Properties of one-nucleon overlap functions for $A \geq 16$ double-closed-shell nuclei in the source-term approach

N. K. Timofeyuk

Department of Physics, University of Surrey, Guildford, Surrey GU2 7XH, United Kingdom

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One-nucleon overlap functions required for various nucleon-removal calculations must be solutions of the inhomogeneous equation with a source term whose shape and strength are determined by nuclear wave functions and the effective interaction of the removed nucleon with the nucleons in the remainder nucleus. A number of previous works has reported calculations of the source term and the overlap functions for a range of $A \leq 16$ nuclei. It was shown that the source-term approach gives reasonable results for spectroscopic factors and asymptotic normalization coefficients and, in particular, it explains the phenomenon of reduction of spectroscopic factors observed in knockout reactions. In the present article, the source-term approach has been extended to $A \geq 16$ nuclei. To achieve this, a new technique has been developed which excludes the spurious center-of-mass motion within the translation-invariant oscillator shell model. Applications using this technique are performed for double-closed-shell nuclei, both stable and away from β stability, such as $^{16,24}\text{O}$, $^{40,48,60}\text{Ca}$, $^{56,78}\text{Ni}$, $^{100,132}\text{Sn}$, and ^{208}Pb . The spectroscopic factors and asymptotic normalization coefficients for one-nucleon removal and addition have been calculated. They are compared to experimental values and results from other microscopic models when available.

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I. INTRODUCTION

Overlap functions are important quantities lying at the interface between nuclear structure and nucleon-removal reactions aimed to study this structure. Their calculation appears to be deceptively simple at first and, indeed, they are easily obtained in approaches employing uncorrelated single-particle wave functions. However, it is very difficult to reproduce the most important property of the overlap function—its asymptotic behavior—in these approaches, and, in fact, in any microscopic approaches, unless this behavior is introduced by hand using, for example, R -matrix ideas. The correct asymptotic behavior is crucial for predicting the cross sections of transfer reactions and the direct proton capture reactions in stars. Over the past decade, another problem arising in calculations of overlap functions within microscopic mean-field type approaches has attracted the attention of the nuclear reactions community. It has been noticed that the total normalization of the overlap integral, the spectroscopic factor, obtained from analysis of the cross sections of nucleon knockout reactions, is often smaller than that suggested by theoretical calculations. The spectroscopic factors are overestimated by about 40–50% in double-closed-shell nuclei like ^{16}O , $^{40,48}\text{Ca}$, and ^{208}Pb [1] but this overestimation can be even higher for removal of deeply bound nucleons from nuclei away from β stability [2]. This phenomenon challenges, in particular, the concept of filled nucleon orbitals in closed-shell nuclei and, in general, the theory of overlap integral calculations. It has been shown in Refs. [3,4] that a possible way to counter these challenges is to use the source-term approach (STA).

The STA is based on a fundamental requirement that one nucleon overlap function $I(\mathbf{r}) = \langle \Psi_{A-1} | \Psi_A \rangle$ should be a solution of the inhomogeneous equation

$$(T_{\text{rel}} + V_C^0 - \varepsilon)I(\mathbf{r}) = U(\mathbf{r}), \quad (1)$$

where $T_{\text{rel}} = \hat{T}_A - \hat{T}_{A-1}$, $\varepsilon = E_A - E_{A-1}$ is the separation energy, and $\hat{T}_i$ and E_i are the kinetic energy operator and the total energy of nucleus i , while V_C^0 is the Coulomb interaction of the removed nucleon with the point charge Z_{A-1} . The right-hand side of Eq. (1) is the source term $U(\mathbf{r})$,

$$U(\mathbf{r}) = \langle \Psi_{A-1} | V_{A-1} - V_A + V_C^0 | \Psi_A \rangle, \quad (2)$$

with V_i being the potential energy operator for nucleus i . The separation energy ε is normally known from experiment. The only quantity then needed to calculate the overlap function is $U(\mathbf{r})$. The radial part $I_{lj}(r)$ of the solution $I(\mathbf{r})$ of Eq. (1) will automatically have the correct asymptotic behavior [5]

$$I_{lj}(r) \approx C_{lj} \frac{W_{-\eta, l+1/2}(2\kappa r)}{r}, \quad r \rightarrow \infty, \quad (3)$$

where $\eta = Z_{A-1}Z_N e^2 \mu / \hbar^2 \kappa$; Z_{A-1} and Z_N is the charge of $A-1$ and the removed nucleon N , respectively; $\kappa = (2\mu\varepsilon/\hbar^2)^{1/2}$; μ is the reduced mass; W is the Whittaker function; and C_{lj} is the asymptotic normalization coefficient (ANC). The asymptotic behavior will be correct even if the wave functions Ψ_A and Ψ_{A-1} used to model $U(\mathbf{r})$ do not have the correct asymptotic behavior. $I_{lj}(r)$ can then be used to obtain the spectroscopic factor:

$$S_{lj} = \int_0^\infty dr r^2 I_{lj}^2(r). \quad (4)$$

The simplest way to calculate the source term $U(\mathbf{r})$ is to use traditional oscillator shell model for the wave functions Ψ_A and Ψ_{A-1} and some effective interaction $V_{A-1} - V_A = \sum_{i=1}^A V_{iA}$. It was found in Refs. [3–7] that phenomenologically choosing for V_{iA} one of the M3Y interactions [8], namely the one that reproduced the oscillator matrix elements derived from the NN -scattering data in Ref. [9], gives reasonable results for ANCs and spectroscopic factors in $A \leq 16$ nuclei. However,

most of interest in modern experiments is concentrated in the area of heavier nuclei where the STA has not yet been applied.

In Refs. [3–7] the source term $U(\mathbf{r})$ has been calculated using a supermultiplet scheme and translation-invariant fractional parentage expansions derived in Ref. [10]. For $A \leq 16$, the coefficients in these expansions are expressed via geometrical quantities such as $3j$ and $9j$ coefficients for the SU(3) and SU(4) groups. These quantities are not readily available for nuclei above the $0p$ shell. On the other hand, the supermultiplet scheme is not often used for $A \geq 16$. The shell-model computer codes, widespread in the nuclear physics community, employ the M scheme instead. Adapting these codes for source-term calculations requires developing a new technique to calculate matrix elements arising in this approach.

In the present article, a new technique to calculate $U(\mathbf{r})$ has been developed that can be used in the M -scheme calculations with oscillator single-particle wave functions. An important feature of this technique is that it allows translation invariance to be accounted for, which helps to get rid of spurious center-of-mass contamination in low-lying intruder states. The details of this new technique are given in Sec. II with the partial expansion for $U(\mathbf{r})$ presented in Sec. III. The new technique is applied in Sec. IV to double-closed-shell nuclei, lying both on and away from the β stability. The results obtained are summarized in Sec. V. The exact expressions for matrix elements arising in the STA are given in the Appendix.

II. LINKING THE TRANSLATION-INVARIANT SOURCE TERM TO MATRIX ELEMENTS OF THE CONVENTIONAL SHELL MODEL

All reaction theories that employ one-nucleon overlap functions assume that the latter are functions of $\mathbf{r} = \mathbf{r}_N - \mathbf{r}_{A-1}$, where $\mathbf{r}_N$ and $\mathbf{r}_{A-1}$ are the radius vectors of the removed nucleon N and the center-of-mass of $A-1$, respectively, or, in other words, $\mathbf{r}$ is the last Jacobi coordinate in the A -body system. So to get this property correctly, translation-invariant wave functions should be employed in overlap function calculations. This can be straightforward in microscopic models based on harmonic oscillator but is very difficult in other approaches. Although it is expected that the center-of-mass corrections decrease with increasing A , their role may not be negligible in low-lying intruder states, near the island of inversion and, generally, in any states influenced by particle-hole excitations to the shells with higher excitation energies.

In the present paper, the wave functions Ψ_A^{TI} and Ψ_{A-1}^{TI} of two neighboring nuclei A and $A-1$ are assumed to come from the translation-invariant oscillator shell model. They are defined in normalized Jacobi coordinates

$$\xi_i = \sqrt{\frac{i}{i+1}} \left(\frac{1}{i} \sum_{j=1}^i \mathbf{r}_j - \mathbf{r}_{i+1} \right), \quad (5)$$

$$\mathbf{X}_A = \sqrt{A} \mathbf{R}_A = \frac{1}{\sqrt{A}} \sum_{i=1}^A \mathbf{r}_i,$$

for which $\xi_{A-1} \equiv \xi = -\mathbf{r}$. The introduction of normalized Jacobi coordinates conserves the oscillator frequency in

various transformations and, thus, is convenient. The source term $U(\mathbf{r})$ that enters nucleon-removal calculations is related to the $U(\xi_{A-1})$ calculated in normalized Jacobi coordinates as

$$U(\mathbf{r}) = \left(\frac{A-1}{A} \right)^{3/4} \tilde{U}(-\xi_{A-1}), \quad (6)$$

where

$$\begin{aligned} \tilde{U}(\xi_{A-1}) &= \langle \Psi_{A-1}^{TI} | \hat{V} | \Psi_A^{TI} \rangle \\ &= \int d\{\xi\}_{A-2} \Psi_{A-1}^{TI\dagger}(\xi_1, \dots, \xi_{A-2}) \\ &\quad \times \hat{V} \Psi_A^{TI}(\xi_1, \dots, \xi_{A-1}). \end{aligned} \quad (7)$$

The integration is done here over all $A-2$ Jacobi coordinates of nucleus $A-1$.

