On Fractional Moments of Multilook Polarimetric Whitening Filter for Polarimetric SAR Data

Salman Khan, *Student Member, IEEE*, and Raffaella Guida, *Member, IEEE*

Abstract—Many multivariate statistical distributions have been derived using the well known product model to stochastically model polarimetric SAR (PolSAR) data. One important factor in their utilisation is the estimation of their texture parameters. Recently, it has been shown that the method of matrix log cumulants (MoMLC) for multilook PolSAR statistical distributions results in estimators with low bias and variance properties. This method is becoming increasingly popular, and can be regarded as state-of-the-art. However, some distributions (e.g. $\mathcal{G}$ distribution) do not have closed form matrix log cumulant (MLC) expressions making the application of MoMLC a challenge. It is therefore desirable to have alternative parameter estimation methods. In this paper, we propose a new estimation method based on fractional moments (FM) of multilook polarimetric whitening filter (MPWF). This results in estimators with mean squared error even lower than the MoMLC based estimators. Also, the mathematical expressions of the estimators are computationally less complicated than MoMLC based estimators. The proposed estimators can be easily derived for all commonly occurring multilook PolSAR distributions, but have been only given for $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{G}^0$ distributions in this paper. Comparisons are made with other known estimators for these distributions using simulated and real PolSAR data. For real data, formal goodness-of-fit (GoF) testing, based on MLC, has been used to assess the fitting accuracy of $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{G}^0$ models using different estimators.

Index Terms—synthetic aperture radar (SAR), radar statistics, multivariate fractional moments, multilook polarimetric whitening filter, polarimetric G distribution.

NOMENCLATURE

ENL	Equivalent number of looks.
FM	Fractional moments.
GIG	Generalized inverse gaussian.
GoF	Goodness of fit.
LC	Log cumulants.
LLF	Log likelihood function.
MC	Multilook complex.
MFM	Multivariate fractional moments.
ML	Maximum likelihood.
MLC	Matrix log cumulants.
MLM	Matrix log moments.
MoLC	Method of log cumulants.
MoMFM	Method of multivariate fractional moments.
MoMLC	Method of matrix log cumulants.
MPWF	Multilook polarimetric whitening filter.
MSE	Mean squared error.
MT	Mellin transform.
PolSAR	Polarimetric synthetic aperture radar.
SAR	Synthetic aperture radar.
SC	Singlelook complex.
SCM	Sample covariance matrix.
TLC	Texture log cumulants.

I. INTRODUCTION

SYNTHETIC Aperture Radar (SAR) data are inherently statistical in nature [1]. This manifests itself in SAR images in the form of *speckle* noise, characterised by random bright and dark spots. Speckle statistics have been well studied in literature [1]. Under certain assumptions [1], [2], the return from a singlelook complex (SC) SAR acquisition follows a zero mean complex gaussian distribution. The gaussian counterparts for singlelook amplitude and intensity are rayleigh and exponential distributions, respectively. In case of multilook gaussian data, the amplitude is square root of gamma, while the intensity is gamma (γ) distributed. In the polarimetric (multi-channel) case, SC speckle follows a multivariate zero mean complex gaussian distribution [3], while multilook complex (MC) speckle follows a matrix-variate scaled complex Wishart distribution [3]. The assumption of gaussian statistics generally holds true, however, many areas in a SAR image like urban, forests, and rough seas show non-gaussian behaviour [4]–[6].

Non-gaussian statistical modeling has traditionally been achieved by employing the *scalar texture product model* or simply the product model [1]. This model assumes that the return signal is a product of two statistically independent random variables, namely, *texture* and *speckle*. Texture is a positive random variable and can be intuitively understood as being analogous to the spatial variation of the normalised radar cross-section, included to statistically model the reflectivity of the scene. We note here that some classical distributions have also been proposed in literature to model single-channel non-gaussianity e.g. Weibull, log normal, and Nakagami-Rice without using the product model [7]. However, the product model framework has been particularly useful to derive compound distributions for SC and MC PolSAR data. Then, the resultant single-channel compound distributions arise as special cases when the dimension is one. The most important texture distributions proposed in literature are γ , inverse gamma (γ^{-1}), generalised inverse gaussian (GIG), Fisher ($\mathcal{F}$), beta (β), and inverse beta (β^{-1}), resulting in the compound distributions $\mathcal{K}$, $\mathcal{G}^0$, $\mathcal{G}$, Kummer- $\mathcal{U}$, $\mathcal{W}$, and $\mathcal{M}$, respectively (see [2], [8]–[11]).

An important factor for the utilisation of these compound distributions is the estimation of corresponding texture pdf parameters. The estimators can generally be classified as mono-pol and polarimetric. The former estimate parameters for each mono-pol channel and average the resulting mono-pol estimates to compute multivariate estimates. The latter utilise fully polarimetric information for estimation, and have been shown to exhibit better bias and variance properties [12]–[14]. Amongst the mono-pol estimators, the ones based on

Frery's fractional moments (FM) [4] and Nicolas' method of log cumulant (MoLC) [15] show similar bias and variance properties [12]. Amongst polarimetric estimators, one type is based on variance of multilook polarimetric whitening filter (MPWF) [16], [17], proposed by Doulgeris et al. [18], [19]. Arguably, one of the most significant developments, in multivariate estimators, has been the application of Mellin transform (MT) to multilook PolSAR data, by Anfinson et al. [12], resulting in method of matrix log cumulants (MoMLC) for multilook PolSAR data. This second type of multivariate estimators exhibit even better bias and variance properties than the former [12], [14].

The MoMLC often results in estimators, which require computation of higher order derivatives of special functions [2], [12]. Mostly, these derivatives have well documented and easily implemented closed form expressions. Then, the MoMLC simply involves simultaneously solving a system of second and higher order matrix log cumulant (MLC) equations (or minimising an objective function based on multiple MLCs). We can identify at least one compound distribution, namely $\mathcal{G}$ distribution, which does not have closed form MLC expressions [14]. This is because its corresponding texture pdf's (GIG) log cumulants (LC) involve first and higher order derivatives of modified Bessel function of the second kind with respect to its order. These derivatives do not have closed form expressions. Although, in [14], it has been shown by the authors that derivatives up to the eighth order can still be efficiently computed numerically with reasonably high accuracy. Hence the MoLCs estimation can be applied to the $\mathcal{G}$ distribution. However, it can be argued that it is always desirable to have closed form expressions for higher accuracy, robustness, and reliability. This is precisely the objective of the current contribution.

In this paper, Doulgeris' multivariate estimators, based on product model decomposition of MPWF, have been revisited. However, instead of using variance, emphasis is given to the use of centralised FMs. FMs have, so far, only been applied to mono-pol estimators. They exhibit low variance and bias properties similar to Nicolas' mono-pol estimators based on MoLC (see [12]). For this purpose, statistical distributions of the speckle part (gaussian) in the product model decomposition of MPWF have been derived. The origins of MPWF speckle distribution can also be traced back to an earlier contribution [16], where the same pdf expression was obtained, although using a different derivation. We observe that this allows us to derive multivariate estimators using centralised FMs of MPWF, intuitively called *multivariate fractional moments* (MFM). The corresponding estimation method is referred to as method of multivariate fractional moments (MoMFM). We further point out that these estimators can be derived in closed form for all the above mentioned texture pdfs, including the GIG pdf. Also, they are computationally efficient and easier to implement as they only require computation of special functions present in the corresponding texture pdf and not their derivatives.

This paper is organised as follows. In Section II, we present the scalar texture product model for SC and MC PolSAR data. Section III describes the MPWF, its product model

decomposition, and its speckle statistics. The ubiquitous PolSAR texture distributions, and their corresponding multilook compound distributions are listed in Section IV. In Section V, MoMFM based estimators of MPWF are compared to various known estimators including MLC based estimators for multilook polarimetric $\mathcal{K}$, $\mathcal{G}^0$, and $\mathcal{G}$ distributions. Application of the proposed estimators to real data is presented in Section VI. Finally, some conclusions are drawn in Section VII, and possible future work is proposed.

II. THE PRODUCT MODEL

The product model takes different but complimentary forms for SC and MC PolSAR data formats. The reason is that the MC PolSAR data contains the second order moments of the scattering coefficients of SC PolSAR data within a multilook window [20]. Let us represent an arbitrary texture pdf as $p_\tau(\tau)$ and make the assumption that the speckle random variate is normalized so that the texture random variable contains the scale information as well. In the following, we briefly explain the product model for SC and MC PolSAR data.

