A Sphere Decoder with Approximate QR Decomposition for Frequency-Selective Channels

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Abstract—This paper presents a method to significantly reduce the preprocessing complexity of the sphere decoder (SD) in frequency-selective channels. The method consists of calculating an approximate QR Decomposition (AQRD) of the channel matrix, making use of its special Toeplitz and block-Topelitz structure in single and multiple-antenna frequency-selective channels, respectively. The AQRD obtains the QR decomposition of a small submatrix of the channel matrix and extends that result to the rest of the matrix, resulting in a considerable complexity reduction compared to the original full QR Decomposition (FQRD). Simulation results show that, despite the lower complexity of the AQRD, it causes only a small bit error rate (BER) performance degradation in the SD.

Index Terms—sphere decoder (SD), approximate QR Decomposition (AQRD), frequency-selective channel.

I. INTRODUCTION

The sphere decoder (SD) has been recently proposed as a means of achieving optimal maximum likelihood (ML) performance in the detection of single input-single output (SISO) and multiple input-multiple output (MIMO) systems in frequency-selective channels [1], [2]. However, for large block sizes and, especially, at high signal to noise ratio (SNR), the required full QR Decomposition (FQRD) of the channel matrix has a critical effect in the overall complexity of the algorithm [2]. In this paper, an approximate QR Decomposition (AQRD) is proposed to significantly reduce the preprocessing complexity of the SD, without greatly affecting the performance. It makes use of the special Toeplitz structure of the channel matrix which results in the convergence of the elements of its FQRD for large block sizes [3]. Although that convergence property has been previously applied to the elements of a Cholesky factorization in the context of joint zero forcing (ZF) detection in time division-code division multiple access (TD-CDMA) systems [4], [5], its application to the preprocessing stage of the SD in frequency-selective channels has not been previously studied.

We consider a MIMO system with M transmit and N receive antennas, denoted as $M \times N$, in a block-fading frequency-selective propagation environment. The

channel between transmitter j and receiver i is modelled as a linear finite impulse response (FIR) filter with L independent symbol-spaced taps, denoted as $\mathbf{h}_{i,j} = [h_{i,j}(0), h_{i,j}(1), \dots, h_{i,j}(L-1)]^T$, where $h_{i,j}(l) \sim \mathcal{CN}(0, \sigma_l^2)$ for $i = 1, \dots, N$, $j = 1, \dots, M$ and σ_l is set according to the power delay profile (PDP) of the channel for $l = 0, \dots, L-1$, such that $\mathbb{E}[\mathbf{h}_{i,j}^H \mathbf{h}_{i,j}] = 1$. For notation purposes, we consider the transmission of MT symbols over a span of T symbol-spaced time instants, resulting in a block size T , with $T > L$. The MT -vector of transmitted symbols per block is denoted as $\mathbf{x} = [x_1, x_2, \dots, x_{MT}]^T$, where the first M symbols represent the symbols transmitted from all M antennas at time instant $k = 0$, and subsequently for $k = 1, \dots, T-1$. The symbols are taken independently from a constellation $\mathcal{O}$ of P points and $\mathbb{E}[|x_i|^2] = 1/M$ for $i = 1, \dots, MT$. In order to avoid inter-block interference, $L-1$ zeros are appended per transmit antenna at the end of each block. Correspondingly, the vector of received symbols $\mathbf{y}$ spans over $K = T + L - 1$ time instants, i.e. $\mathbf{y} = [y_1, y_2, \dots, y_{NK}]^T$, and can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v},$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}(0) & & & & \\ \vdots & \ddots & & & \\ \mathbf{H}(L-1) & \cdots & \mathbf{H}(0) & & \\ & \ddots & \vdots & \ddots & \\ & & \mathbf{H}(L-1) & \cdots & \mathbf{H}(0) \\ & & & \ddots & \vdots \\ & & & & \mathbf{H}(L-1) \end{bmatrix} \quad (1)$$