The translation-invariant oscillator shell-model wave functions can be always represented as

$$\Psi_A^{TI}(\xi_1, \dots, \xi_{A-1}) = \Phi_{000}^{-1}(\mathbf{X}_A) \Phi_A(\mathbf{r}_1, \dots, \mathbf{r}_A), \quad (8)$$

where $\Phi_{000}(\mathbf{X}_A)$ is the wave function of the $0s$ motion of the center of mass,

$$\Phi_{000}(\mathbf{X}_A) = \frac{2}{b^{3/2} \pi^{1/4}} e^{-\frac{X_A^2}{2b^2}} Y_{000}(\hat{\mathbf{X}}_A), \quad (9)$$

b is the oscillator radius, and Φ_A is a linear combination of Slater determinants defined in individual coordinates $\{\mathbf{r}_1, \dots, \mathbf{r}_A\}$. Φ_A can be obtained using the conventional M -scheme shell-model codes in which diagonalization of the matrix of the radius-squared of the center of mass is introduced prior to the diagonalization of the Hamiltonian. In $0\hbar\omega$ spaces such a preliminary diagonalization is not necessary.

Using relation (8) for Ψ_{A-1}^{TI} and Ψ_A^{TI} in Eq. (7) and then multiplying it by $\Phi_{000}(\mathbf{X}_A) \Phi_{000}(\mathbf{X}_{A-1}) e^{i\mathbf{q}\mathbf{r}_A}$ and integrating over $d\mathbf{X}_A d\boldsymbol{\xi}$, we get

$$\begin{aligned} &\int d\mathbf{X}_A d\boldsymbol{\xi} \tilde{U}(\boldsymbol{\xi}) \Phi_{000}(\mathbf{X}_A) \Phi_{000}(\mathbf{X}_{A-1}) e^{i\mathbf{q}\mathbf{r}_A} \\ &= \int d\mathbf{r}_1 \dots d\mathbf{r}_A e^{i\mathbf{q}\mathbf{r}_A} \Phi_{A-1}^\dagger(\mathbf{r}_1, \dots, \mathbf{r}_{A-1}) \hat{V} \Phi_A(\mathbf{r}_1, \dots, \mathbf{r}_A) \\ &\equiv \langle \Phi_{A-1} e^{-i\mathbf{q}\mathbf{r}_A} | \hat{V} | \Phi_A \rangle. \end{aligned} \quad (10)$$

Now the integration over $d\mathbf{X}_A$ in the left-hand side of Eq. (10) can be done using $\mathbf{X}_{A-1} = \sqrt{(A-1)/A} \mathbf{X}_A + \boldsymbol{\xi}/\sqrt{A}$ and $\mathbf{r}_A = \mathbf{X}_A/\sqrt{A} - \sqrt{(A-1)/A} \boldsymbol{\xi}$. This gives

$$\begin{aligned} &\int d\mathbf{X}_A \Phi_{000}(\mathbf{X}_A) \Phi_{000}(\mathbf{X}_{A-1}) e^{i\mathbf{q}\mathbf{X}_A/\sqrt{A}} \\ &= \left(\frac{2A}{2A-1} \right)^{3/2} e^{-\frac{1}{2A-1} \frac{\xi^2}{2b^2}} e^{-\frac{b^2 q^2}{4A-2}} e^{-i\sqrt{\frac{A-1}{A}} \frac{2A}{2A-1} \mathbf{q}\boldsymbol{\xi}}. \end{aligned} \quad (11)$$

Therefore,

$$\begin{aligned} &\langle \Phi_{A-1} e^{-i\mathbf{q}\mathbf{r}_A} | \hat{V} | \Phi_A \rangle \\ &= \left(\frac{2A}{2A-1} \right)^{3/2} e^{-\frac{b^2 q^2}{4A-2}} e^{-\frac{1}{2A-1} \frac{\xi^2}{2b^2}} \int d\boldsymbol{\xi} e^{-i\sqrt{\frac{A-1}{A}} \frac{2A}{2A-1} \mathbf{q}\boldsymbol{\xi}} T(\boldsymbol{\xi}) \\ &= \left(\frac{2A}{2A-1} \right)^{3/2} e^{-\frac{b^2 q^2}{4A-2}} e^{-\frac{1}{2A-1} \frac{\xi^2}{2b^2}} T\left(-\sqrt{\frac{A-1}{A}} \frac{2A}{2A-1} \mathbf{q}\right), \end{aligned} \quad (12)$$

where

$$T(\xi) = e^{-\frac{1}{2A-1}\frac{\xi^2}{2b^2}} \tilde{U}(\xi) \quad (13)$$

and $T(q)$ is the Fourier transform of $T(\xi)$. Therefore,

$$T(q) = \left(\frac{2A-1}{2A}\right)^{3/2} e^{\frac{(2A-1)}{8A(A-1)}q^2b^2} \langle \Phi_{A-1} e^{i\frac{2A-1}{2A}\sqrt{\frac{A}{A-1}}qr_A} | \hat{V} | \Phi_A \rangle. \quad (14)$$

Using the inverse Fourier transform

$$T(\xi) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{-i\mathbf{q}\xi} T(\mathbf{q}), \quad (15)$$

we get the relation between the source term defined in Jacobi coordinates and the model oscillator wave functions defined in individual coordinates:

$$\tilde{U}(\xi) = -\frac{1}{(2\pi)^3} (\alpha+1)^{-3/2} e^{\frac{\alpha\xi^2}{2b^2}} \times \int d\mathbf{q} e^{i\mathbf{q}\xi} e^{\beta q^2 b^2} \langle \Phi_{A-1} e^{-i\gamma \mathbf{q}r_A} | \hat{V} | \Phi_A \rangle. \quad (16)$$

In Eq. (16), the direction of integration over $\mathbf{q}$ has been changed and the coefficients α , β , and γ are defined as

$$\alpha = \frac{1}{2A-1}, \quad \beta = \frac{(2A-1)}{8A(A-1)}, \quad \gamma = \frac{2A-1}{2A} \sqrt{\frac{A}{A-1}}. \quad (17)$$

At $A \rightarrow \infty$, $\alpha \rightarrow 0$, $\beta \rightarrow 0$, $\gamma \rightarrow 1$, and $\xi \rightarrow -\mathbf{r}$ so Eq. (16) reduces to

$$\tilde{U}(\xi) = -\frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\xi} \langle \Phi_{A-1} e^{-i\mathbf{q}r_A} | \hat{V} | \Phi_A \rangle, \quad (18)$$

which, after integration over $d\mathbf{q}$, becomes

$$U(\mathbf{r}) = \tilde{U}(-\xi) = \langle \Phi_{A-1} \delta(\mathbf{r}_A - \mathbf{r}) | \hat{V} | \Phi_A \rangle \\ = \int d\mathbf{r}_1 \dots d\mathbf{r}_{A-1} \Phi_{A-1}^\dagger(\mathbf{r}_1, \dots, \mathbf{r}_{A-1}) \hat{V} \Phi_A(\mathbf{r}_1, \dots, \mathbf{r}). \quad (19)$$

The expression (19) was used in the early version of the STA introduced by Pinkston and Satchler [11] which did not take translation invariance into account but could use nonoscillator single-particle wave functions. The contributions from the valence nucleons only were taken into account in their approach and the effective NN potential has been chosen in such a way to provide the same spectroscopic factors as those obtained by evaluating directly the overlap $\langle \Phi_{A-1} | \Phi_A \rangle$. According to Ref. [4], such a procedure is not justified because the binding energy calculations and source-term calculations probe different components of the effective NN potentials.

III. PARTIAL-WAVE EXPANSION OF $U(\mathbf{r})$

To solve the inhomogeneous equation (1) the radial part $U_{lj}(\mathbf{r})$ of the source term $U(\mathbf{r})$ should be supplied to its right-hand side. This radial part is obtained from $\tilde{U}_{lj}(\xi)$ for which

the following expression is valid:

$$\tilde{U}_{lj}(\xi) = - \sum_{m_j M_{A-1}} (j m_j J_{A-1} M_{A-1} | J_A M_A) \\ \times \int d\hat{\xi} \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\xi} \\ \times \langle \Phi_{J_{A-1} M_{A-1}} f_{lj m_j \tau}(\xi, \mathbf{q}, \mathbf{r}_A) | \hat{V} | \Phi_{J_A M_A} \rangle. \quad (20)$$

Here J_i and (M_i) are the total spin and its projection of nucleus i , l and j are the orbital and total angular momentum of the removed nucleon, and

$$f_{lj m_j \tau}(\xi, \mathbf{q}, \mathbf{r}_A) = (\alpha+1)^{-3/2} e^{\frac{\alpha\xi^2}{2b^2} + \beta q^2 b^2 - i\gamma \mathbf{q}r_A} \\ \times \sum_{m\sigma} \left(l m \frac{1}{2} \sigma | j m_j \right) Y_{lm}(\hat{\xi}) \chi_{\sigma\tau}(A), \quad (21)$$

where $\chi_{\sigma\tau}$ is spin-isospin wave function of the removed nucleon with spin (isospin) projection σ (τ). Since the nuclear wave functions are antisymmetric and the operator $\hat{V}$ is symmetric, then

$$\langle \Phi_{J_{A-1} M_{A-1}} f_{lj m_j \tau}(\xi, \mathbf{q}, \mathbf{r}_A) | \sum_{i=1}^{A-1} \hat{V}_{i,A} | \Phi_{J_A M_A} \rangle \\ = (A-1) \langle \Phi_{J_{A-1} M_{A-1}} f_{lj m_j \tau}(\xi, \mathbf{q}, \mathbf{r}_A) | \hat{V}_{A-1,A} | \Phi_{J_A M_A} \rangle, \quad (22)$$

assuming that only two-body terms of the effective NN potentials contribute to the source term.