A. Singlelook Complex Polarimetric Scattering Vector

An implicit assumption of the scalar texture product model applied to SC PolSAR data is that the texture in all polarimetric channels is not only identically distributed but also completely correlated. This is a simplification of a more general form of the product model usually referred to as the *multitexture product model* (see [21], [22]), which we do not consider in this paper. The SC polarimetric scattering vector is defined as:

$$\mathbf{k} = [s_{hh} \ s_{hv} \ s_{vh} \ s_{vv}]^T, \quad (1)$$

where s_{xy} represents the complex scattering coefficient with x as transmit, y as receive polarization (h-horizontal, v-vertical), and $[\cdot]^T$ represents the transpose. The SC polarimetric product model is given by:

$$\mathbf{k} = \sqrt{\tau} \mathbf{x}, \quad (2)$$

where $\mathbf{x}$ is a d dimensional speckle vector, which follows a zero mean multivariate complex gaussian distribution, denoted as $\mathbf{x} \sim \mathcal{N}_d^c(\mathbf{0}, \mathbf{\Gamma})$, where $\mathbf{\Gamma} = \mathbf{E}\{\mathbf{x}\mathbf{x}^H\}$ is the covariance matrix of $\mathbf{x}$, and $(\cdot)^H$ represents the Hermitian i.e. conjugate transpose. The pdf is given by [3]:

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^d |\mathbf{\Gamma}|} \exp(-\mathbf{x}^H \mathbf{\Gamma}^{-1} \mathbf{x}), \quad (3)$$

where $|\cdot|$ represents the determinant, and $\mathbf{\Gamma}$ is computed using the sample covariance matrix (SCM), $\mathbf{\Sigma}$, as follows:

$$\begin{aligned} \mathbf{\Sigma} &= \mathbf{E}\{\mathbf{k}\mathbf{k}^H\} = \mathbf{E}\{\tau\} \mathbf{E}\{\mathbf{x}\mathbf{x}^H\} = \mathbf{E}\{\tau\} \mathbf{\Gamma}, \\ \Rightarrow \mathbf{\Gamma} &= \frac{\mathbf{\Sigma}}{\mathbf{E}\{\tau\}}. \end{aligned} \quad (4)$$

It is scale normalized as in [18], such that:

$$|\mathbf{\Gamma}| = 1, \quad (5)$$

$$\Rightarrow \mathbf{E}\{\tau\} = |\mathbf{\Sigma}|^{\frac{1}{d}}, \quad (6)$$

and ensures that $\mathbf{\Gamma}$ contains only polarimetric covariance structure information, while the scale is shifted into the texture random variable.

TABLE I
A SUMMARY OF POLSAR TEXTURE DISTRIBUTIONS AND THE CORRESPONDING COVARIANCE MATRIX COMPOUND DISTRIBUTIONS.

Texture Distribution			Compound Distribution		Ref.
Symbol	$p_{\tau}(\tau)$	$\mathbf{E}\{\tau^{\nu}\}$	Symbol	$p_{\mathbf{C}}(\mathbf{C})$	
$\text{GIG}(\tau; \underline{\alpha}, \underline{\omega}, s)$	$\frac{\tau^{\alpha-1} \exp\left(-\frac{\omega}{2}\left(\frac{s}{\tau} + \frac{\tau}{s}\right)\right)}{2s^{\alpha} K_{\alpha}(\omega)}$ $\tau, \omega, s \in \mathbb{R}^+, \alpha \in \mathbb{R}$	$s^{\nu} \frac{K_{\alpha+\nu}(\omega)}{K_{\alpha}(\omega)}$	$\mathcal{G}(\mathbf{C}; \underline{\alpha}, \underline{\omega}, s, L, \Gamma)$	$\frac{L^{Ld} \mathbf{C} ^{L-d} \left(\frac{2L \text{Tr}(\Gamma^{-1} \mathbf{C}) + \omega s}{\omega/s}\right)^{\frac{\alpha-Ld}{2}}}{\Gamma_d(L) \Gamma ^L s^{\alpha} K_{\alpha}(\omega)} \times$ $K_{\alpha-Ld} \left(\sqrt{\frac{\omega(2L \text{Tr}(\Gamma^{-1} \mathbf{C}) + \omega s)}{s}}\right)$	[10], [14]
$\gamma(\tau; \underline{\alpha}, s)$	$\frac{\tau^{\alpha-1} \exp\left(-\frac{\tau}{s}\right)}{s^{\alpha} \Gamma(\alpha)}$ $\tau, \alpha, s \in \mathbb{R}^+$	$s^{\nu} \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}$	$\mathcal{K}(\mathbf{C}; \underline{\alpha}, s, L, \Gamma)$	$\frac{2L^{Ld} \mathbf{C} ^{L-d} (L \text{Tr}(\Gamma^{-1} \mathbf{C}))^{\frac{\alpha-Ld}{2}}}{\Gamma_d(L) \Gamma ^L s^{\frac{\alpha+Ld}{2}} \Gamma(\alpha)} \times$ $K_{\alpha-Ld} \left(2\sqrt{\frac{L \text{Tr}(\Gamma^{-1} \mathbf{C})}{s}}\right)$	[9], [23]
$\gamma^{-1}(\tau; \underline{\lambda}, s)$	$\frac{\tau^{-\lambda-1} \exp\left(-\frac{s}{\tau}\right)}{s^{-\lambda} \Gamma(\lambda)}$ $\tau, s \in \mathbb{R}^+, \lambda > 1$	$s^{\nu} \frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)}$	$\mathcal{G}^0(\mathbf{C}; \underline{\lambda}, s, L, \Gamma)$	$\frac{L^{Ld} \mathbf{C} ^{L-d} \Gamma(Ld+\lambda) (L \text{Tr}(\Gamma^{-1} \mathbf{C}) + s)^{-\lambda-Ld}}{\Gamma_d(L) \Gamma ^L s^{-\lambda} \Gamma(\lambda)}$	[9], [23]
$\beta(\tau; \underline{\alpha}, \underline{\lambda}, s)$	$\frac{\tau^{\alpha-1} (s-\tau)^{\lambda-\alpha-1}}{s^{\lambda-1} B(\alpha, \lambda-\alpha)}$ $\alpha, \lambda, s \in \mathbb{R}^+, \lambda > \alpha,$ $0 < \tau \leq s$	$s^{\nu} \frac{B(\alpha+\nu, \lambda-\alpha)}{B(\alpha, \lambda-\alpha)}$	$\mathcal{W}(\mathbf{C}; \underline{\alpha}, \underline{\lambda}, s, L, \Gamma)$	$\frac{ \mathbf{C} ^{L-d} \left(\frac{L}{s}\right)^{\frac{\alpha+Ld-1}{2}} (\text{Tr}(\Gamma^{-1} \mathbf{C}))^{\frac{\alpha-Ld-1}{2}}}{\Gamma_d(L) \Gamma ^L} \times$ $\frac{\Gamma(\lambda)}{\Gamma(\alpha)} \exp\left(-\frac{L \text{Tr}(\Gamma^{-1} \mathbf{C})}{2s}\right) \times$ $\mathcal{W}_{\frac{Ld+\alpha+1-2\lambda}{2}, \frac{Ld-\alpha}{2}}\left(\frac{L \text{Tr}(\Gamma^{-1} \mathbf{C})}{s}\right)$	[2], [9]
$\beta^{-1}(\tau; \underline{\alpha}, \underline{\lambda}, s)$	$\frac{\tau^{-\lambda} (\tau-s)^{\lambda-\alpha-1}}{s^{-\alpha} B(\alpha, \lambda-\alpha)}$, $\tau \geq s$, $\alpha, \lambda > 1$, $\lambda \geq \alpha$, $s \in \mathbb{R}^+$	$s^{\nu} \frac{B(\alpha-\nu, \lambda-\alpha)}{B(\alpha, \lambda-\alpha)}$	$\mathcal{M}(\mathbf{C}; \underline{\alpha}, \underline{\lambda}, s, L, \Gamma)$	$\frac{ \mathbf{C} ^{L-d} \left(\frac{L}{s}\right)^{\frac{Ld-\lambda}{2}} (-\text{Tr}(\Gamma^{-1} \mathbf{C}))^{\frac{-Ld-\lambda}{2}}}{\Gamma_d(L) \Gamma ^L} \times$ $\frac{B(\alpha+Ld, \lambda-\alpha)}{B(\alpha, \lambda-\alpha)} \exp\left(-\frac{L \text{Tr}(\Gamma^{-1} \mathbf{C})}{2s}\right) \times$ $\mathcal{M}_{\frac{\lambda-Ld-2\alpha}{2}, \frac{Ld+\lambda-1}{2}}\left(-\frac{L \text{Tr}(\Gamma^{-1} \mathbf{C})}{s}\right)$	[2], [9]

B. Multilook Complex Polarimetric Covariance Matrix

The MC PolSAR data is in a matrix-variate format, which contains real valued second order moments of scattering coefficients along the diagonal and the complex valued second order moments in the off-diagonal. Mathematically, the polarimetric multilooking operation is given by:

$$\mathbf{C} = \frac{1}{L} \sum_{l=1}^L \mathbf{k}_l \mathbf{k}_l^H, \quad (7)$$

where $L \geq d$, and $\mathbf{C} \in \Omega_+ \subset \mathbb{C}^{d \times d}$ is the multilook polarimetric covariance matrix, defined on the cone, Ω_+ , of positive definite complex Hermitian matrices. We have also strictly assumed that the scattering vectors of each independent look, $\mathbf{k}_l$, are zero mean. This is a reasonable assumption for distributed targets without dominant scatterers. The utility of this assumption will become apparent in the next section.

