick $\sigma_k \in M^0$ such that $u_i(\beta, \phi^*(\beta), \sigma_{k,j}) \leq u_i(\sigma) - \varepsilon$ and $\sigma_{k,j} \rightarrow \sigma_j$. We have that $\tau_{\delta_{(\beta, \phi^*(\beta))}, \sigma_j}(\{x \in X : u_i \text{ is discontinuous at } x\}) = \sigma_j(\{(\beta, \phi^*(\beta))\}) = 0$, and hence $u_i(\beta, \phi^*(\beta), \sigma_{k,j}) \rightarrow u_i(\beta, \phi^*(\beta), \sigma_j)$. Thus, we get $u_i(\sigma) - \varepsilon < u_i(\beta, \phi^*(\beta), \sigma_j) \leq u_i(\sigma) - \varepsilon$, a contradiction. This contradiction shows that condition (*) is sufficient for G to be admissibly secure.

In what follows we verify that condition $(*)$ hold. Let $i = 1$ without loss of generality, $\varepsilon > 0$ and $\sigma \in M_1 \times M_2^0$. Let $A_2 = \{p \in P : \sigma_2(\{(p, \phi^*(p))\}) > 0\}$ be the set of atoms of σ_2 and note that A_2 is countable.

If $u_1(\sigma) \leq 0$, pick $\beta \in [c'(0), p^*] \setminus A_2$ and note that $u_1(\beta, \phi^*(\beta), \sigma_2) \geq 0$. Thus, we may assume that $u_1(\sigma) > 0$.

There exists $\alpha \in P$ such that $u_1(\alpha, \phi^*(\alpha), \sigma_2) \geq u_1(\sigma)$ since σ_1 is a probability measure and using Lemma 5. As $\phi^*(\alpha) = 0$ if $\alpha \leq c'(0)$ by Lemma 2.2, it follows that $\alpha > c'(0)$. Moreover, we may take $\alpha \leq p^*$. Indeed, if $\alpha > p^*$, it follows by Lemma 7 that α can be replaced with p^* .

Let $h : [0, \alpha]^2 \rightarrow [0, d(0)]$ be defined by setting, for each $p_1, p_2 \in [0, \alpha]$,

$$h(p_1, p_2) = \min \left\{ \frac{d(p_2) - \phi^*(p_2)}{d(p_2)} d(p_1), \phi^*(p_1) \right\}.$$

We have that

$$(p_1, p_2) \mapsto p_1 h(p_1, p_2) - c(h(p_1, p_2)) : [0, \alpha]^2 \rightarrow \mathbb{R}$$

and m are uniformly continuous. Hence, let $\delta > 0$ be such that $|p - \alpha| < \delta$ implies that

$$|[ph(p, p_2) - c(h(p, p_2))] - [\alpha h(\alpha, p_2) - c(h(\alpha, p_2))]| < \varepsilon \text{ for all } p_2 \in [0, \alpha] \quad (2)$$

and

$$|m(p) - m(\alpha)| < \varepsilon. \quad (3)$$

We may take δ to be such that $\alpha - \delta > c'(0)$.

Choose $\beta \in (\alpha - \delta, \alpha)$ such that $\beta \notin A_2$. Hence, $\beta \in [c'(0), p^*] \setminus A_2$. Let $\gamma(p_2) = u_1(\beta, \phi^*(\beta), p_2, \phi^*(p_2)) - u_1(\alpha, \phi^*(\alpha), p_2, \phi^*(p_2))$ for each $p_2 \in [c'(0), p^*]$. We consider four cases.

Case 1: $p_2 < \beta$. We then have that $|\gamma(p_2)| = |[\beta h(\beta, p_2) - c(h(\beta, p_2))] - [\alpha h(\alpha, p_2) - c(h(\alpha, p_2))]| < \varepsilon$ by (2).

Case 2: $\beta < p_2 < \alpha$. Then $u_1(\alpha, \phi^*(\alpha), p_2, \phi^*(p_2)) = \alpha h(\alpha, p_2) - c(h(\alpha, p_2)) \leq \alpha \phi^*(\alpha) - c(\phi^*(\alpha)) < \beta \phi^*(\beta) - c(\phi^*(\beta)) + \varepsilon = u_1(\beta, \phi^*(\beta), p_2, \phi^*(p_2)) + \varepsilon$ by (3).

Case 3: $p_2 = \alpha$. Let $\phi_i \leq \phi^*(\alpha) \leq d(\alpha)$ be such that $u_1(\alpha, \phi^*(\alpha), \alpha, \phi^*(\alpha)) = \alpha \phi_i - c(\phi_i)$. Then, exactly as in Case 2 above, we obtain that $\gamma(p_2) > -\varepsilon$.