The shell-model wave functions $\Phi_{J_{A-1} M_{A-1}}$ and $\Phi_{J_A M_A}$ in Eq. (22) are linear combinations of Slater determinants. To evaluate the matrix element (22), we separate a single-particle wave function $\phi_{\alpha_1}(\mathbf{r}_{A-1})$ from $\Phi_{J_{A-1} M_{A-1}}$ and a product $\phi_{\alpha_2}(\mathbf{r}_{A-1})\phi_{\alpha_3}(\mathbf{r}_A)$ from $\Phi_{J_A M_A}$, where $\alpha_i \equiv \{n_i l_i j_i m_{j_i} \tau_i\}$ is the set of quantum numbers describing the single-particle state i . Equation (22) then will reduce to calculations of linear combinations of the following two-body matrix elements:

$$U_{lj m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi) = \frac{1}{(2\pi)^3} \int d\hat{\xi} \int d\mathbf{q} e^{i\mathbf{q}\xi} \\ \times \int d\mathbf{r}_1 d\mathbf{r}_2 \phi_{\alpha_1}^\dagger(\mathbf{r}_1) f_{lj m_j \tau}^\dagger(\xi, \mathbf{q}, \mathbf{r}_2) \hat{V}(\mathbf{r}_1, \mathbf{r}_2) \\ \times [\phi_{\alpha_2}(\mathbf{r}_1)\phi_{\alpha_3}(\mathbf{r}_2) - \phi_{\alpha_3}(\mathbf{r}_1)\phi_{\alpha_2}(\mathbf{r}_2)], \quad (23)$$

so

$$\tilde{U}_{lj}(\xi) = - \sum_{m_j M_{A-1}} (j m_j J_{A-1} M_{A-1} | J_A M_A) \\ \times \sum_{\alpha_1 \alpha_2 \alpha_3} A_{\alpha_1} B_{\alpha_2 \alpha_3} U_{lj m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi), \quad (24)$$

where A_{α_1} and $B_{\alpha_2 \alpha_3}$ are the expansion coefficients generated in the usual shell-model codes. All technical details concerning the calculations of $U_{lj m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi)$ are given in the Appendix.

IV. APPLICATION TO DOUBLE-CLOSED-SHELL NUCLEI

In this article, the new approach has been applied to double-closed-shell nuclei which are described by only one

Slater determinant made, in other words, by the independent particle model (IPM). The Slater determinants were made of single-particle oscillator wave functions with the oscillator radius of $b = \sqrt{\hbar/m\omega}$, where m is the nucleon mass and $\hbar\omega = 41A^{-1/3} - 25A^{-2/3}$ MeV. The effective NN potential used here is the same one as in previous publications, Refs. [3,4], namely the M3Y potential from Ref. [8] that reproduces the oscillator matrix elements derived from NN -scattering data in Ref. [9]. A new code has been written to calculate the source term. It has been tested by replacing the radial parts of the NN interaction by unity or r^2 for which alternative analytic expressions were derived. For ^{16}O , the results of calculations have been compared to those obtained with the old code based on a translation-invariant fractional parentage expansion, designed for p -shell nuclei and used in Refs. [3–5]. For central interactions the overlap functions, obtained with the old and the new code, agree very well. However, it has been found that the contribution from the spin-orbit force has not been treated properly in the old code. Extensive testing of the new code has been done to make sure that the contribution from the spin-orbit code is now correctly calculated. Below, corrected results for ^{16}O are presented.

The source term $U_{ij}(r)$ generates the radial overlap function $I_{ij}(r)$ from the inhomogeneous equation obtained from partial-wave expansion of Eq. (1):

$$\left\{ \varepsilon + \frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right] - V_c^0(r) \right\} I_{ij} = U_{ij}. \quad (25)$$

This equation is solved by means of the Green's function technique [5]. The experimental value of the separation energy ε is used. The norm of $I_{ij}(r)$ gives the spectroscopic factor S_{ij} while its behavior at large distance provides ANC C_{ij} . The correct radial behavior for $I_{ij}(r)$ is guaranteed.

Using oscillator single-particle wave functions to generate the source term allows the influence of the center-of-mass effects to be quantified. For this reason, the spectroscopic factors, rms radii of the overlaps, and the ANCs calculated both in the translation-invariant model and in the $A \rightarrow \infty$ limit are shown in Table I. It is well known that the center-of-mass correction to the spectroscopic factor calculated in traditional oscillator shell model is given by a factor of $[A/(A-1)]^{2n+l}$.

TABLE I. Spectroscopic factors S_{ij} , rms radii $\langle r^2 \rangle^{1/2}$ of the overlaps (in fm) and the squared ANCs C_{ij}^2 in (fm^{-1}) obtained with translation-invariant source term (CM removed) and in the $A \rightarrow \infty$ limit (no CM removal).

A	A-1	l_j	CM removed			No CM removal		
			S_{ij}	$\langle r^2 \rangle^{1/2}$	C_{ij}^2	S_{ij}	$\langle r^2 \rangle^{1/2}$	C_{ij}^2
^{16}O	^{15}N	$p_{1/2}$	1.45	2.867	182	1.29	2.834	151
^{24}O	^{23}O	$s_{1/2}$	1.66	3.087	10.3	1.47	2.903	8.39
^{40}Ca	^{39}K	$d_{3/2}$	2.90	3.774	5322	2.69	3.749	4646
^{41}Ca	^{40}Ca	$f_{7/2}$	0.52	4.207	4.40	0.47	4.184	3.81
^{133}Sn	^{132}Sn	$f_{7/2}$	0.68	5.213	0.424	0.65	5.196	0.397
^{209}Pb	^{208}Pb	$s_{1/2}$	1.13	5.282	1.02	1.10	5.281	0.904

For example, in ^{16}O and ^{209}Pb it leads to an increase of the spectroscopic factor for neutron removal from the last shell by 6.7 and 1.9%, respectively. It turns out that the influence of the center-of-mass effects on the source term, and, as a result, on the spectroscopic factors and ANCs, is stronger. Thus, the center-of-mass effects increase the spectroscopic factors for the same nuclei by 12 and 2.7%, respectively. Their influence of the ANCs is even stronger, 21 and 13% for ^{16}O and ^{209}Pb , respectively. The center-of-mass effects also slightly increase the rms radii of the overlap functions. This means that the center-of-mass removal should be properly done in the source-term calculations. This could be important, especially when configurations with particle-hole excitations are present in the nuclear states of interests, in particular, in intruder states. In what follows, the center of mass has been removed in all calculations except in those where the residual nucleus does not have the minimal number of the oscillator quanta (such as removal of the $2s_{1/2}$ proton from ^{208}Pb) because for these cases the wave function of the residual nucleus $A-1$ cannot be factorized by Eq. (8). Such cases occur in medium and large mass regions where the center-of-mass influence is not very strong.

The source term can be presented as a sum of contributions from different orbital momentum λ relative two-body NN motion. The contribution from the relative S -wave motion dominates for all cases. This is illustrated in Fig. 1 for the case of $d_{5/2}$ proton removal from ^{208}Pb . The p and d waves give noticeable contributions as well but the contributions from $\lambda > 4$ can be neglected. Since the dominant contribution comes from the s wave where noncentral interactions are absent their influence on the source term is small. In particular, the contribution from the tensor force is negligible for all the cases studied here.

The spectroscopic factors, the rms radii and the ANCs squared obtained from I_{ij} for a range of double-closed-shell nuclei are shown in Table II. For each case it is shown whether the center-of-mass has been removed. Also shown is the ratio

FIG. 1. (Color online) The contributions from different NN orbital momentum λ to the central part of the source term $U_{ij}(r)$ calculated for the $d_{5/2}$ proton removal from ^{208}Pb .

TABLE II. Spectroscopic factors, rms radii of the overlaps (in fm) and the squared ANCs in (fm⁻¹) for the overlaps $\langle A|A-1 \rangle$ obtained in the STA. Also shown are the quantum numbers (lj) of the removed nucleon, the excitation energy E_x of A or $A-1$, the separation energy ε of the removed nucleon (both in MeV), and the spectroscopic factor S_{lj}^{IPM} of the independent particle model. In each case it is stated whether the center-of-mass has been removed.