Another implicit assumption in the MC PolSAR case is that the texture is not only identically distributed between all diagonal and off-diagonal elements of $\mathbf{C}$, but is also completely correlated. Hence the product model for the MC polarimetric case can be defined as:

$$\mathbf{C} = \tau \mathbf{X}, \quad (8)$$

where $\mathbf{X}$ is a random speckle matrix. It has been shown that $\mathbf{Y} = L\mathbf{X}$ follows a complex Wishart distribution, denoted as $\mathbf{Y} \sim \mathcal{W}_d^{\mathbb{C}}(L, \Gamma)$, where $\Gamma = \mathbf{E}\{\mathbf{X}\} = \mathbf{E}\{\mathbf{Y}\}/L$ is the speckle covariance matrix. It can be readily derived that $\mathbf{X}$ follows a scaled complex Wishart distribution, denoted as $\mathbf{X} \sim \mathcal{W}_d^{\mathbb{C}}(L, \Gamma) |J_{\mathbf{Y} \rightarrow \mathbf{X}}|$, where $|J_{\mathbf{Y} \rightarrow \mathbf{X}}| = L^{d^2}$ is the Jacobian determinant of the transformation $\mathbf{Y} = L\mathbf{X}$ [24]. The pdf of

$\mathbf{X}$ is given by:

$$p_{\mathbf{X}}(\mathbf{X}) = \frac{L^{Ld} |\mathbf{X}|^{L-d}}{\Gamma_d(L) |\Gamma|^L} \exp\{-L \text{Tr}(\Gamma^{-1} \mathbf{X})\}, \quad (9)$$

where $\text{Tr}(\cdot)$ is the matrix trace operator, and $\Gamma_d(L)$ is the multivariate gamma function of the complex kind, defined as:

$$\Gamma_d(L) = \pi^{d(d-1)/2} \prod_{i=0}^{d-1} \Gamma(L-i), \quad (10)$$

and $\Gamma(\cdot)$ is a standard Euler Gamma function. If we assume that the texture varies spatially on a scale larger than the multilook window, with L independent looks, then the covariance matrix, Γ , can be computed using the SCM, Σ , as follows:

$$\begin{aligned} \Sigma &= \mathbf{E}\{\mathbf{C}\} = \mathbf{E}\{\tau\} \mathbf{E}\{\mathbf{X}\} = \mathbf{E}\{\tau\} \Gamma \\ \Rightarrow \Gamma &= \frac{\Sigma}{\mathbf{E}\{\tau\}}, \end{aligned} \quad (11)$$

and is scale normalized in the same way as it was done in (5), and (6).

An estimate of number of looks called the equivalent number of looks (ENL), can be computed over a homogeneous region in a PolSAR image. For polarimetric data, many estimators for L have been proposed (see [25]). However, we have chosen to use the LCs based Maximum Likelihood (ML) estimator, proposed in [25], because of its superior accuracy. Over a homogeneous area, $\hat{L}$ can be obtained as a solution of the following implicit equation:

$$\mathbf{E}\{\ln |\mathbf{C}|\} = \psi_d^{(0)}(\hat{L}) + \ln |\hat{\Sigma}| - d \ln \hat{L}, \quad (12)$$

where $\psi_d^{(\nu)}(\cdot)$ is the ν -th order multivariate polygamma function of the complex kind, $\psi_d^{(\nu)}(z) = \sum_{i=0}^{d-1} \psi^{(\nu)}(z-i)$, and $\psi^{(\nu)}(\cdot)$ is the well known polygamma function. Note that $\hat{L}$ is not necessarily an integer in this case.

III. MULTILOOK POLARIMETRIC WHITENING FILTER

The main objective of proposing MPWF filter was to optimally reduce the speckle occurring in PolSAR images for improved processing. The MPWF can be defined as [16], [17]:

$$\begin{aligned} M &= \text{Tr}(\Sigma^{-1} \mathbf{C}) \\ &= \frac{1}{L} \sum_{l=1}^L \mathbf{k}_l^H \Sigma^{-1} \mathbf{k}_l. \end{aligned} \quad (13)$$

We note that M is scale invariant due to normalisation by Σ , which can be estimated using the unnormalised SCM estimator, $\hat{\Sigma}$, described in the previous section. Further, we reiterate that the scattering vectors of each independent l -th look, $\mathbf{k}_l$, are assumed to be strictly zero mean. This ensures that each l -th quadratic term $\mathbf{k}_l^H \Sigma^{-1} \mathbf{k}_l$ is already centred in a statistical sense and hence independent of origin. Consequently, this also means that M , i.e. the average over L independent looks, is also centred. This conclusion is very significant as we can directly infer that the raw moments of M are also its central moments under the zero mean assumption. Thus, M represents a scale and origin invariant quantity.

Let us now apply the product model decomposition to M :

$$\begin{aligned} M &= \frac{\tau}{\mathbf{E}\{\tau\}} \frac{1}{L} \sum_{l=1}^L \mathbf{x}_l^H \Gamma^{-1} \mathbf{x}_l \\ &= \frac{\tau}{\mathbf{E}\{\tau\}} \text{Tr}(\Gamma^{-1} \mathbf{X}), \end{aligned} \quad (14)$$

and analyse the distribution of the speckle part i.e. $\frac{1}{L} \sum_{l=1}^L \mathbf{x}_l^H \Gamma^{-1} \mathbf{x}_l$. Using Theorem 2.2 from [26], and the variate relationships of chi-square distribution from [23], we observe that:

$$\begin{aligned} \sum_{l=1}^L \mathbf{x}_l^H \Gamma^{-1} \mathbf{x}_l &\sim \frac{\chi^2(2Ld)}{2} \\ &\sim \gamma(Ld, 1), \end{aligned} \quad (15)$$

$$\Rightarrow \frac{1}{L} \sum_{l=1}^L \mathbf{x}_l^H \Gamma^{-1} \mathbf{x}_l \sim \gamma\left(Ld, \frac{1}{L}\right), \quad (16)$$

where $\chi^2(d)$ is the central chi-square distribution with d degrees of freedom, and $\gamma(d, \mu)$ is the gamma distribution with shape parameter d and scale parameter μ . The same relation has also been derived in the earlier contribution on MPWF by Lopès et al. (1998) [16], where the authors used the Fourier kind characteristic function of $\text{Tr}(\Gamma^{-1} \mathbf{X})$.

We can now list the ν -th moments of M :

$$\mathbf{E}\{M^\nu\} = \frac{\mathbf{E}\{\tau^\nu\}}{\mathbf{E}\{\tau\}^\nu} \left(\frac{1}{L}\right)^\nu \frac{\Gamma(Ld + \nu)}{\Gamma(Ld)}, \quad (17)$$

where $\nu \in \mathbb{R}^+$ and can thus acquire fractional values. When this is the case, we term the moments in (17), the MFM of MPWF. The use of MFM is a novel contribution of this research. It should be noted that the ratio $\mathbf{E}\{\tau^\nu\}/\mathbf{E}\{\tau\}^\nu$ is scale invariant. This is true even for FMs of any arbitrary texture pdf, $p_\tau(\tau)$, if they exist. For all the texture pdfs listed in Section I, FMs exist.

Fig. 1. Theoretical TLCs of various models in 2D TLC diagram.

IV. POLSAR TEXTURE DISTRIBUTIONS

We present here, a summary of the PolSAR texture distributions and their corresponding compound distributions mentioned earlier in Section I. However, we begin by first describing a method of visualising the shape flexibility of texture pdfs. This graphical method is based on a two dimensional (2D) texture LC (TLC) diagram, which is becoming increasingly popular because of its ability to intuitively visualise textures in PolSAR imagery [2], [9], [12], [14].

The 2D TLC diagram plots the third TLC, κ_3 , along the x-axis, against the second TLC, κ_2 , along the y-axis. For the texture distributions mentioned earlier, the theoretical TLC equations can be derived, and have been extensively studied in literature [2], [9], [14], [15]. The second and higher TLCs are independent of scale and can thus give valuable insight into texture shapes which can be attained under a given model. It should be mentioned here, that the dimensions of the manifold spanned by a texture distribution in 2D TLC diagram is equal to the number of shape parameters of the distribution. Consequently, γ and γ^{-1} are represented by lines, and GIG, $\mathcal{F}$, β , and β^{-1} are represented as surfaces. The degenerate textureless case, $\delta(\tau - 1)$ (Dirac delta), is represented by a point.

Figure 1 shows the complete coverage of TLC diagram under various texture distribution models [9]. Interestingly, GIG and Fisher models occupy the same space in 2D TLC diagram [14]. Since GIG does not have closed form LC expressions (although numerically they can still be computed [14]), the contribution of this paper is more justified for GIG compared to Fisher. Hence, for the remainder of this paper, we intentionally choose to analyse GIG instead of Fisher. Nevertheless, similar analysis can be carried out for Fisher.