and

$$\mathbf{H}(l) = \begin{bmatrix} h_{1,1}(l) & \cdots & h_{1,M}(l) \\ \vdots & \ddots & \vdots \\ h_{N,1}(l) & \cdots & h_{N,M}(l) \end{bmatrix}$$

for $l = 0, \dots, L-1$. The vector $\mathbf{v} = [v_1, v_2, \dots, v_{NK}]^T$, contains independent complex Gaussian noise samples where $\mathbb{E}[|v_i|^2] = N_0 = \sigma^2$ for $i = 1, \dots, NK$. Finally,

we assume that the channel is perfectly known at the receiver and that $N \geq M$ ¹.

A. Sphere Decoder

The main idea behind the SD is to reduce the complexity of the maximum likelihood detector (MLD), given by

$$\hat{\mathbf{x}}_{\text{ML}} = \arg \min_{\mathbf{x} \in \mathcal{O}^{MT}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2,$$

by searching over only the vectors $\mathbf{x}$ that satisfy

$$\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \leq R^2, \quad (2)$$

i.e. searching over the lattice vectors $\mathbf{H}\mathbf{x}$ that lie within a hypersphere of radius R around the received vector $\mathbf{y}$ and achieving the same ML performance [6]. During the preprocessing stage, the SD makes use of the FQRD decomposition of $\mathbf{H} = \tilde{\mathbf{Q}}[\mathbf{R}^T \mathbf{0}]^T$ [7], where $\tilde{\mathbf{Q}}$ is an $NK \times NK$ unitary matrix, $\mathbf{R}$ is an $MT \times MT$ upper triangular matrix, and $\mathbf{0}$ is a 0-matrix of the appropriate size. Partitioning $\tilde{\mathbf{Q}} = [\mathbf{Q} \mathbf{Q}']$ into two submatrices $\mathbf{Q}$ and $\mathbf{Q}'$, containing the first MT columns and the last $NK - MT$ columns of $\tilde{\mathbf{Q}}$, respectively, and removing constant terms, the metric calculation in (2) can be rewritten as

$$\|\mathbf{Q}^H \mathbf{y} - \mathbf{R}\mathbf{x}\|^2 \leq R^2. \quad (3)$$

Due to the upper triangular matrix $\mathbf{R}$, the solution to (3) can be obtained recursively by performing a constrained *depth-first* tree search through a tree with MT levels where P branches originate from each node on the tree [6].

One important problem of the SD in frequency-selective channels is that the size of $\mathbf{H}$ increases with the block size, an effect that is not present in the flat-fading case. Thus, there is a dramatic increase in the complexity of the FQRD with the block size. That effect is more noticeable at high SNR, where the preprocessing complexity can become the dominant factor in the overall complexity of the SD [2]. In order to overcome that problem, an AQRD is proposed in the next section to considerably reduce the preprocessing complexity of the SD.

II. APPROXIMATE QR DECOMPOSITION

In order to introduce the AQRD, we take into account that the channel matrix $\mathbf{H}$ in (1) has a block-Toeplitz structure with $N \times M$ blocks. Thus, the matrices $\mathbf{R}$ and

$\mathbf{Q}$ in (3) can be written as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}^{(1)}(0) & \cdots & \mathbf{R}^{(L)}(L-1) & \mathbf{0} \\ & \ddots & \vdots & \vdots \\ & & \mathbf{R}^{(L)}(0) & \mathbf{R}^{(T)}(L-1) \\ & & & \ddots \\ \mathbf{0} & & & \mathbf{R}^{(T)}(0) \end{bmatrix}, \quad (4)$$

with $M \times M$ blocks $\mathbf{R}^{(n)}(l)$ for $l = 0, \dots, L-1$ and $n = l+1, \dots, T$, for a given l , and as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^{(1)}(L-1) & \cdots & \mathbf{Q}^{(T)}(T+L-2) \\ \vdots & \ddots & \vdots \\ \mathbf{Q}^{(1)}(0) & & \mathbf{Q}^{(T)}(L-1) \\ & \ddots & \vdots \\ \mathbf{0} & & \mathbf{Q}^{(T)}(0) \end{bmatrix}, \quad (5)$$

with $N \times M$ blocks $\mathbf{Q}^{(n)}(m)$ for $n = 1, \dots, T$ and $m = 0, \dots, n+L-2$, respectively.