A	$A-1$	lj	E_x	ε	CM removed?	S_{lj}^{IPM}	S_{lj}	$\langle r^2 \rangle^{1/2}$	C_{lj}^2	$S_{lj}/S_{lj}^{\text{IPM}}$
¹⁶ O	¹⁵ N	$p_{1/2}$	0.0	12.13	Yes	2.133	1.45	2.867	182	0.68
		$p_{3/2}$	6.32	18.45	Yes	4.267	2.61	2.770	1159	0.61
¹⁶ O	¹⁵ O	$p_{1/2}$	0.0	15.66	Yes	2.133	1.39	2.825	116	0.65
		$p_{3/2}$	6.18	21.84	Yes	4.267	2.55	2.746	728	0.60
¹⁷ O	¹⁶ O	$d_{5/2}$	0.0	4.14	Yes	1.129	0.60	3.548	0.45	0.53
		$s_{1/2}$	0.87	3.27	Yes	1.129	0.57	2.710	1.76	0.51
¹⁷ F	¹⁶ O	$d_{5/2}$	0.0	0.60	Yes	1.129	0.65	3.736	0.58	0.58
		$s_{1/2}$	0.50	0.10	Yes	1.129	0.63	3.971	2101	0.56
²⁴ O	²³ O	$s_{1/2}$	0.0	3.6	Yes	2.177	1.66	3.087	10.3	0.76
²⁴ O	²³ N	$p_{1/2}$	0.0	26.6	Yes	2.087	1.18	2.915	6049	0.57
²⁵ F	²⁴ O	$d_{5/2}$	0.0	15.10	Yes	1.085	0.46	3.444	252	0.42
⁴⁰ Ca	³⁹ Ca	$d_{3/2}$	0.0	15.64	Yes	4.208	2.71	3.698	821	0.64
		$s_{1/2}$	2.47	18.11	Yes	2.104	1.34	3.199	3.34×10^3	0.64
⁴⁰ Ca	³⁹ K	$d_{3/2}$	0.0	8.33	Yes	4.208	2.90	3.774	5322	0.69
		$s_{1/2}$	2.52	10.85	Yes	2.104	1.15	3.137	1.68×10^4	0.55
⁴¹ Ca	⁴⁰ Ca	$f_{7/2}$	0.0	8.362	Yes	1.077	0.52	4.207	4.40	0.49
		$p_{3/2}$	1.94	6.42	Yes	1.077	0.55	3.278	17.14	0.51
⁴¹ Sc	⁴⁰ Ca	$f_{7/2}$	0.0	1.085	Yes	1.077	0.57	4.331	290	0.53
⁴⁸ Ca	⁴⁷ K	$s_{1/2}$	0.0	15.81	Yes	2.086	1.38	3.600	1.15×10^5	0.66
		$d_{3/2}$	0.36	16.17	Yes	4.172	2.70	3.793	5.66×10^4	0.65
		$d_{5/2}$	3.43	19.24	Yes	6.258	4.21	3.785	1.85×10^5	0.67
⁴⁸ Ca	⁴⁷ Ca	$f_{7/2}$	0.0	9.95	Yes	8.523	4.330	4.266	106	0.51
⁴⁹ Ca	⁴⁸ Ca	$p_{3/2}$	0.0	5.15	Yes	1.064	0.67	3.692	15.1	0.63
		$p_{1/2}$	2.02	3.12	Yes	1.064	0.71	3.862	3.77	0.67
⁴⁹ Sc	⁴⁸ Ca	$f_{7/2}$	0.0	9.63	Yes	1.064	0.49	4.260	1353	0.46
⁵⁶ Ni	⁵⁵ Co	$f_{7/2}$	0.0	7.17	Yes	8.444	4.59	4.355	219	0.54
⁵⁶ Ni	⁵⁵ Ni	$f_{7/2}$	0.0	16.64	Yes	8.444	4.18	4.277	2279	0.49
⁵⁷ Ni	⁵⁶ Ni	$p_{3/2}$	0.0	10.25	Yes	1.055	0.59	3.804	241	0.56
⁵⁷ Cu	⁵⁶ Ni	$p_{3/2}$	0.0	0.70	Yes	1.055	0.70	4.351	1.77×10^8	0.64
¹³² Sn	¹³¹ In	$g_{9/2}$	0.0	15.71	Yes	8.247	5.29	5.416	9.03×10^8	0.64
		$p_{1/2}$	0.30	16.01	No	2.0	1.47	4.770	2.41×10^9	0.74
		$p_{3/2}$	1.29	17.00	No	4.0	2.94	4.820	4.15×10^9	0.74
¹³² Sn	¹³¹ Sn	$d_{3/2}$	0.0	7.31	No	4.0	3.22	5.066	936	0.81
		$h_{11/2}$	0.07	7.38	Yes	12.47	6.45	5.855	23.4	0.52
		$s_{1/2}$	0.33	7.31	No	2.0	1.68	4.795	2029	0.84
		$d_{5/2}$	1.66	8.97	No	6.0	4.88	5.052	4202	0.81
		$g_{7/2}$	2.43	9.75	No	8.0	6.10	5.433	406	0.76
¹³³ Sn	¹³² Sn	$f_{7/2}$	0.0	2.42	Yes	1.038	0.68	5.213	0.424	0.66
		$p_{3/2}$	0.85	3.27	Yes	1.038	0.72	4.418	7.79	0.69
		$p_{1/2}$	1.36	3.78	Yes	1.038	0.66	4.309	8.84	0.64
		$f_{5/2}$	2.00	4.42	Yes	1.038	0.48	4.653	2.98	0.47
¹³³ Sb	¹³² Sn	$g_{7/2}$	0.0	9.63	Yes	1.038	0.57	5.436	3.87×10^7	0.56
²⁰⁸ Pb	²⁰⁷ Tl	$s_{1/2}$	0.0	8.00	No	2.0	1.49	5.600	4.21×10^{14}	0.75
		$d_{3/2}$	0.35	8.35	No	4.0	2.90	5.177	2.35×10^{14}	0.73
		$h_{11/2}$	1.35	9.35	Yes	12.29	5.77	6.320	1.06×10^{13}	0.47
		$d_{5/2}$	1.67	9.67	No	6.0	4.41	5.717	4.15×10^{14}	0.74
		$g_{7/2}$	3.47	11.47	No	8.0	4.92	5.893	1.09×10^{13}	0.62
²⁰⁸ Pb	²⁰⁷ Pb	$p_{1/2}$	0.0	7.37	No	2.0	1.62	5.357	3835	0.81
²⁰⁹ Pb	²⁰⁸ Pb	$g_{9/2}$	0.0	3.94	Yes	1.029	0.64	5.820	1.97	0.62
		$i_{11/2}$	0.78	3.16	Yes	1.029	0.63	6.449	3.12×10^{-3}	0.61
		$d_{5/2}$	1.57	2.37	Yes	1.029	0.77	5.335	3.14	0.75
		$s_{1/2}$	2.03	1.90	Yes	1.029	1.13	5.282	1.02	1.10
		$g_{7/2}$	2.49	1.45	Yes	1.029	0.65	5.840	0.011	0.63
		$d_{3/2}$	2.54	1.40	Yes	1.029	0.79	5.325	0.365	0.77
²⁰⁹ Bi	²⁰⁸ Pb	$h_{9/2}$	0.0	3.80	Yes	1.029	0.50	6.331	3.86×10^{14}	0.49

TABLE III. Spectroscopic factors, given as ratios to $(jm_j J_{A-1} M_{A-1} | J_A M_A)^2$, obtained from the $(e, e'p)$ knockout [12] and in the present approach. Also shown are results obtained in other microscopic approaches: correlated basis functions method (CBFM) [13], self-consistent Green's function method (SCGFM) [14,15], and coupled-cluster method (CCM) [16].

Nucleus A	J_{A-1}^π	Expt.	Present	CBFM	SCGFM	CCM
^{16}O	$\frac{1}{2}^-$	0.64 ± 0.07	0.73	0.89	0.77	~ 0.9
	$\frac{3}{2}^-$	0.56 ± 0.06	0.65	0.89	0.72	~ 0.9
^{40}Ca	$\frac{3}{2}^+$	0.65 ± 0.05	0.73	0.85	~ 0.8	
	$\frac{1}{2}^+$	0.52 ± 0.04	0.58	0.87	~ 0.8	
^{48}Ca	$\frac{1}{2}^+$	0.54 ± 0.04	0.69	0.84	0.36	
	$\frac{3}{2}^+$	0.57 ± 0.04	0.68	0.86	0.59	
	$\frac{5}{2}^+$	0.11 ± 0.02	0.71	0.85		
^{208}Pb	$\frac{1}{2}^+$	0.49 ± 0.05	0.74	0.85		
	$\frac{3}{2}^+$	0.58 ± 0.06	0.72	0.83		
	$\frac{5}{2}^+$	0.49 ± 0.05	0.73	0.83		
	$\frac{7}{2}^+$	0.26 ± 0.03	0.61	0.82		
	$\frac{11}{2}^+$	0.57 ± 0.06	0.48	0.82		

of the spectroscopic factors S_{lj} to the spectroscopic factors S_{lj}^{IPM} obtained in IPM by direct evaluation of the overlap integral between Slater determinants of nuclei A and $A - 1$.

A. Removing one nucleon from a closed shell nucleus

The spectroscopic factors S_{lj}^{STA} for one-nucleon removal from closed-shell nuclei obtained with IPM wave functions are smaller than the original IPM spectroscopic factors S_{lj}^{IPM} . For stable double-closed-shell nuclei these spectroscopic factors are compared to those determined from the $(e, e'p)$ knockout in Table III where predictions of other microscopic models are shown as well (to facilitate comparison with other models S_{lj}^{STA} are divided by $(J_{A-1} M_{A-1} | J_A M_A)^2$ in Table III). Most S_{lj}^{STA} are just above the upper limits of the experimental values but by no more than 10%, except for the $^{48}\text{Ca} \rightarrow ^{47}\text{K}(\frac{5}{2}^+) + p$ and $^{208}\text{Pb} \rightarrow ^{207}\text{Tl}(\frac{7}{2}^+) + p$ cases, where the experimental spectroscopic factors have abnormally low values. In practically all cases, except the $^{48}\text{Ca} \rightarrow ^{47}\text{K}(\frac{3}{2}^+) + p$ knockout, S_{lj}^{STA} are much closer to the experimental data than the spectroscopic factors obtained in other microscopic approaches such as correlated basis function method [13], self-consistent Green's functions method (SCGFM) [14,15], and coupled-cluster method [16] (see Table III). For ^{16}O , the ANC is also available from the $^{15}\text{N}(^3\text{He}, d)^{16}\text{O}$ reaction study [17]. Its value, updated in [4], is $C_{\text{exp}}^2 = 175 \pm 29 \text{ fm}^{-1}$. The C^2 value from STA, equal to 182 fm^{-1} , agrees with the experimental value very well.