Table I provides a summary of PolSAR texture distributions and their corresponding covariance matrix compound distributions. The texture pdfs listed are GIG, γ , γ^{-1} , β , and β^{-1} along with their corresponding compound distributions of $\mathcal{G}$, $\mathcal{K}$, $\mathcal{G}^0$, $\mathcal{W}$, and $\mathcal{M}$, respectively. For each model, the following information is included: 1) symbol of texture distribution along with the shape parameters underlined to distinguish them from the scale parameter, 2) texture pdf and the domain of its parameters, 3) moment equation, 4) symbol of MC polarimetric covariance matrix distribution, and 5) its pdf

expression, $p_{\mathbf{C}}(\mathbf{C})$. Some special functions occurring in the definition of the texture and compound distributions should be introduced: 1) $K_{\nu}(\cdot)$ is the modified Bessel function of second kind with order ν , 2) $B(\cdot)$ is the well known beta function, 3) $\mathcal{M}$ is the Whittaker M function, and 4) $\mathcal{W}$ is the Whittaker W function. Finally, the scale invariant $\mathbf{E}\{\tau^{\nu}\}/\mathbf{E}\{\tau\}^{\nu}$ ratios follow trivially from the texture moment equations and are, therefore, omitted. Note that $p_{\mathbf{C}}(\mathbf{C})$ can be readily derived using Bayes' theorem as recently done in [9], and historically done in various earlier contributions (see e.g. [2], [8], [10]).

V. ESTIMATOR PERFORMANCE ANALYSIS

We have utilised Monte Carlo methods to evaluate the performance of the proposed estimators by comparing their bias, variance, and mean squared error (MSE) with previously known estimators. This has been done for MC polarimetric $\mathcal{K}$, $\mathcal{G}^0$, and $\mathcal{G}$ distributions. The $\mathcal{K}$ and $\mathcal{G}^0$ are the most widely used models, while $\mathcal{G}$ does not have closed form LCs and thus alternative estimation methods for its texture parameters are highly desirable. Similar analyses can be performed for $\mathcal{M}$ and $\mathcal{W}$ distributions, but have not been included in this contribution.

A. $\mathcal{K}$ Distribution

The MoMFM based estimator for the texture parameter of $\mathcal{K}$ distribution, denoted as $\hat{\alpha}_{\mathcal{K}1}$ ¹, can be obtained as the solution of the following implicit equation:

$$\mathbf{E}\{M^{\nu}\} = \frac{\Gamma(\hat{\alpha}_{\mathcal{K}1} + \nu) \Gamma(Ld + \nu)}{\hat{\alpha}_{\mathcal{K}1}^{\nu} \Gamma(\hat{\alpha}_{\mathcal{K}1}) L^{\nu} \Gamma(Ld)}, \quad (18)$$

where the moments of γ texture pdf in Table I have been combined in (17). We have used $\nu = \frac{1}{8}$ in our MoMFM estimator. The particular FM chosen is arbitrary, but generally lower FMs exhibit lower variance.

We can now list other known estimators for the $\mathcal{K}$ distribution used in comparison with our MoMFM estimator above. Amongst the polarimetric estimators, we first introduce Doulgeris' estimator [19]:

$$\hat{\alpha}_{\mathcal{D}} = \frac{d(Ld + 1)}{L \text{Var}\{M\} - d}, \quad (19)$$

where $\text{Var}\{\cdot\}$ is the statistical variance. This estimator is similar to our MoMFM estimator in (18) for $\nu=2$, with the only difference being that we use raw moments of M , which, we argue, have already been centred because of the zero mean assumption. Although Doulgeris et al. also used the zero mean assumption, they still explicitly performed centring. We have found exactly the same texture shape parameter values for real data, and also the same biases, variances, and MSE for different sample sizes when using our estimator for $\nu=2$ and Doulgeris' estimator on simulated data.

Now, we define two estimators based on second and higher order theoretical MLCs of $\mathcal{K}$ distribution, proposed by Anfin- sen et al. [12]:

$$\kappa_{\nu>1}\{\mathbf{C}\} = d^{\nu} \psi^{(\nu-1)}(\alpha) + \psi_d^{(\nu-1)}(L), \quad (20)$$

¹The subscript represents the authors who proposed the estimator, and may include a number if there are more than one estimators from the authors.

where $\kappa_{\nu}\{\mathbf{C}\}$ is the ν -th order theoretical MLC, and is replaced by the sample MLC, denoted by $\langle \kappa_{\nu}\{\mathbf{C}\} \rangle$, for estimation purposes. The ν -th order sample MLCs require computing sample matrix log moments (MLM) up to the same order. The sample MLMs are defined as $\mu_{\nu}\{\mathbf{C}\} = \mathbf{E}\{(\ln |\mathbf{C}|)^{\nu}\}$, which are then combined, using simple moment to cumulant transformation equations, to obtain sample MLCs. The transformation equations up to the eighth order can be found in [27]. We note here that, just like the TLCs, the second and higher order MLCs are independent of scale.

The first MLC based estimator, denoted as $\hat{\alpha}_{\mathcal{A}1}$, is based on implicit solution of second order MLC equation [12]:

$$\langle \kappa_2\{\mathbf{C}\} \rangle = d^2 \psi^{(1)}(\hat{\alpha}_{\mathcal{A}1}) + \psi_d^{(1)}(L). \quad (21)$$

The second MLC based estimator, denoted as $\hat{\alpha}_{\mathcal{A}2}$, is based on multiple MLCs and can be defined as [12], [27]:

$$\hat{\alpha}_{\mathcal{A}2} = \arg \left\{ \min_{\alpha} \{d_m^2\} \right\}, \quad (22)$$

where d_m^2 is the squared Mahalanobis distance given by:

$$d_m^2 = (\langle \boldsymbol{\kappa} \rangle - \boldsymbol{\kappa})^T \mathbf{K}^{-1} (\langle \boldsymbol{\kappa} \rangle - \boldsymbol{\kappa}), \quad (23)$$

which contains the sample MLC vector $\langle \boldsymbol{\kappa} \rangle = [\langle \kappa_2, \kappa_3 \rangle]^T$, the mean MLC vector $\boldsymbol{\kappa} = \mathbf{E}\{\langle \boldsymbol{\kappa} \rangle\} = [\kappa_2, \kappa_3]^T$, and the covariance matrix

$$\mathbf{K} = \text{Cov}\{\langle \boldsymbol{\kappa} \rangle\} = \begin{bmatrix} \kappa_4 + 2\kappa_2^2 & \kappa_5 + 6\kappa_2\kappa_3 \\ \kappa_5 + 6\kappa_2\kappa_3 & \kappa_6 + 9\kappa_2\kappa_4 + 9\kappa_3^2 + 6\kappa_2^3 \end{bmatrix}. \quad (24)$$

For a given sample, the minimisation is performed by varying $\boldsymbol{\kappa}$ and $\mathbf{K}$, both of which depend on α through the theoretical MLCs in (20).

The next polarimetric estimator, $\hat{\alpha}_{\mathcal{K}2}$, is based on numerically maximizing the log likelihood function (LLF) of polarimetric $\mathcal{K}$ distribution. It was proposed by the authors (Khan et al.) in [11] for SC case and has been extended to MC case here. The LLF is given by:

$$\ell(\hat{\alpha}_{\mathcal{K}2}, \hat{\mathbf{s}}_{\mathcal{K}2} | \mathbf{S}, L, \Gamma) = N \left[\ln 2 - \ln \Gamma(\hat{\alpha}_{\mathcal{K}2}) - \left(\frac{\hat{\alpha}_{\mathcal{K}2} + Ld}{2} \right) \ln \hat{\mathbf{s}}_{\mathcal{K}2} \right] + \sum_{i=1}^N \left[\left(\frac{\hat{\alpha}_{\mathcal{K}2} - Ld}{2} \right) \ln L(q_i) + \ln \left(K_{\hat{\alpha}_{\mathcal{K}2} - Ld} \left(2 \sqrt{\frac{Lq_i}{\hat{\mathbf{s}}_{\mathcal{K}2}}} \right) \right) \right], \quad (25)$$

where $\mathbf{S} = \{\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_N\}$ is a given sample of polarimetric covariance matrices, and $q_i = \text{Tr}(\Gamma^{-1} \mathbf{C}_i)$ for $i \in \{1, 2, \dots, N\}$. The negative of the LLF in (25) can be minimized for $\hat{\alpha}_{\mathcal{K}2}$. At each iteration of minimizer, the scale parameter $\hat{\mathbf{s}}_{\mathcal{K}2}$ is computed using the first moment equation of γ texture distribution in Table I with the mean of texture given in (6). The minimization algorithm used is the Nelder-Mead Simplex [28], which is a well known direct search method for multidimensional minimization of an objective function. It attempts to minimize the real valued objective function without utilizing any derivative information, and is only granted to converge to a global minimum in one dimension [28]. In this paper, a relative convergence criterion of the negative LLF value less than 10^{-14} is used as the stopping condition in

Fig. 2. Estimator bias (left), variance (middle), and MSE (right) for $\mathcal{K}$ distribution texture parameter α as a function of sample size at $L=10$, and $\alpha=10$.

the Simplex algorithm. Even with such a strict convergence criterion, it is observed that for simulated MC polarimetric data with $\alpha=10$, and $L=10$, a single run of the algorithm always converges for sample size greater than or equal to 32.