Case 4: $p_2 > \alpha$. We then have that $|\gamma(p_2)| = |m(\beta) - m(\alpha)| < \varepsilon$ by (3).

It then follows that $u_1(\beta, \phi^*(\beta), \sigma_2) > u_1(\alpha, \phi^*(\alpha), \sigma_2) - \varepsilon$ since $\sigma_2 \in M_2^0$ and $(\beta, \phi^*(\beta)) \notin A_2$. As $u_1(\alpha, \phi^*(\alpha), \sigma_2) \geq u_1(\sigma)$, it follows that $u_1(\beta, \phi^*(\beta), \sigma_2) > u_1(\sigma) - \varepsilon$, establishing condition (*). This completes the proof of the lemma. $\square$

Theorem 2 now follows by combining Lemmas 9 and 10, Remark 4 and Theorem 1.

Remark 11. Using a result in Carmona and Podczeck (2018), we now show that the set of limit admissible equilibria is independent of the choice of the allocation ϕ that determines players' payoff functions in (1), in the following sense. Let $\tilde{\phi}$ be a measurable selection of Φ such that the conclusion of Lemma 4 holds. Furthermore, let $u_i^{\tilde{\phi}}$ be defined by replacing ϕ with $\tilde{\phi}$ in (1) for each $i = 1, 2$ and let $G_B^{\tilde{\phi}} = (X_1, X_2, u_1^{\tilde{\phi}}, u_2^{\tilde{\phi}})$. Then the closure of the set of undominated strategies of player i in $G_B^{\tilde{\phi}}$ is still C_i for each $i = 1, 2$; this holds because the specific definition of ϕ was only used to establish Lemma 4. More importantly, $L(G_B^{\tilde{\phi}}) = L(G_B)$.

To see the above, let $\tilde{X}_i = \{(p_i, s_i) \in X_i : s_i = \phi^*(p_i)\}$ for each $i = 1, 2$, $\tilde{G}_B = (\tilde{X}_1, \tilde{X}_2, u_1, u_2)$ and $\tilde{G}_B^{\tilde{\phi}} = (\tilde{X}_1, \tilde{X}_2, u_1^{\tilde{\phi}}, u_2^{\tilde{\phi}})$. Note that $\tilde{X}_i \subseteq X_i$ and, hence, $\tilde{M}_i \subseteq M_i$ for each $i = 1, 2$ and $\tilde{M} \subseteq M$. Carmona and Podczeck (2018, Section 5.1) have shown that $E(\tilde{G}_B) = E(\tilde{G}_B^{\tilde{\phi}})$. For each $G \in \{G_B, G_B^{\tilde{\phi}}\}$, we clearly have that $E(G) \cap \tilde{M} \subseteq E(\tilde{G})$; furthermore, we have that $E(\tilde{G}) \subseteq E(G) \cap \tilde{M}$ due to Lemma 5.⁶ As $M^0 \subseteq \tilde{M}$, we then obtain that $L(G_B^{\tilde{\phi}}) = E(G_B^{\tilde{\phi}}) \cap M^0 = E(\tilde{G}_B^{\tilde{\phi}}) \cap M^0 = E(\tilde{G}_B) \cap M^0 = E(G_B) \cap M^0 = L(G_B)$.

⁶To see this, let $\sigma \in E(\tilde{G})$, $i \in \{1, 2\}$ and $\sigma'_i \in M_i$. We clearly have that $\sigma \in \tilde{M}$. Define $f_i : X_i \rightarrow \tilde{X}_i$ by setting, for each $(p_i, s_i) \in X_i$, $f_i(p_i, s_i) = (p_i, \phi^*(p_i))$. Then, for each $(p_i, s_i, p_j, x_j) \in X$, $u_i(p_i, s_i, p_j, x_j) \leq u_i(f_i(p_i, s_i), p_j, x_j)$ by Lemma 5. Hence, $u_i(\sigma'_i, \sigma_j) \leq \int_{X_i} u_i(f_i(p_i, s_i), \sigma_j) d\sigma'_i(p_i, s_i) = \int_{\tilde{X}_i} u_i(p_i, s_i, \sigma_j) d\sigma'_i \circ f_i^{-1}(p_i, s_i) = u_i(\sigma'_i \circ f_i^{-1}, \sigma_j)$. As $\sigma'_i \circ f_i^{-1} \in \tilde{M}_i$ and $\sigma \in E(\tilde{G})$, we have that $u_i(\sigma'_i \circ f_i^{-1}, \sigma_j) \leq u_i(\sigma)$. Thus, $u_i(\sigma'_i, \sigma_j) \leq u_i(\sigma)$ and, therefore, $\sigma \in E(G)$.

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