Away from stability, new magic numbers appear where nuclear structure can be modelled by the IPM wave functions. Recently, ^{24}O has been suggested to have a doubly magic

structure [18,19]. The spectroscopic factor for $s_{1/2}$ neutron removal from ^{24}O has been determined from the one-neutron knockout reaction $^{12}\text{C}(^{24}\text{O}, ^{23}\text{O})$ at $E = 920 \text{ MeV/A}$ [19]. It is equal to 1.74 ± 0.19 being smaller than the IPM value of 2 (or $S_{\text{IPM}} = 2.18$ if the center-of mass is removed). The STA gives the value of 1.66 for this spectroscopic factor, in agreement with the experimental value. To achieve this reduction within the shell model all the sd -shell model space should be included. In this case, $S_{\text{SM}}(\text{SDPF-M}) = 1.769$ and $S_{\text{SM}}(\text{USDB}) = 1.810$ [19]. The *ab initio* coupled-cluster theory predicts for this spectroscopic factor the value between 1.83 and 1.84 [20] (corresponding to their normalized spectroscopic factors of 0.916–0.918). The STA approach predicts asymmetry in neutron-proton ^{24}O spectroscopic factors: the proton spectroscopic factor $S_{\text{STA}} = 1.18$ is much more strongly suppressed with respect to the IPM value. A similar result is obtained in the *ab initio* coupled-cluster theory where the spectroscopic factor for the $p_{1/2}$ proton removal is between 1.21 and 1.30.

For another double-closed-shell nucleus away from stability, ^{56}Ni , the STA predicts that its neutron and proton spectroscopic factors are only 49 and 54% of the IPM value. This reduction is stronger than that predicted in the SCGFM [21]. The latter gives 72 and 73% of the IPM value for $^{56}\text{Ni} \rightarrow ^{55}\text{Ni} + n$ and $^{56}\text{Ni} \rightarrow ^{55}\text{Co} + p$, respectively. Both approaches predict neutron-proton asymmetry in SF reduction; however, the STA suggests stronger asymmetry than the SCGFM does. The STA also predicts large asymmetry in spectroscopic factor reduction for another known doubly magic nucleus away from stability, ^{132}Sn : S_{lj}^{STA} is 80% of the IPM value for the $d_{3/2}$ neutron removal to the ground state of ^{131}Sn and 64% of the IPM value for $g_{9/2}$ proton removal to the ground state of ^{131}In . However, for proton removal to excited states of ^{131}In this reduction is smaller. It is interesting that the dispersive-optical model suggests that the spectroscopic factor for the $g_{9/2}$ orbital in ^{132}Sn is 56% of the IPM value [22], which is similar to the STA predictions.

Three other double-closed-shell nuclei are expected to exist away from the stability, ^{60}Ca , ^{78}Ni , and ^{100}Sn (the latter has been already seen). Their masses (and, therefore, nucleon separation energies) are as yet unknown. Therefore, in the present work the ratio $S_{\text{STA}}/S_{\text{IPM}}$ is shown in Fig. 2 for different lj as a function of separation energy. One can see that this ratio always increases at small separation energies although at a different rate for different lj . Given that the realistic spectroscopic factor should be smaller than S_{IPM} , in other words, $S_{\text{STA}}/S_{\text{IPM}} \leq 1$, one can speculate that ^{60}Ca should be bound by at least 2 MeV if its last (and most weakly bound) nucleon is on the $p_{1/2}$ orbit. A noticeable reduction of the strength of the $g_{9/2}$ orbital is expected in ^{78}Ni . For ^{100}Sn , the $S_{\text{STA}}/S_{\text{IPM}}$ values corresponding to the separation energies obtained from systematics are shown by open squares in Fig. 2. It is expected that the spectroscopic factor for the $g_{9/2}$ orbital is strongly reduced, being only 55 and 48% of the IPM value for protons and for neutrons, respectively. This would represent significant symmetry breaking in mirror $g_{9/2}$ spectroscopic factors, by about 16%, if the systematic separation energies are confirmed experimentally. However, this difference is much smaller as compared to what could

FIG. 2. (Color online) The ratio S_{STA}/S_{IPM} as a function on nucleon separation energy calculated for nucleon knockout from the double-closed-shell nuclei ^{60}Ca , ^{78}Ni , and ^{100}Sn .

be expected for the $\varepsilon_p - \varepsilon_n \sim 15$ MeV from the systematics for the spectroscopic factor reduction observed in knockout reactions in Ref. [2]: One could have easily expected a factor of 2 between the neutron and proton spectroscopic factors in ^{100}Sn .

B. Adding one nucleon to a closed-shell nucleus

The most surprising result obtained for adding one nucleon to a closed-shell nucleus is the small value for the ^{17}O and ^{17}F spectroscopic factors (see Table II). The properties of the $\langle ^{17}\text{O} | ^{16}\text{O} \rangle$ overlap are known from the simultaneous accurate analysis of the sub-barrier peripheral reaction $^{16}\text{O}(^{17}\text{O}, ^{17}\text{O})^{16}\text{O}$ and elastic magnetic electron scattering on ^{17}O [23], where both the ANC and the spectroscopic factor of ^{17}O have been determined. The $S_{\text{exp}} = 1.03 \pm 0.07$ from this study is 91% of the center-of-mass-corrected IPM value, $S_{\text{IPM}} = 1.13$, while the STA predicts only 53% of this value. Moreover, the STA gives a smaller prediction for the squared ANC of ^{17}O , $C^2 = 0.45 \text{ fm}^{-1}$, as compared to $C_{\text{exp}}^2 = 0.68 \pm 0.03 \text{ fm}^{-1}$ from Ref. [23].¹ These small values originate due to an insufficiently strong source term. It should be remembered that in the present approach the effective NN interactions are assumed to be the same for all nuclei. This is not necessarily true: effective interactions that take into account coupling to missing model spaces can be A dependent and, moreover, they can contain contributions from three-body, four-body, etc., terms. It is possible that three-body contributions, which arise due to the interaction of the removed nucleon with a pair of nucleons in the remainder could be sensitive to whether the removed nucleon is a single valence nucleon outside the closed

shell or is one of the nucleons in a completely occupied shell of an even-even nucleus.

Adding a nucleon to the doubly magic nucleus ^{40}Ca gives similarly small spectroscopic factors in ^{41}Ca and ^{41}Sc , 43 and 57% of the IPM value, respectively. However, it is not clear to what extent their values are reduced with respect to experimental ones (if reduced at all). The problem is that spectroscopic factors for these nuclei have been studied from transfer reactions which are mainly sensitive to the asymptotic region of the overlap function so spectroscopic factors strongly depend on the assumption about the shape of the radial overlap functions $I_{ij}(r)$ employed in the analysis. In the most recent $^{40}\text{Ca}(d, p)^{41}\text{Ca}$ study [24], the spectroscopic factor determined from the experimental data varied from 1.3 to 0.27 when the radius of the potential well of a transferred neutron has been changed from 1.1 to 1.5 fm. For a standard choice of $r_0 = 1.25$ fm the extracted spectroscopic factor is either 0.64 or 0.70 depending on the reaction theory used to calculate the transfer cross sections. The corresponding squared ANC is about 4 fm^{-1} which is not far away from the STA prediction of 4.4 fm^{-1} . Another $^{40}\text{Ca}(d, p)^{41}\text{Ca}$ study [25], in which the geometry of the interaction potentials has been based on modern Hartree-Fock calculations, also gives a small spectroscopic factor for ^{41}Ca , $S_{\text{exp}} = 0.73 \pm 0.04$. The STA prediction for this spectroscopic factor, $S_{f7/2}^{\text{STA}} = 0.52$, is smaller than both experimental values quoted above.

Adding a neutron to the doubly magic ^{48}Ca results in the STA spectroscopic factor which is 64% of the IPM value. The $^{48}\text{Ca}(d, p)^{49}\text{Ca}$ reaction study using information about the ^{49}Ca ANC derived from the $^{48}\text{Ca}(n, \gamma)^{49}\text{Ca}$ capture gives even smaller value, $S_{\text{exp}} = 0.53 \pm 0.11$ [26], while another (d, p) study that used the Hartree-Fock geometry for transferred nucleon overlap function leads to $S_{\text{exp}} = 0.74 \pm 0.08$ [25]. The STA value of 0.67 is somewhere in between the two experimental values; however, the STA predictions for ANCs are definitely low, $C^2 = 15.1 \text{ fm}^{-1}$ as compared to $C_{\text{exp}}^2 = 32.1 \pm 3.2 \text{ fm}^{-1}$ for the ground state of ^{49}Ca and $C^2 = 3.77 \text{ fm}^{-1}$ as compared to $C_{\text{exp}}^2 = 9.30 \pm 0.93 \text{ fm}^{-1}$ for its first excited state from Ref. [26]. The small ANC values in the STA can be the consequence of using the oscillator single-particle $1p_{3/2}$ wave functions where one node is present. Because of the node, the cancellations between internal and external contributions to $I_{ij}(r)$ from the source term may become important while the oscillator wave functions used may not guarantee the right proportion between external and internal contributions.