Finally, we present the only mono-pol estimator used in our comparison. This estimator, denoted as $\hat{\alpha}_F$, was proposed by Frery et al. [29]. It is based on combining half and quarter moments of mono-pol intensity I , and is given by:

$$\frac{\Gamma^2(\hat{\alpha}_F + \frac{1}{4})}{\Gamma(\hat{\alpha}_F) \Gamma(\hat{\alpha}_F + \frac{1}{2})} \frac{\Gamma^2(L + \frac{1}{4})}{\Gamma(L) \Gamma(L + \frac{1}{2})} - \frac{\langle I^{\frac{1}{4}} \rangle^2}{\langle I^{\frac{1}{2}} \rangle} = 0, \quad (26)$$

which can be solved to obtain an estimate of $\hat{\alpha}_F$ for each mono-pol intensity channel. The final estimate results from the average of the mono-pol estimates.

Figure 2 shows the absolute value of bias, variance, and MSE of all the estimators after 5000 Monte Carlo simulations with $\alpha=10$, and $L=10$. The absolute value has been taken since the biases of $\hat{\alpha}_{A2}$ estimator at certain sample sizes were found to be negative. We can clearly observe that the mono-pol estimator, $\hat{\alpha}_F$, performs the worst. Amongst the polarimetric estimators, $\hat{\alpha}_D$, has the worst bias, variance, and MSE properties. The MLC based estimators ($\hat{\alpha}_{A1}$, $\hat{\alpha}_{A2}$) generally exhibit the lowest bias. The $\hat{\alpha}_{A2}$ estimator has overall the lowest bias but it is computationally expensive. The bias of MoMFM and MLE based estimators ($\hat{\alpha}_{K1}$, $\hat{\alpha}_{K2}$, respectively) is only slightly higher than that of $\hat{\alpha}_{A1}$. Interestingly, $\hat{\alpha}_{K1}$, $\hat{\alpha}_{K2}$ also show the lowest variance. The MSE, which is a sum of the variance and squared bias, provides a more comprehensive means of assessing the performance of an estimator. The better overall performance of $\hat{\alpha}_{K1}$, and $\hat{\alpha}_{K2}$ estimators manifests through lower MSE compared to all other estimators. Note, $\hat{\alpha}_{K2}$ estimator is computationally very expensive as it is directly dependent on the sample size. It should be emphasized that the $\hat{\alpha}_{K1}$ estimator outperforms $\hat{\alpha}_D$ because of utilising MFM instead of higher moments. This highlights the advantage of using MFM. We have also observed similar results at other values of α , and L .

B. $\mathcal{G}^0$ Distribution

We can readily list the counterparts of K1, D, A1, A2, K2, and F estimators for the polarimetric $\mathcal{G}^0$ distribution. The

MoMFM based estimator, $\hat{\lambda}_{K1}$, is given by:

$$\mathbf{E}\{M^\nu\} = \frac{\Gamma(\hat{\lambda}_{K1} - \nu)(\hat{\lambda}_{K1} - 1)^\nu \Gamma(Ld + \nu)}{\Gamma(\hat{\lambda}_{K1}) L^\nu \Gamma(Ld)}, \quad (27)$$

where we again use $\nu = \frac{1}{8}$. The Douglgeris' estimator, $\hat{\lambda}_D$, is given by:

$$\hat{\lambda}_D = \frac{2L \text{Var}\{M\} + d(Ld - 1)}{L \text{Var}\{M\} - d}. \quad (28)$$

The next two estimators are based on second and higher order theoretical MLCs of $\mathcal{G}^0$ distribution, proposed by Anfinset et al. [2], [12]:

$$\kappa_{\nu>1}\{\mathbf{C}\} = (-d)^\nu \psi^{(\nu-1)}(\lambda) + \psi_d^{(\nu-1)}(L). \quad (29)$$

Using the above equation at $\nu=2$, we can define, $\hat{\lambda}_{A1}$, as the solution of:

$$\langle \kappa_2\{\mathbf{C}\} \rangle = d^2 \psi^{(1)}(\hat{\lambda}_{A1}) + \psi_d^{(1)}(L). \quad (30)$$

The second MLC based estimator, $\hat{\lambda}_{A2}$, can be obtained in exactly the same way as done in the previous section for $\mathcal{K}$ distribution (see (22)-(24)). However, the theoretical MLCs for $\mathcal{G}^0$ distribution (29) are instead used.

The MLE based estimator, $\hat{\lambda}_{K2}$, is again an extension of the one proposed by the authors for SC case in [11]. It is based on maximising the LLF:

$$\begin{aligned} \ell(\hat{\lambda}_{K2}, \hat{s}_{K2} | \mathbf{S}, L, \mathbf{\Gamma}) &= N \left[\ln(\Gamma(Ld + \hat{\lambda}_{K2})) - \ln(\Gamma(\hat{\lambda}_{K2})) \right. \\ &\quad \left. + \hat{\lambda}_{K2} \ln \hat{s}_{K2} \right] + \sum_{i=1}^N \left[(-\hat{\lambda}_{K2} - Ld) \ln(Lq_i + \hat{s}_{K2}) \right]. \end{aligned} \quad (31)$$

In practice, the negative of LLF is minimized using Simplex algorithm with the same stopping criterion as the one used for the $\mathcal{K}$ distribution. Again, a single run of the algorithm always converged for $\lambda=10$, and $L=10$ with sample size greater than or equal to 32.

Finally, Frery's FM based mono-pol estimator for $\mathcal{G}^0$, $\hat{\lambda}_F$, can be defined as:

$$\frac{\Gamma^2(\hat{\lambda}_F - \frac{1}{4})}{\Gamma(\hat{\lambda}_F) \Gamma(\hat{\lambda}_F - \frac{1}{2})} \frac{\Gamma^2(L + \frac{1}{4})}{\Gamma(L) \Gamma(L + \frac{1}{2})} - \frac{\langle I^{\frac{1}{4}} \rangle^2}{\langle I^{\frac{1}{2}} \rangle} = 0. \quad (32)$$

Fig. 3. Estimator bias (left), variance (middle), and MSE (right) for $\mathcal{G}^0$ distribution texture parameter λ as a function of sample size at $L=10$, and $\lambda=10$.

Figure 3 shows the absolute value of bias, variance, and MSE of all the estimators of $\mathcal{G}^0$ texture parameter after 6000 Monte Carlo simulations with $\lambda=10$, and $L=10$. The results observed are very similar to the $\mathcal{K}$ distribution case. One exception is that the bias of $\hat{\lambda}_{K1}$ is not as close to that of $\hat{\lambda}_{A1}$ as it was observed for the corresponding shape parameter estimators ($\hat{\alpha}_{K1}$, $\hat{\alpha}_{A1}$) for the $\mathcal{K}$ distribution. Also, the variance of $\hat{\lambda}_{K1}$ is closer to that of $\hat{\lambda}_{A1}$, although still lower. However, the lower MSE again highlights the overall better performance of $\hat{\lambda}_{K1}$, and $\hat{\lambda}_{K2}$ compared to all other estimators. The $\hat{\lambda}_{K2}$ is computationally very expensive as it is directly dependent on the sample size. It is again emphasized that $\hat{\lambda}_{K1}$ outperforms $\hat{\lambda}_D$ because of utilizing MFM instead of higher moments. We have also observed similar results at other values of λ , and L .

C. $\mathcal{G}$ Distribution

The $\mathcal{G}$ distribution has two texture shape parameters, α and ω . The general form of the equation for the MoMFM based estimator, denoted as $\hat{\alpha}_{K1}$, $\hat{\omega}_{K1}$, is given by:

$$\mathbf{E}\{M^\nu\} = \frac{K_{\hat{\alpha}_{K1}+\nu}(\hat{\omega}_{K1}) K_{\hat{\alpha}_{K1}}^{\nu-1}(\hat{\omega}_{K1}) \Gamma(Ld+\nu)}{K_{\alpha+1}^\nu(\hat{\omega}_{K1}) L^\nu \Gamma(Ld)}, \quad (33)$$

where the moments of GIG texture pdf in Table I have been combined in (17). Since we are estimating two parameters, two equations derived from (33), for $\nu=\frac{1}{8}, \frac{1}{4}$, can be solved simultaneously to estimate the parameters.

The MLC based estimator, denoted as $\hat{\alpha}_{A1}$, $\hat{\omega}_{A1}$, is the one we have proposed in [14]. It results from the simultaneous solution of:

$$\langle \kappa_2\{\mathbf{C}\} \rangle = d^2 \ln K_{\hat{\alpha}_{A1}}^{(2)}(\hat{\omega}_{A1}) + \psi_d^{(1)}(L), \quad (34)$$

$$\langle \kappa_3\{\mathbf{C}\} \rangle = d^3 \ln K_{\hat{\alpha}_{A1}}^{(3)}(\hat{\omega}_{A1}) + \psi_d^{(2)}(L), \quad (35)$$

where $\ln K_\alpha^{(\nu)}(\omega)$ is the ν -th derivative, with respect to order, of the logarithm of modified Bessel function of the second kind. No special function to compute $\ln K_\alpha^{(\nu)}(\omega)$ exists. In [14], we have numerically computed these derivatives up to the eighth order with reasonable accuracy, using the extended Neville's algorithm (see [14] for further details).