Using the fact that $\mathbf{R}$ can also be obtained through the Cholesky factorization of the hermitian block-Toeplitz Gram matrix $\mathbf{G} = \mathbf{H}^H \mathbf{H}$, it has been previously shown that

$$\lim_{n \rightarrow \infty} \mathbf{R}^{(n)}(l) = \mathbf{R}(l), \quad (6)$$

i.e. each one of the L sequences $\mathbf{R}^{(n)}(l)$ in n converges to a given value $\mathbf{R}(l)$ [3], [8]. Further, as a consequence of (6), it can be directly shown that

$$\lim_{n \rightarrow \infty} \mathbf{Q}^{(n)}(m) = \mathbf{Q}(m), \quad (7)$$

i.e. the sequences $\mathbf{Q}^{(n)}(m)$ converge in n to a given value $\mathbf{Q}(m)$, and

$$\lim_{m \rightarrow \infty} \mathbf{Q}^{(n)}(m) = \mathbf{0}, \quad (8)$$

i.e. the sequences $\mathbf{Q}^{(n)}(m)$ converge in m to the 0-matrix [9].

The AQRD proposed in this section makes use of the above convergence results to obtain the FQRD of $\mathbf{H}$ in the following three steps:

- 1) Initially, a FQRD is performed on the submatrix of $\mathbf{H}$ resulting from selecting its first MB columns, with $L < B < T$. The output of that FQRD corresponds to the first MB columns of $\mathbf{R}$ and $\mathbf{Q}$, i.e. $\mathbf{R}^{(n)}(l)$ for $l = 0, \dots, L-1$ and $n = l+1, \dots, B$ and $\mathbf{Q}^{(n)}(m)$ for $n = 1, \dots, B$ and $m = 0, \dots, n+L-2$, respectively.
- 2) The remaining $M(T-B)$ columns of $\mathbf{R}$ are obtained using (6) and setting

$$\mathbf{R}^{(n)}(l) = \mathbf{R}^{(B)}(l)$$

for $n = B+1, \dots, T$ and $l = 0, \dots, L-1$.

¹Although the discussion in this paper concentrates on the MIMO case, the alternative proposed here also applies to the SISO case, where $\mathbf{H}$ is Toeplitz instead of block-Toeplitz.

- 3) The $M(T - B)$ columns of $\mathbf{Q}$ are obtained, first using (7) to set

$$\mathbf{Q}^{(n)}(m) = \mathbf{Q}^{(B)}(m)$$

for $n = B + 1, \dots, T$ and $m = 0, \dots, B + L - 2$, and then using (8) to set

$$\mathbf{Q}^{(n)}(m) = \mathbf{0}$$

for $n = B + 1, \dots, T$ and $m = B + L - 1, \dots, T + L - 2$.

Thus, the FQRD of $\mathbf{H}$ is obtained approximating the last $T - B$ block-columns of $\mathbf{R}$ ($\mathbf{Q}$) by the B -th block-column of $\mathbf{R}$ ($\mathbf{Q}$), performing a one-block down-shift from block to block and introducing $\mathbf{0}$ in the top-right non-computed blocks of $\mathbf{Q}$. If we approximate the total number of real operations required by the FQRD of $\mathbf{H}$ using the modified Gram-Schmidt (MGS) method as $N_F \approx 8NL(MT)^2$ [10], the use of the AQRD reduces that figure to $N_A \approx 8NL(MB)^2$, where $B < T$. Similar reductions in complexity can be obtained by applying the above convergence results to other QR methods like the Schur algorithm, optimized for Toeplitz or block-Toeplitz matrices [11], [12].