Similar problems occur for adding one neutron to the doubly magic ^{208}Pb . The spectroscopic factors for large orbital momentum, $l = 4$ and 6 , are 62% of the IPM value, which is the same as for removing a proton from the $0g_{7/2}$ from ^{208}Pb . Adding a neutron into the $2d_{5/2}$ and $2d_{3/2}$ shells provides similar spectroscopic factors as removing a proton from the $1d_{5/2}$ and $1d_{3/2}$ shells. Adding a neutron to the $3s_{1/2}$ shell gives a spectroscopic factor that is even larger than the IPM one. At the same time, the ratio between C_{STA}^2 and the ANC-squared C_{exp}^2 derived from sub-Coulomb transfer in Ref. [27] strongly depends on the number of nodes (see Table IV). For the $3s_{1/2}$ state (three nodes) the STA predictions are abnormally low. With a decreasing number of nodes the STA predictions

¹The ANC $\bar{C}$ in Ref. [23] is related to C adopted in the present article by the following relation: $C^2 = 2\kappa \bar{C}^2$.

TABLE IV. The ANCs squared C_{ij}^2 (in fm⁻¹) for the (²⁰⁸Pb|²⁰⁹Pb) overlap calculated for several excited states in ²⁰⁹Pb in the STA in comparison to the C_{exp}^2 values obtained from peripheral transfer reactions in Ref. [27].

E_x	lj	C_{STA}^2	C_{exp}^2
0	$g_{9/2}$	1.97	2.25 ± 0.11
0.78	$i_{11/2}$	3.12×10^{-3}	$(1.56 \pm 0.02) \times 10^{-3}$
1.57	$d_{5/2}$	3.14	10.33 ± 0.49
2.09	$s_{1/2}$	1.02	36.1 ± 2.0
2.49	$g_{7/2}$	0.011	0.016 ± 0.001
2.54	$d_{3/2}$	0.37	2.02 ± 0.12

become closer to C_{exp}^2 . For the nodeless case of $i_{11/2}$ the STA overestimates the ANC-squared by a factor of 2, which could be a consequence of a very large rms radius of $I_{lj}(r)$ predicted by the STA as the spectroscopic factor is small. All this is consistent with the suggestion that correct behavior of the single-particle wave functions in the surface area could be very important for overlap calculations in heavy nuclei.

Away from stability, adding one nucleon to the doubly magic ⁵⁶Ni gives the spectroscopic factor of 0.59 which agrees very well with the value $S_{\text{exp}} = 0.58 \pm 0.11$ derived from the neutron knockout from ⁵⁷Ni [28]. It is also close to another theoretical prediction, $S = 0.65$, obtained within the SCGFM [21]. One again, the STA predicts a larger asymmetry in the spectroscopic factor reduction, $S = 0.59$ for adding one neutron and $S = 0.70$ for adding one proton to ⁵⁶Ni. The SCGFM does not support this asymmetry, and the mirror spectroscopic factors of ⁵⁷Ni and ⁵⁷Cu are similar: 0.65 and 0.66, respectively.

Recently, adding one neutron to the neutron-rich doubly magic ¹³²Sn has been studied in the (d, p) reaction using the distorted-wave Born approximation [29]. This reaction has been reanalyzed in Ref. [30] using the adiabatic distorted-wave approach to take deuteron breakup into account. The spectroscopic factors for all excited states of ¹³³Sn obtained in this study are around 1.1. The STA approach gives a much lower value, around 0.7. It also gives the ANC-squared $C^2 = 0.42$ fm⁻¹ for the ground state of ¹³³Sn which is half the experimental value $C_{\text{exp}}^2 = 0.82 \pm 0.07$ fm⁻¹ but for the first excited state it predicts a larger ANC, $C^2 = 7.79$ fm⁻¹, as compared to the experimental value of 6.5 ± 0.05 fm⁻¹. These STA values should be influenced by the nodes in the overlap functions $I_{lj}(r)$ obtained with the single-particle oscillator wave functions, similar to the ⁴⁹Ca case.

Finally, adding a proton to the double-closed-shell neutron-rich nuclei ²⁴O, ⁴⁸Ca, and ¹³²Sn results in spectroscopic factors that are more strongly reduced from the IPM values than those obtained for proton removal from these nuclei (this statement refers to the ground states obtained by adding or removing a proton). Thus, the spectroscopic factors for adding a proton to ²⁴O and removing a proton from ²⁴O are 42 and 57% of the IPM values, respectively. An even bigger gap is seen in ⁴⁸Ca: 46 and 66% for adding and removing a proton, respectively. A slightly smaller difference occurs in ¹³²Sn with 56% for adding and 64% for removing a proton.

V. SUMMARY AND CONCLUSIONS

In the present article, the source-term approach to calculate one-nucleon overlap functions, previously applied to $A \leq 16$, has been extended to heavier nuclei. New analytical expressions have been derived for a source term which takes translation invariance of the wave functions into account. Although these expressions have been obtained for single-particle oscillator wave functions, they can be naturally generalized for those cases where the total wave function is factorized into the harmonic oscillator center-of-mass motion and an arbitrary internal wave function independent of this motion (such as arising in a microscopic cluster model or the coupled-cluster method). The calculations performed for double-closed-shell nuclei revealed that removal of the spurious center-of-mass motion in the source term has a considerably larger effect on spectroscopic factors and ANCs than that due to the well-known center-of-mass correction $[A/(A-1)]^{(2n+l)}$ in the oscillator shell model. Thus, for the lightest nucleus considered here, ¹⁶O, the center-of-mass removal increases the STA spectroscopic factor by 12% and the ANC by 21% as compared to a 6.7% correction to the IPM value that follows from the simple $[A/(A-1)]^{(2n+l)}$ formula.

With the NN potential chosen in the present calculations, the STA predicts a reduction of the spectroscopic factors with respect to the IPM values for all the cases considered over a wide range of A . This is remarkable because the current calculations implicitly assume that the effective interaction between the last (removed) nucleon and the nucleons in the remainder are mass and state independent. In reality, these interactions should be mass- and/or state-dependent in the same way that the effective matrix elements of the Hamiltonian are. A closer look reveals an odd-even effect in the ratio $S_{\text{STA}}/S_{\text{IPM}}$ that measures the spectroscopic factor reduction. This ratio is normally larger in even- A nuclei than in odd- A nuclei (see Fig. 3). It appears that this ratio is slightly larger with respect to experimental values (where available) for even nuclei and is somewhat smaller for odd nuclei. These systematic even-odd effects could signal the presence of the three-body effective force. Anyway, both for even and odd A the ratio $S_{\text{STA}}/S_{\text{IPM}}$ has a similar dependence on the difference in proton and neutron separation energies, $\Delta S = S_p - S_n$ (or $\Delta S = S_n - S_p$), as seen in Fig. 3. This dependence shows an asymmetry in proton-neutron spectroscopic factor reductions away from stability. However, the slope of this dependence is flatter than that observed in knockout reactions in Ref. [2].

The STA provides the correct asymptotic decrease of the overlap functions since the experimental separation energies are always employed in the inhomogeneous equation. Therefore, these overlaps can be used in one-nucleon-removal reaction calculations. However, comparison to experimental ANCs in the few cases available has shown that, with the current choice of the single-particle (oscillator) wave functions, reasonable ANC values are obtained only for overlaps either without nodes or with only one node. The more nodes are present in the overlap function the stronger the ANCs obtained in the STA are suppressed with respect to the experimental values. This suggests that the single-particle wave functions with more realistic shapes than that given by the

FIG. 3. (Color online) Ratio of the spectroscopic factors obtained from STA to the IPM values as a function of difference between proton and neutron experimental separation energies ΔS shown separately for (a) even and (b) odd A .

oscillator model should be used in the STA calculations. While it is simple to use arbitrary shapes of the single-particle wave functions in the nontranslation shell-invariant model, removal of the spurious center-of-mass motion presents a challenge for this case. For very heavy nuclei one may hope that the proper choice of shapes can be more important than the treatment of the center-of-mass motion, while the situation should be opposite for the very light nuclei. Further development of the STA is needed to address these issues.

Finally, the STA has a great potential to generate the overlap functions for nucleon-removal calculations. In the present article it has been applied only to the double-closed-shell nuclei. The generalization of the STA for open-shell nuclei is straightforward. It can be introduced into major existing shell-model codes to give the opportunity to predict the overlap functions for any nuclei studied in modern experiments.

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APPENDIX

The two-body NN potential in Eq. (22) is assumed to have contributions from the central (c), spin-orbit (so), and tensor (t) parts:

$$\begin{aligned} \hat{V}(\mathbf{r}_1, \mathbf{r}_2) = & \sum_{S,T=0,1} V_{ST}^c(r_{12}) \hat{P}_S(1,2) \hat{P}_T(1,2) \\ & + \sum_{T=0,1} [(L \cdot S) V_T^{so}(r_{12}) + \hat{S}_{12} V_T^t(r_{12})] \hat{P}_T(1,2), \end{aligned} \quad (\text{A1})$$

where $\hat{P}_S$ ($\hat{P}_T$) is the projection operator into a two-nucleon state with spin S (isospin T); L and S are the operator of the orbital momentum and spin of the NN pair, respectively; and $\hat{S}_{12} = 3(\sigma_1 \mathbf{r}_{12})/r_{12}^2 - (\sigma_2 \sigma_1)$, $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$.