The MLE based estimator, denoted as $\hat{\alpha}_{K2}$, $\hat{\omega}_{K2}$, is the one

we have proposed in [14]. It is based on maximising the LLF:

$$\begin{aligned} \ell(\hat{\alpha}_{K2}, \hat{\omega}_{K2}, \hat{s}_{K2} | \mathbf{S}, L, \mathbf{\Gamma}) = & N \left[-\hat{\alpha}_{K2} \ln(\hat{s}_{K2}) - \ln\{K_{\hat{\alpha}_{K2}}(\hat{\omega}_{K2})\} \right] \\ & + \sum_{i=1}^N \left[\left(\frac{\hat{\alpha}_{K2} - Ld}{2} \right) \left[\ln(2Lq_i + \hat{\omega}_{K2}\hat{s}_{K2}) - \ln\left(\frac{\hat{\omega}_{K2}}{\hat{s}_{K2}}\right) \right] \right. \\ & \left. + \ln \left[K_{\hat{\alpha}_{K2}-Ld} \left(\sqrt{\frac{\hat{\omega}_{K2}}{\hat{s}_{K2}}} (2Lq_i + \hat{\omega}_{K2}\hat{s}_{K2}) \right) \right] \right]. \end{aligned} \quad (36)$$

The negative of LLF is minimized using Simplex algorithm with the same stopping criterion as the one used for $\mathcal{K}$, and $\mathcal{G}^0$ distributions. Also, a single run of the algorithm always converged for $\alpha=10$, $\omega=10$, and $L=10$ with sample size greater than or equal to 32.

Finally, we list the mono-pol estimator, denoted as $\hat{\alpha}_F$, $\hat{\omega}_F$, based on combining the first moment and fractional moments of mono-pol intensity [14]. It results from simultaneously solving:

$$\frac{K_{\hat{\alpha}_F+\frac{1}{4}}^2(\hat{\omega}_F)}{K_{\hat{\alpha}_F}(\hat{\omega}_F) K_{\hat{\alpha}_F+\frac{1}{2}}(\hat{\omega}_F)} \frac{\Gamma^2(L+\frac{1}{4})}{\Gamma(L)\Gamma(L+\frac{1}{2})} - \frac{\langle I^{\frac{1}{4}} \rangle^2}{\langle I^{\frac{1}{2}} \rangle} = 0, \quad (37)$$

$$\frac{K_{\hat{\alpha}_F+\frac{1}{2}}^2(\hat{\omega}_F)}{K_{\hat{\alpha}_F}(\hat{\omega}_F) K_{\hat{\alpha}_F+1}(\hat{\omega}_F)} \frac{\Gamma^2(L+\frac{1}{2})}{\Gamma(L)\Gamma(L+1)} - \frac{\langle I^{\frac{1}{2}} \rangle^2}{\langle I \rangle} = 0. \quad (38)$$

Figure 4 shows the absolute value of bias, variance, and MSE of all estimators of $\mathcal{G}$ distribution texture parameters after 5030 Monte Carlo simulations with $\alpha=10$, $\omega=10$, and $L=10$. The absolute value has been taken since most of the biases were found to be negative with only a few exceptions. For the F estimator, at sample sizes 32 and 64, the biases and variances were too high and, therefore, have been omitted. Even at a larger sample size, the mono-pol estimator shows the highest bias. Amongst the polarimetric estimators, the A1 estimator has the highest bias for α , and for ω its estimator bias is only slightly lower than the K2 estimator bias. Also, the variance of the A1 estimator for α , and ω is higher than both the K1 and K2 estimators. It can be easily seen that the K1 estimator has the lowest bias. Also, the variance of the K1 estimator is very similar to the K2 estimator, which generally shows the lowest variance, with only a few exceptions. Analyzing

Fig. 4. Estimator absolute value of bias (left), variance (middle), and MSE (right) for $\mathcal{G}$ distribution texture parameters α (top row) and ω (bottom row) as a function of sample size at $L=10$. True value: $\alpha=10$, and $\omega=10$.

the MSE results, it is observed that the K1 and K2 estimators exhibit the lowest and very similar MSE, followed by a higher MSE of the A1 estimator, and generally the worst MSE of the mono-pol F estimator. Therefore, the K1 and K2 estimators perform better by exhibiting relatively lower bias, variance, and MSE characteristics compared to other known estimators. However, the K2 estimator suffers from high computational complexity as it is directly dependent on the sample size. Further, the A1 estimator requires numerical computation of $\ln K_{\alpha}^{(\nu)}(\omega)$, which is not very desirable. Similar results have been observed at other values of α , ω , and L .

VI. APPLICATION TO REAL POLSAR DATA

Let us now apply our new estimator to real data. We have chosen two PolSAR datasets for this purpose. The first dataset has been acquired using TerraSAR-X quad-pol mode over Amsterdam. It has been further multilooked to 7.5 ENLs. The second dataset is tripolar, and has been acquired using an airborne S-band SAR demonstrator operated by Astrium, U.K. The essence of this campaign was to explore the potential of S-band in SAR technology keeping in perspective NovaSAR-S, a spaceborne S-band SAR sensor due for launch in 2015. This dataset has been multilooked to 5.25 ENLs. Before proceeding to the statistical analysis of these two datasets, let us briefly present an overview of assessing the Goodness-of-fit (GoF) of the compound models (listed in Table I) to PolSAR data.

A. Goodness of Fit using Log Cumulants

GoF testing for PolSAR statistical models has been traditionally done by comparing single channel amplitude/intensity

histograms to the corresponding single-channel forms of the models. This procedure is repeated for each channel to complete GoF assessment for multivariate PolSAR data. These methods are limited in the sense that they do not utilise the complete polarimetric information contained in the covariance matrix. Recently, GoF procedures using multiple MLCs, specially developed for multivariate PolSAR data, have been proposed [27]. These offer a truly polarimetric approach, where a single test statistic is obtained for the multivariate PolSAR data. Also, this approach captures more statistical information as it computes GoF statistic using multiple MLCs. Let us list the main results of this GoF testing scheme in the simple hypothesis case i.e. when the model parameters are considered known (for more details see [27]).

Let $\langle \kappa \rangle$ be a p dimensional vector of sample MLCs of selected orders $\{v_1, v_2, \dots, v_p\}$:

$$\langle \kappa \rangle = [\langle \kappa_{v_1} \rangle, \langle \kappa_{v_2} \rangle, \dots, \langle \kappa_{v_p} \rangle]^T, \quad (39)$$

with mean vector κ , defined as:

$$\mathbf{E}\{\langle \kappa \rangle\} = \kappa = [\kappa_{v_1}, \kappa_{v_2}, \dots, \kappa_{v_p}]^T. \quad (40)$$

It was shown in [27] that for sample size n :

$$\sqrt{n}(\langle \kappa \rangle - \kappa) \xrightarrow{\mathcal{D}} \mathcal{N}_p(\mathbf{0}, \mathbf{K}), \quad (41)$$

where $\mathbf{K}$ is the scaled covariance matrix, given by:

$$\mathbf{K} = n\mathbf{E}\{(\langle \kappa \rangle - \kappa)(\langle \kappa \rangle - \kappa)^T\}. \quad (42)$$

The mean vector κ is constructed using the corresponding p population MLCs of the hypothesized model. The $\mathbf{K}$ matrix

Fig. 5. (Top) A portion of TerraSAR-X polarimetric image over Amsterdam (7.5 equivalent number of looks) shown in false color Pauli decomposition © DLR 2012. (Middle) Four sample images extracted over different homogeneous areas. (Bottom) Texture log cumulant diagram showing sample log cumulants and color-coded bootstrapped samples of the extracted areas.

requires MLCs up to order $2v_{\max} = 2 \cdot \max\{v_1, v_2, \dots, v_p\}$. The equation to construct $\mathbf{K}$ matrix is given in the appendix of [27]. When only the second and third order MLCs are used to form κ , the expression for the corresponding $\mathbf{K}$ matrix is given by (24).

Now, a test statistic, Q_p , can be formulated as:

$$Q_p = n (\langle \kappa \rangle - \kappa)^T \mathbf{K}^{-1} (\langle \kappa \rangle - \kappa), \quad (43)$$

which uses p sample MLCs. It was shown in [27], that Q_p follows a χ^2 distribution with p degrees of freedom:

$$Q_p \xrightarrow{D} \chi^2(p). \quad (44)$$

Therefore, statistical testing can be done and the corresponding p value can be computed. It should be noted that the number of MLCs required by GoF test is at least one more than the number of texture shape parameters [27]. We have used second, third, and fourth MLCs to construct the mean vector κ , and therefore MLCs up to the eighth order are required to construct the $\mathbf{K}$ matrix.

B. Results

Figures 5 and 6 show the statistical analysis on Amsterdam and Pembrokeshire datasets. The results have been arranged in a manner similar to [14], [27] for consistency and clarity. The first row in both figures shows a subset image, displayed in false color Pauli decomposition [1]. Four different samples (256 pixels each), highlighted as tiny color-coded squares in the subset images, are extracted from each dataset. The zoomed sample areas are shown in the middle row. They

TABLE II
GoF P VALUES (%) OF AMSTERDAM DATA.