The convergence result in (6) has been previously used to propose an approximate Cholesky factorization of a Hermitian block-Toeplitz matrix in the context of ZF joint detection of TD-CDMA systems [4], [5]. In this paper, instead of considering sub-optimal detectors, the AQRD is applied to the preprocessing stage of the optimal SD in frequency-selective MIMO channels. In addition, given that the QR decomposition is needed, this paper includes the convergence results of the blocks of $\mathbf{Q}$, describing the extension procedure to obtain the full matrix $\mathbf{Q}$.

Fig. 1. Expected value of $F_R(n)$ in a 2×2 system with $T = 50$.

III. SIMULATION RESULTS

This section shows the convergence of the FQRD of $\mathbf{H}$ and the BER performance of the SD when the FQRD is replaced by the AQRD. In order to show the convergence of the FQRD of $\mathbf{H}$, we define, as a distance measure, the stochastic processes $F_R(n) = \{\|\mathbf{R}^{(n)}(0) - \mathbf{R}^{(T)}(0)\|_F : n = 1, \dots, T\}$ and $F_Q(m) = \{\|\mathbf{Q}^{(T)}(m)\|_F : m = 0, \dots, T + L - 2\}$, where $\|\cdot\|_F$ denotes the Frobenius norm². Fig. 1 shows the expected value of $F_R(n)$ in a 2×2 system with block size $T = 50$ for a different number of taps L . Both uniform and exponential PDPs have been considered. In the former, $\sigma_l^2 = 1/L$ for $l = 0, \dots, L - 1$, while in the latter, $\sigma_l^2 = e^{-l}/\sigma_h^2$ for $l = 0, \dots, L - 1$ with $\sigma_h^2 = \sum_{l=0}^{L-1} e^{-l}$. It can be observed how the expected value decreases with n , i.e. $\mathbf{R}^{(n)}(0)$ is a better approximation of $\mathbf{R}^{(T)}(0)$ as n increases, especially in the exponential PDP case. Based on that, the AQRD can select the parameter B so that the average $\|\mathbf{R}^{(B)}(0) - \mathbf{R}^{(T)}(0)\|_F$ is below an arbitrary threshold. If L increases, the parameter B also needs to increase to maintain the distance constraint. However, the effect is less noticeable in the exponential PDP case, given that the additional taps have an exponentially decreasing energy. It has also been observed that if B is selected outside the end tail of the average $F_R(n)$, its value remains almost unchanged with the increase of T , for a given L and distance constraint. That indicates that the number of taps is the determinant factor in the choice of B . Those last results have not been included due to space constraints.

Fig. 2. Expected value of $F_Q(m)$ in a 2×2 system with $T = 50$.

Fig. 2 shows the expected value of $F_Q(m)$ in the same scenario. For the range of selectable values of m ($m = L - 1, \dots, T + L - 2$), it can be observed how the

²The stochastic processes $F_R(n)$ and $F_Q(m)$ have been selected, due to space constraints, as illustrative examples of the convergence of $\mathbf{R}^{(n)}(l)$ and $\mathbf{Q}^{(n)}(m)$ in n and of $\mathbf{Q}^{(n)}(m)$ in m , respectively.

Fig. 3. BER performance of the SD and the ASD as a function of the E_b/N_0 in a 2×2 system with $T = 15$, $L = 3$ and uniform PDP.

average $F_Q(m)$ decreases with m , approaching its limit 0. Thus, the parameter B can be selected to keep the average $\|\mathbf{Q}^{(T)}(B+L-2)\|_F$ below an arbitrary threshold. As in Fig. 1, B would need to increase with L to keep the same distance constraint, with a less noticeable effect in the exponential PDP case. It is worth noting, though, that, for $m \leq L-2$ in the exponential PDP case, the average $F_Q(m)$ initially increases, reaches a maximum and then decreases. Therefore, in the exponential PDP case, B needs to be selected so that $B+L-2$ is within the region of monotonic convergence of the average $F_Q(m)$, a condition which is always guaranteed since, by definition, $B > 0$. Finally, it has also been observed that, for a given L , B remains almost unchanged with the increase of T for the same distance constraint, provided that $B+L-2$ is outside the end tail of the average $F_Q(m)$.