1. Central interaction

To evaluate $U_{l j m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi)$ for the central interaction, we use the fact that for arbitrary nuclear single-particle wave functions $\phi_{n l j m_j \tau}(\mathbf{r}) = \phi_{n l j}(\mathbf{r}) [Y_l(\hat{\mathbf{r}}) \otimes \chi_{\frac{1}{2}}^j(\tau)]$ their products can be presented as

$$\begin{aligned} & \phi_{n_1 l_1 j_1 m_1 \tau_1}(\mathbf{r}_1) \phi_{n_2 l_2 j_2 m_2 \tau_2}(\mathbf{r}_2) \\ & = \sum_{J M_J T M_T} (j_1 m_1 j_2 m_2 | J M_J) \left(\frac{1}{2} \tau_1 \frac{1}{2} \tau_2 \middle| T M_T \right) \\ & \quad \times \chi_{T M_T}(1,2) \sum_{L S} \hat{j}_1 \hat{j}_2 \hat{L} \hat{S} \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix} \\ & \quad \times |\phi_{n_1 l_1}(\mathbf{r}_1) \phi_{n_2 l_2}(\mathbf{r}_2)(L) : \chi_{S(1,2) : J M_J} \rangle, \end{aligned} \quad (\text{A2})$$

where $\hat{i} = \sqrt{2i+1}$ and $\chi_{S M_S T M_T}(1,2)$ is the spin-isospin wave function of the two nucleons with the total spin S and isospin T while M_S and M_T are the projections of S and T , respectively. Then

$$\begin{aligned} & U_{l j m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi) \\ & = \sum_{J M_J T M_T} (j_1 m_1 j m_j | J M_J) (j_2 m_2 j_3 m_3 | J M_J) \\ & \quad \times \left(\frac{1}{2} \tau_1 \frac{1}{2} \tau \middle| T M_T \right) \left(\frac{1}{2} \tau_2 \frac{1}{2} \tau_3 \middle| T M_T \right) \sum_{L S} \hat{j}_1 \hat{j}_2 \hat{j}_3 \hat{L}^2 \hat{S}^2 \\ & \quad \times (1 - (-)^{S+T} (-)^{l_2+l_3+L}) \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l & \frac{1}{2} & j \\ L & S & J \end{Bmatrix} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{Bmatrix} \\ & \quad \times \frac{1}{(2\pi)^3} \int d\hat{\xi} \int d\mathbf{q} e^{i\mathbf{q}\cdot\hat{\xi}} \langle \phi_{n_1 l_1}(\mathbf{r}_1) f_l(\hat{\xi}, \mathbf{q}, \mathbf{r}_2) : L || \\ & \quad \times V_{ST}^{(c)} || \phi_{n_2 l_2}(\mathbf{r}_1) \phi_{n_3 l_3}(\mathbf{r}_2) - \phi_{n_3 l_3}(\mathbf{r}_1) \phi_{n_2 l_2}(\mathbf{r}_2) : L \rangle. \end{aligned} \quad (\text{A3})$$

Below, the oscillator single-particle wave functions with the oscillator radius b were always used,

$$\phi_{nl}(r) = \frac{1}{b^{3/2}} \sqrt{\frac{2n!}{\Gamma(n+l+\frac{3}{2})}} \left(\frac{r}{b}\right)^l e^{-\frac{r^2}{2b^2}} L_n^{l+1/2} \left(\frac{r^2}{b^2}\right), \quad (\text{A4})$$

where $L_n^{l+1/2}$ is the generalized Laguerre polynomial. To evaluate the reduced matrix element in Eq. (A3) in the oscillator basis the new variables are first introduced:

$$\mathbf{x} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}; \quad \mathbf{X} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{\sqrt{2}}. \quad (\text{A5})$$

In these coordinates,

$$|\phi_{n_2 l_2}(\mathbf{r}_1) \phi_{n_3 l_3}(\mathbf{r}_2) : LM_L\rangle = \sum_{\nu \lambda N \Lambda L M_L} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle |\phi_{\nu \lambda}(\mathbf{x}) \phi_{N \Lambda}(\mathbf{X}) : LM_L\rangle, \quad (\text{A6})$$

where $\langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle$ is the Talmi-Moshinsky transformation coefficient for particles with equal masses [31]. After substitution of (A6) into Eq. (A3) the Talmi-Moshinsky transformation is applied to the product $\phi_{n_1 l_1 m_1}^*(\mathbf{r}_1) \phi_{N \Lambda M}(\mathbf{X})$. New coordinates are introduced,

$$\mathbf{z} = \sqrt{\frac{2}{3}} \mathbf{r}_1 - \frac{1}{\sqrt{3}} \mathbf{X} = \frac{1}{\sqrt{3}} \mathbf{x}, \quad \mathbf{Z} = \frac{1}{\sqrt{3}} \mathbf{r}_1 + \sqrt{\frac{2}{3}} \mathbf{X} = \sqrt{\frac{3}{2}} \mathbf{X} + \frac{1}{\sqrt{6}} \mathbf{x}, \quad (\text{A7})$$

in which

$$\begin{aligned} |\phi_{n_1 l_1}(\mathbf{r}_1) \phi_{N \Lambda}(\mathbf{X}) : L' M'_L\rangle &= \sum_{n' l' N' \Lambda'} \langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle |\phi_{n' l'}(\mathbf{z}) \phi_{N' \Lambda'}(\mathbf{Z}) : L' M'_L\rangle \\ &= \sum_{n' l' N' \Lambda'} \langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle \left| \phi_{n' l'} \left(\frac{1}{\sqrt{3}} \mathbf{x} \right) \phi_{N' \Lambda'} \left(\sqrt{\frac{3}{2}} \mathbf{X} + \frac{1}{\sqrt{6}} \mathbf{x} \right) : L' M'_L \right\rangle, \end{aligned} \quad (\text{A8})$$

where $\langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle$ is the Talmi-Moshinsky transformation coefficient for particles with masses 2 and 1. Taking into account Eq. (A8), we then integrate over $d\mathbf{X}$ using

$$\int d\mathbf{X} e^{i\gamma \mathbf{q} \mathbf{X} / \sqrt{2}} \phi_{N' \Lambda' M'} \left(\sqrt{\frac{3}{2}} \mathbf{X} + \sqrt{\frac{1}{6}} \mathbf{x} \right) = \left(\frac{2}{3} \right)^{3/2} e^{-i \frac{1}{3\sqrt{2}} \gamma \mathbf{q} \mathbf{x}} \phi_{N' \Lambda' M'} \left(\frac{\gamma \mathbf{q}}{\sqrt{3}} \right), \quad (\text{A9})$$

where $\phi_{N' \Lambda' M'}(\frac{\gamma \mathbf{q}}{\sqrt{3}})$ is the oscillator wave function in the momentum representation,

$$\phi_{N' \Lambda' M'}(\mathbf{q}) = (2\pi)^{3/2} (-)^{N'} i^{\Lambda'} \phi_{N' \Lambda'}(q) Y_{\Lambda' M'}(\hat{\mathbf{q}}), \quad (\text{A10})$$

and $\phi_{N' \Lambda'}(q)$ is calculated from Eq. (A4) making the substitution $b \rightarrow 1/b$. We next integrate over $\mathbf{q}$ and make summation over all projections of angular momenta and obtain the final expression for the central part of $U_{l_j m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi)$:

$$U_{l_j m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi) = \sum_{J M_J} (j_2 m_2 j_3 m_3 | J M_J) (j_1 m_1 j m_j | J M_J) \sum_{T M_T} \left(\frac{1}{2} \tau_1 \frac{1}{2} \tau \middle| T M_T \right) \left(\frac{1}{2} \tau_2 \frac{1}{2} \tau_3 \middle| T M_T \right) U_{l_j, STJ}^{n_1 l_1 j_1 n_2 l_2 j_2 n_3 l_3 j_3}(\xi), \quad (\text{A11})$$

where

$$\begin{aligned} U_{l_j, STJ}^{n_1 l_1 j_1 n_2 l_2 j_2 n_3 l_3 j_3}(\xi) &= \frac{1}{2\pi^2} \left(\frac{4\pi}{3\alpha + 3} \right)^{3/2} \hat{l} \hat{j} \hat{j}_1 \hat{j}_2 \hat{j}_3 e^{\frac{\alpha \xi^2}{2b^2}} \sum_{LS} (-)^{l+L} \hat{L} \hat{S}^2 \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l & \frac{1}{2} & j \\ L & S & J \end{Bmatrix} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{Bmatrix} \\ &\times \sum_{\nu \lambda N \Lambda} (1 - (-1)^{S+T+\lambda}) \hat{\lambda} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle \sum_{n' l' N' \Lambda' L' M'} \hat{\lambda}' \hat{l}' \hat{L}'^2 \langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle \\ &\times (\lambda' 0 | 0 | \Lambda' 0) (l' 0 | \lambda 0 | \lambda' 0) W(\lambda l' l \Lambda'; \lambda' L') W(l_1 L' L \lambda; \Lambda l) (-)^{\frac{\lambda' + \Lambda' + l}{2} + N'} \int_0^\infty dq q^2 j_l(q\xi) \phi_{N' \Lambda'} \left(\frac{\gamma q}{\sqrt{3}} \right) e^{\beta q^2 b^2} \\ &\times \int_0^\infty dx x^2 j_{\lambda'} \left(\frac{4\gamma q x}{3\sqrt{2}} \right) \phi_{n' l'} \left(\frac{x}{\sqrt{3}} \right) V_{ST}^c(\sqrt{2}x) \phi_{\nu \lambda}(x). \end{aligned} \quad (\text{A12})$$

For the Coulomb interaction, $U_{l_j m_j \tau}^{\alpha_1 \alpha_2 \alpha_3}(\xi)$ is given by Eqs. (A11) and (A12) in which $V_{ST}^c(\sqrt{2}x)$ should be substituted by $\delta_{M_T, -1} e^2 / (\sqrt{2}x)$. Equation (A11) can also be used to calculate the matrix element $\langle \Psi_{A-1} | V_C^0 | \Psi_A \rangle$ by making the substitution $V_{ST}^c(\sqrt{2}x) \rightarrow V_C^0(\xi)$.