	Urban A	Urban B	Vegetation	Water
$\mathcal{G}$ Distribution				
$\mathcal{G}_{A1}$	64.09	23.66	88.39	59.42
$\mathcal{G}_F$	0	0	0	0
$\mathcal{G}_{K1}$	0.44	3.12	1.12	0.71
$\mathcal{G}_{K2}$	0.08	10.15	1.04	0.65
$\mathcal{G}^0$ Distribution				
$\mathcal{G}_{A1}^0$	2.37	52.97	87.75	58.11
$\mathcal{G}_{A2}^0$	7.03	55.08	89.60	63.84
$\mathcal{G}_F^0$	0	0	0	0
$\mathcal{G}_{K1}^0$	0	3.12	1.25	0.79
$\mathcal{G}_{K2}^0$	0.02	10.15	1.21	0.78
$\mathcal{G}_D^0$	0.68	2.17	1.10	0.85
$\mathcal{K}$ Distribution				
$\mathcal{K}_{A1}$	13.16	0.03	85.55	57.86
$\mathcal{K}_{A2}$	23.04	0.39	87.27	63.52
$\mathcal{K}_F$	0	0	0	0
$\mathcal{K}_{K1}$	16.09	0.09	1.03	0.67
$\mathcal{K}_{K2}$	5.32	0.29	1.05	0.65
$\mathcal{K}_D$	12.10	0	1.02	0.80

have been selected such that they are as homogeneous as possible so as to keep the statistics stationary. The bottom row shows TLCs of each sample, plotted using '+' symbol in TLC diagram. For each sample, it also shows multiple color-coded TLCs of bootstrapped samples. These are obtained using 128 bootstrap samples each of size 128 from the 256-pixel sample images. This graphically gives an idea of the statistical variance of sample TLCs. The bootstrapped TLCs are enclosed by 95% confidence ellipses drawn using 2×2 $\mathbf{K}$ matrices, given in eq. (24).

Let us first analyse the TerraSAR-X Amsterdam dataset in fig. 5. We have selected one water (orange), one vegetation (magenta), and two urban samples; urban A (cyan) and urban B (red). The TLCs of water and vegetation samples overlap each other and show almost no texture (gaussian). Urban A sample TLCs lie inside GIG/Fisher domain, while urban B sample TLCs accumulate just outside the γ^{-1} manifold.

Tables II lists the GoF p values (%) obtained for the four selected samples from Amsterdam dataset using the polarimetric $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions. It is important to note that the p values of MLCs based estimators will generally be higher than other estimators. This is because both parameter estimation and GoF testing is performed in the MLC domain. We observe that the mono-pol estimators $\mathcal{G}_F$, $\mathcal{G}_F^0$, and $\mathcal{K}_F$ show a p value of zero for all the four samples. For the water sample, $\mathcal{G}_{A2}^0$ shows the highest p value of 63.84%, followed closely by $\mathcal{K}_{A2}$, $\mathcal{G}_{A1}$, $\mathcal{G}_{A1}^0$, and $\mathcal{K}_{A1}$ at 63.52%, 59.42%, 58.11%, and 57.86%, respectively. The non-MLC polarimetric estimators show very similar p values ranging between 0.65% for $\mathcal{K}_{K2}$, $\mathcal{G}_{K2}$, and 0.85% for $\mathcal{G}_D^0$. For the vegetation sample exactly the same trend in p values is observed for the MLC based estimators. Amongst the non-MLC based polarimetric estimators, the p values range from 1.02% for $\mathcal{K}_D$ to 1.25% for $\mathcal{G}_{K1}^0$. For urban B sample, the highest p values of 55.08% and 52.97% are obtained for $\mathcal{G}_{A2}^0$ and $\mathcal{G}_{A1}^0$, respectively, followed by a value of 23.66% for $\mathcal{G}_{A1}$. Amongst non-MLC based polarimetric estimators the highest value of 10.15% is shared between $\mathcal{G}_{K2}$ and $\mathcal{G}_{K2}^0$, followed by a value of 3.12% shared

Fig. 6. (Top) A portion of Astrium demonstrator SAR polarimetric image over Pembrokeshire (5.25 equivalent number of looks) shown in false color Pauli decomposition © Astrium 2012. (Middle) Four sample images extracted over different homogeneous areas. (Bottom) Texture log cumulant diagram showing sample log cumulants and color-coded bootstrapped samples of the extracted areas.

between $\mathcal{G}_{K1}$ and $\mathcal{G}_{K1}^0$. Finally, for urban A sample the highest p values of 64.09% is obtained for $\mathcal{G}_{A1}$ amongst the MLC based estimators. Whereas, for non-MLC based estimators, the highest p value of 16.09% is obtained for $\mathcal{K}_{K1}$.

Now, let us analyse the S-band Pembrokeshire dataset in fig. 6. We have selected one water (orange), one urban (red), and two vegetation samples; vegetation A (cyan) and vegetation B (magenta). The sample TLCs of water and vegetation A samples overlap each other and show almost no texture (gaussian). Both vegetation B and urban sample TLCs lie inside GIG/Fisher domain close to the γ^{-1} manifold.

Tables III list the GoF p values (%) obtained for the four selected samples from Pembrokeshire dataset using the polarimetric $\mathcal{G}$, $\mathcal{G}^0$, and $\mathcal{K}$ distributions, respectively. For water sample, the MLC based estimators $\mathcal{G}_{A2}^0$, $\mathcal{G}_{A1}$, and $\mathcal{G}_{A1}^0$ show the highest p values of 20.25%, 20.20%, and 20.20%, respectively. Amongst non-MLC based estimators, the mono-pol estimators $\mathcal{G}_F$, $\mathcal{G}_F^0$, and $\mathcal{K}_F$ show reasonably high p values of 19.83%, 5.12%, and 4.92%, respectively, while the remaining polarimetric estimators show very low p values ranging from 0.01% for $\mathcal{K}_D$ to a maximum of 0.24% shared between $\mathcal{G}_{K1}^0$ and $\mathcal{K}_{K2}$. For vegetation B sample, the $\mathcal{K}$ distribution shows the worst fitting for all estimators, with the highest p value of 0.88%. The mono-pol $\mathcal{G}_F$ estimator shows a low p value of 0.10%, while all the other estimators of $\mathcal{G}$ and $\mathcal{G}^0$ distributions show very high p values, with the maximum being 96.11% for $\mathcal{G}_{A1}$, followed by a value of 94.63% for $\mathcal{G}_{K1}$. For vegetation A sample, the mono-pol estimators $\mathcal{G}_F$, $\mathcal{G}_F^0$, and $\mathcal{K}_F$ show very low p values with the highest being 1.96%. All MLC based estimators show p values

TABLE III
GoF P VALUES (%) OF PEMBROKESHIRE DATA.

	Urban	Vegetation A	Vegetation B	Water
$\mathcal{G}$ Distribution				
$\mathcal{G}_{A1}$	48.23	64.25	96.11	20.20
$\mathcal{G}_F$	0.06	0.42	0.10	19.83
$\mathcal{G}_{K1}$	21.44	17.49	94.63	0.22
$\mathcal{G}_{K2}$	29.07	17.98	94.28	0.16
$\mathcal{G}^0$ Distribution				
$\mathcal{G}_{A1}^0$	77.64	64.25	79.02	20.20
$\mathcal{G}_{A2}^0$	81.33	64.55	81.22	20.25
$\mathcal{G}_F^0$	8.51	1.65	75.05	5.12
$\mathcal{G}_{K1}^0$	63.63	17.67	72.42	0.24
$\mathcal{G}_{K2}^0$	52.83	20.48	81.26	0.16
$\mathcal{G}_D^0$	25.09	24.31	20.99	0.02
$\mathcal{K}$ Distribution				
$\mathcal{K}_{A1}$	0	64.25	0.05	20.20
$\mathcal{K}_{A2}$	0	64.54	0.78	20.25
$\mathcal{K}_F$	0	1.96	0.88	4.92
$\mathcal{K}_{K1}$	0	17.27	0.76	0.21
$\mathcal{K}_{K2}$	0.02	17.98	0.37	0.24
$\mathcal{K}_D$	0	23.06	0.15	0.01

of approximately 64%, while the non-MLC based polarimetric estimators show reasonably high p values ranging between 17.27% to 24.31%. Finally, the $\mathcal{K}$ distribution again shows the worst fitting for urban sample for all estimators. Amongst the mono-pol estimators, only $\mathcal{G}_F^0$ shows a slightly higher p value of 8.51%. All the remaining $\mathcal{G}$ and $\mathcal{G}^0$ estimators show high p values, with $\mathcal{G}_{K1}^0$ showing the highest p value of 63.63% amongst non-MLC based polarimetric estimators.

We can draw some important inferences from the above results: 1) the non-MLC based polarimetric estimators generally show higher p values, and therefore better estimation performance, compared to Frery's mono-pol estimators, and 2) amongst these polarimetric estimators the proposed K1 estimators generally exhibit better estimation performance by acquiring higher p values than Doulgeris' estimators, with only a few exceptions. The K2 estimator is computationally expensive and, therefore, only usable for very small sample size.