The performance of the original SD has been compared to that of the SD using the AQRD, i.e. ASD, to show how the latter can approximate the performance of the former with a considerably reduction in preprocessing complexity. Fig. 3 shows the BER performance of the SD and the ASD- B in a 2×2 system with $T = 15$, $L = 3$, uniform PDP and $B = 6, 9$ as a function of the SNR per bit at each receive antenna, $E_b/N_0 = \log_2^{-1}(P)/\sigma^2$. Both 4 and 16-quadrature amplitude modulation (QAM) constellations have been

Fig. 4. BER performance of the SD and the ASD as a function of the E_b/N_0 in a 3×3 system with $T = 10$, 4-QAM and exponential PDP.

used. It can be observed how, in both cases, the ASD can approximate the performance of the SD. However, for the 16-QAM case, the higher constellation order causes the approximation of the AQRD to have a more detrimental effect on the BER, thus requiring a higher value of B to approach the performance of the SD. For the same system, with 4-QAM and at $E_b/N_0 = 14$ dB, Table I shows the average number of real operations of the fixed-complexity preprocessing stage and the variable-complexity tree search stage in SD and the ASD both using the Schnorr-Euchner enumeration. The radius R is initially set to ∞ and updated every time a lattice vector is found inside the hypersphere. The clear reduction in preprocessing complexity only marginally increases the average complexity of the tree search stage, resulting in an overall complexity reduction. It has also been observed that increasing the block size T does not have significant effect on the choice of B for a given system in order to achieve the same relative performance, thus increasing the complexity reduction.

Finally, Fig. 4 shows the effect the number of taps L has on the performance of the ASD in a 3×3 system with $T = 10$, 4-QAM and exponential PDP. In this case, it can be observed how the increase in the number of taps L requires an increase in the value of B if we were to maintain the same relative performance. In particular, it can be observed how when $L = 4$, the performance degradation of the ASD-4 causes its performance to go beyond that of the SD and ASD when $L = 2$, which is a lower-diversity scenario. In addition, it can be seen how the difference in performance between the SD and the ASD-6 increases. Thus, as pointed out at the beginning of this section and in Section II, the number of taps L has a more determinant effect on the choice of B in the ASD than the block size T . It is important to note that due to the errors introduced in the AQRD, the performance of

TABLE I

AVERAGE NUMBER OF REAL OPERATIONS OF THE SD AND THE ASD IN THE SYSTEM OF FIG. 3 WITH 4-QAM AT $E_b/N_0 = 14$ dB.

	SD	ASD-9	ASD-6
Preprocessing stage	41,534	19,010	11,528
Tree search stage	2,239.6	2,330.2	2,768.3
Total	43,773.6	21,340.2	14,296.3

any ASD is expected to have an error floor at high SNR. However, that problem can be overcome in practice by selecting a value of B that places the ASD error floor below the error floor fixed-point quantization error floor.

IV. CONCLUSION

This paper has presented a method to reduce the pre-processing complexity of the SD in frequency-selective single or multiple-antenna channels. The method relies on the convergence of the elements of the QR decomposition of the Toeplitz or block-Toeplitz channel matrix to perform an AQRD, a QR decomposition of a small submatrix of the channel matrix, as opposed to the FQRD originally required by the SD. The size of that submatrix depends on the number of antennas, the block size, the constellation order and the number of taps, with the latter being the most determinant factor in order to achieve a specific level of performance. The resulting ASD approximates the performance of the SD while having a considerably lower preprocessing complexity. In addition, the size of the submatrix can be selected such that the error introduced by the AQRD is lower than that of the fixed-point quantization, effectively removing the effect of the former. This fact and the reduced complexity make the ASD more suited for hardware implementation than the original SD.

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