2. Spin-orbit interaction

To get the contribution from the spin-orbit interaction, we use the following representation:

$$\begin{aligned}
 & V_T^{\text{so}}(r_{12})(\mathbf{L} \cdot \mathbf{S}) \hat{P}_T(1, 2) \phi_{n_2 l_2 j_2 m_2 \tau_2}(\mathbf{r}_1) \phi_{n_3 l_3 j_3 m_3 \tau_3}(\mathbf{r}_2) \\
 &= V_T^{\text{so}}(\sqrt{2}x) \sum_{JM_J T M_T} (j_2 m_2 j_3 m_3 | J M_J) \left(\frac{1}{2} \tau_2 \frac{1}{2} \tau_3 \middle| T M_T \right) \chi_{T M_T}(1, 2) \sum_{LS} \delta_{S,1} \hat{j}_2 \hat{j}_3 \hat{L} \hat{S} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{Bmatrix} \\
 &\times \sum_{\nu \lambda N \Lambda} (-)^{\lambda+\Lambda-L} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle \frac{1}{2} \sum_j \hat{j}^2 (j(j+1) - \lambda(\lambda+1) - S(S+1)) \\
 &\times \sum_{L''} |\phi_{N \Lambda}(\mathbf{X}) \phi_{\nu \lambda}(\mathbf{x})(L'') : \chi_S(1, 2) : J M_J \rangle \hat{L} \hat{L}'' W(\Lambda \lambda J S; L j) W(\Lambda \lambda J S; L'' j). \quad (\text{A13})
 \end{aligned}$$

Inserting Eq. (A13) into Eq. (23) and following the scheme outlined in the previous section we get

$$\begin{aligned}
 U_{lj, STJ}^{n_1 l_1 j_1 n_2 l_2 j_2 n_3 l_3 j_3}(\xi) &= \delta_{S,1} \frac{3}{4\pi^2} \left(\frac{4\pi}{3\alpha+3} \right)^{3/2} \hat{l} \hat{j} \hat{j}_1 \hat{j}_2 \hat{j}_3 e^{\frac{\alpha \xi^2}{2b^2}} \sum_{LL''} (-)^{l+L} \hat{L}^2 \hat{L}''^2 \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l & \frac{1}{2} & j \\ L'' & 1 & J \end{Bmatrix} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & 1 & J \end{Bmatrix} \\
 &\times \sum_{\nu \lambda N \Lambda} (1 - (-1)^{1+T+\lambda}) \hat{\lambda} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle \\
 &\times \sum_{n' l' N' \Lambda' L' \lambda'} \hat{\lambda}' \hat{l}' \hat{L}'^2 \langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle (l' 0 \lambda' 0 | \lambda' 0) W(\lambda' l' \Lambda'; \lambda' L') W(l_1 L' L'' \lambda; \Lambda l) (\lambda' 0 l' 0 | \Lambda' 0) \\
 &\times \sum_{j'} \hat{j}'^2 (j'(j'+1) - \lambda(\lambda+1) - 2) W(\Lambda \lambda J 1; L j') W(\Lambda \lambda J 1; L'' j') (-)^{\frac{\lambda'+\Lambda'+l}{2}+N'} \\
 &\times \int_0^\infty dq q^2 j_l(q\xi) \phi_{N' \Lambda'} \left(\frac{\gamma q}{\sqrt{3}} \right) e^{\beta q^2 b^2} \int_0^\infty dx x^2 j_{\lambda'} \left(\frac{4\gamma q x}{3\sqrt{2}} \right) \phi_{n' l'} \left(\frac{x}{\sqrt{3}} \right) V_T^{\text{so}}(\sqrt{2}x) \phi_{\nu \lambda}(x). \quad (\text{A14})
 \end{aligned}$$

3. Tensor interactions

The contribution from the tensor part of the NN interaction can be obtained using the following representation:

$$\begin{aligned}
 & V_T^t(r_{12}) \hat{S}_{12} \hat{P}_T(1, 2) \phi_{n_2 l_2 j_2 m_2 \tau_2}(\mathbf{r}_1) \phi_{n_3 l_3 j_3 m_3 \tau_3}(\mathbf{r}_2) \\
 &= V_T^t(\sqrt{2}x) \sum_{JM_J T M_T} (j_2 m_2 j_3 m_3 | J M_J) \left(\frac{1}{2} \tau_2 \frac{1}{2} \tau_3 \middle| T M_T \right) \chi_{T M_T}(1, 2) \sum_{LS} \delta_{S,1} \hat{j}_2 \hat{j}_3 \hat{L} \hat{S} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{Bmatrix} \\
 &\times \sum_{\nu \lambda N \Lambda} (-)^{\lambda+\Lambda-L} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle \sum_{\lambda'' L''} |\phi_{N \Lambda}(\mathbf{X}) \phi_{\nu \lambda''}(\mathbf{x})(L'') : \chi_S(1, 2) : J M_J \rangle \\
 &\times \sum_{j'} \hat{L} \hat{L}'' \hat{j}'^2 S_{\lambda \lambda'' j'} W(\Lambda \lambda J S; L j') W(\Lambda \lambda'' J S; L'' j'), \quad (\text{A15})
 \end{aligned}$$

where

$$S_{\lambda \lambda'' j'} = 2\sqrt{6}(-)^{1-j'} \hat{\lambda} \hat{\lambda}'' (\lambda 0 \lambda'' 0 | 20) W(11 \lambda \lambda''; 2 j'). \quad (\text{A16})$$

Inserting Eq. (A15) into Eq. (23) and following the scheme outlined above we get

$$\begin{aligned}
 U_{lj, STJ}^{n_1 l_1 j_1 n_2 l_2 j_2 n_3 l_3 j_3}(\xi) &= \delta_{S,1} \frac{\sqrt{10}}{\pi^2} \left(\frac{4\pi}{\alpha+1} \right)^{3/2} \hat{l} \hat{j} \hat{j}_1 \hat{j}_2 \hat{j}_3 e^{\frac{\alpha \xi^2}{2b^2}} \sum_{LL''} (-)^{l+L} \hat{L}^2 \hat{L}''^2 \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l & \frac{1}{2} & j \\ L'' & 1 & J \end{Bmatrix} \begin{Bmatrix} l_2 & \frac{1}{2} & j_2 \\ l_3 & \frac{1}{2} & j_3 \\ L & 1 & J \end{Bmatrix} \\
 &\times \sum_{\nu \lambda N \Lambda} (1 - (-1)^{1+T+\lambda}) \hat{\lambda} \langle \nu \lambda N \Lambda : L | n_2 l_2 n_3 l_3 : L \rangle \sum_{n' l' N' \Lambda' L' \lambda''} \hat{\lambda}' \hat{\lambda}'' \hat{l}' \hat{L}'^2 \langle n' l' N' \Lambda' : L' | 2:1 | n_1 l_1 N \Lambda : L' \rangle \\
 &\times (l' 0 \lambda'' 0 | \lambda' 0) W(\lambda' l' \Lambda'; \lambda' L') W(l_1 L' L'' \lambda''; \Lambda l) (\lambda' 0 l' 0 | \Lambda' 0) (\lambda 0 20 | \lambda'' 0) W(L'' 2 J 1; L 1) W(2 L \lambda'' \Lambda; L'' \lambda) \\
 &\times (-)^{\frac{\lambda'+\Lambda'+l}{2}+N'} \int_0^\infty dq q^2 j_l(q\xi) \phi_{N' \Lambda'} \left(\frac{\gamma q}{\sqrt{3}} \right) e^{\beta q^2 b^2} \int_0^\infty dx x^2 j_{\lambda'} \left(\frac{4\gamma q x}{3\sqrt{2}} \right) \phi_{n' l'} \left(\frac{x}{\sqrt{3}} \right) V_T^t(\sqrt{2}x) \phi_{\nu \lambda''}(x), \quad (\text{A17})
 \end{aligned}$$

where the following relation has been used:

$$\sum_{j'} \hat{j}^2 S_{\lambda\lambda''j'} W(\Lambda\lambda J1; Lj') W(\Lambda\lambda'' J1; L''j') = 2\sqrt{30}\hat{\lambda}(\lambda 020|\lambda'' 0) W(L''2J1; L1) W(2L\lambda'' \Lambda; L''\lambda). \quad (\text{A18})$$

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