VII. CONCLUSION

In this paper we have proposed a new method for estimating texture parameters of compound distributions based on product model using fractional moments of multilook polarimetric whitening filter (MPWF). For this purpose, we have derived the distribution of the speckle part (gaussian part) of MPWF, which has also been done earlier in [16]. The proposed estimators can be derived for all commonly occurring compound distributions of multilook PolSAR data. Analyses on synthetic data show, that for $\mathcal{K}$ and $\mathcal{G}^0$ models, the variance and MSE of the new estimators is clearly lower, and the bias is only slightly higher, than MLC based estimators. For the $\mathcal{G}$ model, the new estimators noticeably outperform MLC based estimators in terms of bias, variance, and MSE. Moreover, their mathematical expressions are less complicated than MLC based estimators. Therefore, they can serve as a valuable alternative method of parameter estimation, specially for compound distributions which do not have closed form MLC expressions, e.g. the $\mathcal{G}$ distribution. Results on real data

also show that the new estimators compete well with other non-MLC based polarimetric estimators.

In the future, the proposed estimators will be utilised in various PolSAR image processing algorithms like classification, segmentation, and target detection. Also, the theory will be extended to singlelook PolSAR data using polarimetric whitening filter.

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REFERENCES

- [1] J.-S. Lee and E. Pottier, *Polarimetric Radar Imaging: From Basics to Applications*. Boca Raton, FL: CRC Press, Taylor & Francis Group, 2009.
- [2] S. Anfinsen, "Statistical analysis of multilook polarimetric radar images with Mellin transform," Ph.D. dissertation, University of Tromsø, Tromsø, May 2010.
- [3] N. R. Goodman, "Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction)," *Ann. Math. Statist.*, vol. 34, no. 1, pp. 152–177, Mar. 1963.
- [4] A. Frery, H.-J. Muller, C. Yanasse, and S. Sant'Anna, "A model for extremely heterogeneous clutter," *IEEE Trans. Geosci. Remote Sens.*, vol. 35, no. 3, pp. 648–659, May 1997.
- [5] T. Eltoft and K. Hogda, "Non-gaussian signal statistics in ocean SAR imagery," *IEEE Trans. Geosci. Remote Sens.*, vol. 36, no. 2, pp. 562–575, Mar. 1998.
- [6] C. Oliver, "Rain forest classification based on SAR texture," *IEEE Trans. Geosci. Remote Sens.*, vol. 38, no. 2, pp. 1095–1104, Mar. 2000.
- [7] C. Tison, J.-M. Nicolas, F. Tupin, and H. Maitre, "A new statistical model for Markovian classification of urban areas in high-resolution SAR images," *IEEE Trans. Geosci. Remote Sens.*, vol. 42, no. 10, pp. 2046–2057, Oct. 2004.
- [8] J. Lee, D. Schuler, R. Lang, and K. Ranson, "K-distribution for multilook processed polarimetric SAR imagery," in *Proc. IGARSS*, vol. 4, Pasadena, CA, Aug. 1994, pp. 2179–2181.
- [9] L. Bombrun, S. Anfinsen, and O. Harant, "A complete coverage of log-cumulant space in terms of distributions for polarimetric SAR data," in *Proc. PolInSAR*, Frascati, Italy, 2011.
- [10] C. Freitas, A. Frery, and A. Correia, "The polarimetric $\mathcal{G}$ distribution for SAR data analysis," *Environmetrics*, vol. 16, no. 1, pp. 13–31, Feb. 2005.
- [11] S. Khan and R. Guida, "On single-look multivariate $\mathcal{G}$ distribution for PolSAR data," *IEEE J. Sel. Topics Appl. Earth Observations Remote Sens.*, vol. 5, no. 4, pp. 1149–1163, Aug. 2012.
- [12] S. Anfinsen and T. Eltoft, "Application of the matrix-variate Mellin transform to analysis of polarimetric radar images," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 6, pp. 2281–2295, Jun. 2011.
- [13] S. Anfinsen, "On the supremacy of logging," in *Proc. PolInSAR*, Frascati, Italy, 2011.
- [14] S. Khan and R. Guida, "Application of Mellin kind statistics to polarimetric $\mathcal{G}$ distribution for SAR data," *IEEE Trans. Geosci. Remote Sens.*, 2012, ID: TGRS-2012-01212, submitted ver. attached.
- [15] J.-M. Nicolas, "Introduction aux statistiques de deuxième espèce: Application des logs-moments et des logs-cumulants à l'analyse des lois images radar," *Traitement du Signal*, vol. 19, no. 3, pp. 139–167, Jun. 2002.
- [16] A. Lopes and F. Sery, "Optimal speckle reduction for the product model in multilook polarimetric SAR imagery and the Wishart distribution," *IEEE Trans. Geosci. Remote Sens.*, vol. 35, no. 3, pp. 632–647, May 1997.
- [17] G. Liu, S. Huang, A. Torre, and F. Rubertone, "The multilook polarimetric whitening filter (MPWF) for intensity speckle reduction in polarimetric SAR images," *IEEE Trans. Geosci. Remote Sens.*, vol. 36, no. 3, pp. 1016–1020, May 1998.
- [18] A. Doulgeris and T. Eltoft, "Scale mixture of Gaussians modelling of polarimetric SAR data," *EURASIP J. Adv. Signal Process.*, vol. 2010, no. 874592, pp. 1–12, Jan. 2010.
- [19] A. Doulgeris, S. Anfinsen, and T. Eltoft, "Classification with a non-gaussian model for PolSAR data," *IEEE Trans. Geosci. Remote Sens.*, vol. 46, no. 10, pp. 2999–3009, Oct. 2008.
- [20] K. Tragl, "Polarimetric radar backscattering from reciprocal random targets," *IEEE Trans. Geosci. Remote Sens.*, vol. 28, no. 5, pp. 856–864, Sep. 1990.
- [21] T. Eltoft, S. Anfinsen, and A. Doulgeris, "A multitexture model for multilook polarimetric radar data," in *Proc. IGARSS*, Vancouver, BC, Jul. 2011, pp. 1048–1051.
- [22] S. Khan and R. Guida, "The new dual-texture $\mathcal{G}$ distribution for single-look PolSAR data," in *Proc. IGARSS*, Munich, Germany, Jul. 2012., in press.
- [23] C. Forbes, M. Evans, N. Hastings, and B. Peacock, *Statistical Distributions*, 4th ed. Hoboken, NJ: John Wiley & Sons, 2010.
- [24] A. M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Argument*. New York: Springer-Verlag, 1997, ch. 6.
- [25] S. Anfinsen, A. Doulgeris, and T. Eltoft, "Estimation of the equivalent number of looks in polarimetric synthetic aperture radar imagery," *IEEE Trans. Geosci. Remote Sens.*, vol. 47, no. 11, pp. 3795–3809, Nov. 2009.
- [26] N. Giri, "On the complex analogues of T^2 - and R^2 -tests," *Ann. Math. Statist.*, vol. 36, no. 2, pp. 664–670, Apr. 1965.
- [27] S. Anfinsen, A. Doulgeris, and T. Eltoft, "Goodness-of-fit tests for multilook polarimetric radar data based on the Mellin transform," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 7, pp. 2764–2781, Jul. 2011.
- [28] J. Lagarias, J. Reeds, M. Wright, and P. Wright, "Convergence properties of the Nelder-Mead Simplex method in low dimensions," *SIAM Journal of Optimization*, vol. 9, pp. 112–147, 1998.
- [29] A. Frery, A. Correia, and C. da Freitas, "Classifying multifrequency fully polarimetric imagery with multiple sources of statistical evidence and contextual information," *IEEE Trans. Geosci. Remote Sens.*, vol. 45, no. 10, pp. 3098–3109, Oct. 2007.

Salman Khan (S'11) was born in 1982 in Lahore, Pakistan. He received B.S. degree in Computer Sciences from National University of Computer and Emerging Sciences, Pakistan in 2004, the M.S. degree in Electrical Engineering as a Fulbright scholar from University of Central Florida, Orlando, U.S.A. in 2009, and is currently in the fourth year of Ph.D. degree in Electronics Engineering (Remote Sensing Applications group) at the Surrey Space Centre, University of Surrey in Guildford, U.K. His current research interests include Statistical Signal

Processing in polarimetric SAR, and its applications in Pattern Recognition and Target Detection.

He is the recipient of IEEE GRSS 2013 J-STARS Prize Paper Award for the paper titled *On Single-Look Multivariate $\mathcal{G}$ Distribution for PolSAR Data*.

Raffaella Guida (S'04–M'08) was born in Naples, Italy, on October 24, 1975. She received the Laurea degree (cum laude) in Telecommunications Engineering and the Ph.D. degree in Electronic and Telecommunications Engineering from the University of Naples Federico II, Naples, in 2003 and 2007, respectively. In 2006, she was a Guest Scientist with the Department of Photogrammetry and Remote Sensing, Technische Universität München, München, Germany. In 2008, she joined the Surrey Space Centre (SSC), University of Surrey, Guildford, U.K.,

as Lecturer in Satellite Remote Sensing. Today she is still in SSC where she leads the Remote Sensing Applications group. Her main research interests are in the fields of electromagnetics and microwave remote sensing, particularly in simulation and modeling of synthetic aperture radar signals relevant to natural surfaces and urban scenes, new remote sensing mission concepts and applications. She is involved as PI and co-I in many national and European research projects.